Coarse-Graining and Renormalization Group in the Einstein Universe

ALFIO BONANNO

Canadian Institute for Advanced Research Cosmology Program,

Theoretical Physics Institute, University of Alberta,

Edmonton, Alberta, Canada T6G 2J1

and

Institute of Astronomy, University of Catania

Viale Andrea Doria 6, 95125 Catania, Italy

Abstract

The Kadanoff-Wilson renormalization group approach for a scalar self-interacting field theory generally coupled with gravity is presented. An average potential that monitors the fluctuations of the blocked field in different scaling regimes is constructed in a nonflat background and explicitly computed within the loop-expansion approximation for an Einstein universe. The curvature turns out to be dominant in setting the crossover scale from a double-peak and a symmetric distribution of the block variables. The evolution of all the coupling constants generated by the blocking procedure is examined: the renormalized trajectories agree with the standard perturbative results for the relevant vertices near the ultraviolet fixed point, but new effective interactions between gravity and matter are present. The flow of the conformal coupling constant is therefore analyzed in the improved scheme and the infrared fixed point is reached for arbitrary values of the renormalized
I. INTRODUCTION

An outstanding issue that has still to be addressed within the theories of a quantum field in curved spacetime, concerns the understanding of the infrared domain of the models. While powerful techniques (see [1] for a review) clarified the renormalizability properties and the structure of the effective Lagrangians describing the physics of the ultraviolet fixed point, the local character of the methods applied is a serious limitation to studying the infrared scaling behavior of a quantum field in a non-trivial topology.

When the system strongly fluctuates over all the scales, the global topology and, especially in cosmology, the dynamic nature of the background spacetime, play a dominant role in determining long-range features of the system. In this case the physics is characterized by the scaling around the infrared fixed point, and the usual set of renormalizable operators may not be enough to build the low-energy effective Lagrangian.

To describe the change in the physics as the cutoff is lowered, we need to follow the evolution of all the coupling constants, both relevant and nonrelevant, as we approach the infrared region. The evolution of the latter slows down when we leave the UV fixed point towards the infrared. However, they need to be followed as well, since they may have a different scaling law in the deep low-energy domain. In particular the classification of the operators close to the UV or the IR fixed point can be radically different. For instance, in flat space, the ultraviolet and the infrared are separated by a crossover region at the mass gap in a massive theory. Below the mass scale fluctuations are strongly damped and the scaling around the infrared fixed point is monitored only by the mass term. But if the theory is massless, the infrared sector is plagued by divergences indicating that a new set of IR relevant operators is needed. The actual form of this scaling operators can be very complicated in terms of local field variables because, just at the IR fixed point, the physics is essentially nonlocal and therefore the global properties of the spacetime affect dramatically the long range behavior of these models.

In a curved spacetime the theory presents a new characteristic length, the local radius of curvature that usually couples with the field as a new, nonhomogeneous, mass gap \(\frac{1}{2}\xi R(x)\phi^2(x)\) where \(R(x)\) is the scalar of curvature. For this reason fluctuations of the field on regions that are comparable with the local radius of the spacetime are damped and eventually, at energies well below the “gravitational” mass contribution, an infrared fixed point can form even for models that are massless at the bare level. In this case the crossover region is determined by the interplay between matter and gravity. In particular one expects that in the strong curvature regime the gravitational contribution is the leading one in monitoring the scaling properties of the theory and in determining the location of the infrared fixed point.

The Kadanoff-Wilson [2] realization of the renormalization group approach used in statistical mechanics allows one to follow the evolution of an infinite number of coupling constants as the cutoff changes. The natural framework for describing fluctuations at different relevant scales is obtained with the block-spin or “coarse-graining” analysis: An average over a group of spins within a characteristic “hypercube” of dimension \((\Lambda/s)^{-d}\) greater than the original
elementary lattice step $\Lambda^{-1}$, defines the new average field. The Hamiltonian is built in terms of the new “blocked” average field by “integrating out” the non-relevant degrees of freedom between $\Lambda$ and $\Lambda/s$. As consequence, this procedure generates all the effective interactions (see [4] for an introduction). Then by a global rescaling of the fields to the previous value of the cutoff, the action undergoes a renormalization group transformation that governs an infinite set of couplings.

In field theory the renormalization group equations are derived from the scaling behavior of the Green’s functions. The relation between the bare and the renormalized vertex functions,

$$
\Gamma^{(n)}_{B}(x^{1}, \ldots, x^{n}; \lambda^{n}_{B}, \Lambda) = Z^{-n/2}(\lambda^{n}_{B}, \Lambda/\mu)\Gamma^{(n)}_{\text{ren}}(x^{1}, \ldots, x^{n}; \lambda^{n}_{\text{ren}}, \mu)
$$

where $\Lambda$ and $\mu$ are the cutoff and a low-energy scale respectively, can be read either in terms of the renormalized or the bare couplings. In one case, by imposing the invariance of the renormalized vertex functions under the action of the diffeomorphisms generated by the vector field $\Lambda \partial/\partial \Lambda$ in the parameter space, at fixed $\lambda^{n}_{\text{ren}}$ and $\mu$, one obtains the renormalization group equation in the bare scheme. Conversely, the nondependence of the bare vertices from the low energy scale $\mu$ at fixed bare couplings expresses the renormalization group equations for the renormalized quantities. The two schemes can be formulated both in a momentum as well as in a coordinate space representation, but the renormalized scheme has a more clear physical meaning because it follows the evolution of physical, renormalized quantities.

In a nonflat background the loss of Poincaré invariance prevents one from defining a global momentum space. Thus, as suggested by Nelson and Panangaden [5] the ultraviolet flow of a generic interacting quantum field can be analyzed by looking at the scaling behavior of the field under a rescaling of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu}/s$. This can be intuitively regarded as a rescaling of the geodesic distance between two points, and it is equivalent to introducing a local Riemann coordinate when the points in the argument of the Green’s functions become close. The derivation of the renormalization group equations in a curved spacetime strictly resembles the procedure above outlined for the flat case. This is not surprising because the very-high-frequency modes will be increasingly insensitive to the local curvature as the cut-off tends to infinity. But by construction this approach can be reliable only to follow the evolution of the UV relevant coupling constants.

In recent works the intimate connection between renormalization group flow in the Kadanoff-Wilson formulation and the usual concept of renormalizability in field theory has been deeply discussed (see [6]) and several efforts have been done to investigate the possibility of formulating the block-spin approach in the framework of a quantum field theory.

The analogue of the Kadanoff transformation for the field variable $\phi(x)$ is in principle constructed by means of the blocked field, the average of the original field in the characteristic volume $V_{d}$

$$
\phi_{V_{d}} = \frac{1}{V_{d}} \int_{V_{d}} d^{d}x \phi(x).
$$

One is interested in studying the $V_{d}$ dependence of the distribution for the blocked field

$$
\eta(\Phi) = \langle \delta(\Phi - \frac{1}{V_{d}} \int_{V_{d}} d^{d}x \phi(x)) \rangle = \int D[\phi] e^{-S[\phi]} \delta(\Phi - \phi_{V_{d}}).
$$
An equivalent realization can be carried out in a momentum-space by defining the coarse-grained field as

$$\phi_k(x) = \int d^d y \rho_k(x - y) \phi(y)$$

(4)

where the smearing function $\rho_k(x)$ acts as a projector on the low-energy region of momenta $q < k$ and in the configuration space rapidly decays for $1/k < |x - y|$. This latter alternative scheme turns out to be more suited to a perturbative analysis. In fact, by using an O(4) smearing function of the kind (4), Wetterich [7] formulates an “average” effective action describing the fluctuations of the field average as the “observational” region $1/k^4$ changes (see [8]). In a more recent work Liao and Polonyi [9,10] explicitly construct the Kadanoff transformation and calculate the renormalized action non perturbatively, with the loop-expansion. Their approach has the advantage of generating the renormalized action fully, with either relevant and irrelevant terms as well, and clarifies in what respect the two formulations of the renormalization group in statistical mechanics and field theory are equivalent.

A realization of the “coarse-grained” effective action in curved spacetime was first given by Hu [11] to discuss the back reaction problem in the inflationary cosmological models, and by Sinha and Hu [12] to examine the validity of the minisuperspace approximation in quantum cosmology.

In this work a formulation of the Kadanoff-Wilson renormalization group approach in curved space in the case of slowly varying background manifold is presented. The approach, borrowed from [7,9], represents an improvement of the perturbative traditional scheme. Explicit expressions of the blocked action are given, in the case of the Einstein universe, within the loop-expansion approximation at one loop by means of the generalized [13] $\zeta$-function method. When the expansion in the kinetic energy is possible, then a local potential $U_n(\Phi)$ governs fluctuations that have been averaged in the characteristic space volume $\Omega_n \sim (a/n)^3$ - here $a$ is the radius of the Universe. It is found that the distribution of the block variables

$$\eta_n(\Phi) = \langle \delta(\Phi - \phi_n(x)) \rangle \approx e^{-\Omega_n U_n(\Phi)},$$

(5)

where $t$ is the time extent of the system, is strongly peaked around two non symmetric vacua as the “observational” space volume becomes small enough. This confirms the flat space result of [9] supporting the conjecture that the field distribution presents a domain structure even in the symmetric phase. We find that in the strong curvature regime this effect is more pronounced and the crossover scale approaches the infrared as the curvature increases.

The renormalization group analysis is employed by a differential equation for the local potential. The behavior of the $\beta$ functions near the UV fixed point reproduces the perturbative flow for the relevant vertices. But the renormalization group approach based on the coarse-graining procedure traces the evolution of all the new interaction terms arising from the coupling of the field with gravity as we move towards the low energy region. In particular, the flow of the conformal coupling is analyzed in the improved scheme near the infrared end of the UV region. In particular, the infrared fixed point for a massive model is present for arbitrary renormalized coupling strength, and $\xi_R = 1/6$ is not an infrared attractor.

The organization of the work is the following. In Sec. I the general lines of the blocking transformation in the case of static, spatially almost homogeneous spacetime are presented.
In Sec. II the specific case of a closed Einstein Universe is treated in detail. In Sec. III the renormalization group equation is formulated and the infrared scaling around the UV fixed point is discussed. Sec. IV is devoted to the conclusions while the relevant details of the computation are given in the Appendix.

II. BLOCKING TRANSFORMATION

Let the Euclidean section Ω of the spacetime be a compact, boundaryless, Riemannian manifold. The Euclidean bare action for the matter field reads:

\[ S[\phi(x)] = \int d\Omega(x) \left\{ \frac{1}{2}(\nabla_\mu \phi(x))^2 + V(\phi(x)) \right\} \]

where the bare potential is given by

\[ V(\phi(x)) = \frac{1}{2} m_B^2 \phi^2(x) + \frac{1}{2} \xi_B R(x) \phi(x)^2 + \frac{1}{4!} \lambda_B \phi^4(x) \]

\( d\Omega(x) = g^{1/2} d^d x \) is the invariant four-volume, \( g \) is the determinant of the Euclidean metric on the manifold, \( \phi \) is a real scalar field and \( R \) is the Ricci scalar. The Laplace-Beltrami operator acting on scalars is

\[ \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}g^{\mu \nu} \partial_\nu). \]

The spectrum of this operator is discrete; the eigenfunctions \( \psi_n(x) \) form a complete set in the space of the square integrable functions \( L^2(\Omega) \), and satisfy

\[ \nabla_\mu \nabla^{\mu} \psi_n(x) = -\omega_n^2 \psi_n(x), \]

with the orthonormality condition

\[ \int_\Omega d\Omega(x) \psi_n(x) \psi_m^*(x) = \delta_{nm} \]

where the asterisk means complex conjugation. A generic element in \( L^2(\Omega) \) reads

\[ \phi(x) = \sum_{n=0}^{+\infty} \epsilon_n[\phi] \psi_n(x), \]

where in this concise notation the summation is understood even in subspaces spanned by degenerate eigenfunctions (if present). \( n = 0 \) corresponds to the homogeneous mode, constant function on \( \Omega \).

We define a new field, analogous to the block-spin variable introduced in statistical mechanics, \( \phi'(x) = \phi_n(x) \), in the following way:

\[ \phi_n(x) = \int d\Omega(x') \rho_n(x, x') \phi(x'), \]

where
\[ \rho_n(x, x') = \sum_{m=0}^{m=n} \psi_m(x) \psi^*_m(x') \]  

(13)

is a generalized smearing function that introduces a sharp cutoff on the short distance modes. In general we could have chosen any smooth test function that, compatibly with the generic group of symmetries of our spacetime, is almost constant within a given volume \( \Omega_n \) where the field is averaged, and rapidly decays outside that region. It should be stressed that different realizations of the Kadanoff transformation could in principle be achieved by means of different smearing functions but the scaling properties near the fixed points will be independent of how the modes have been eliminated. We note that \( \phi_n(x) \to \phi(x) \) when \( n \to \infty \) and, conversely, \( \phi_n(x) \to \phi_0 \) when \( n \to 0 \), being \( \phi_0 \) the homogeneous mode. Moreover, under integration \( \rho_n(x, x') = \rho_n(x', x) \).

The average action is therefore defined as:

\[
e^{-S_n[\phi'(x)\cdot]} = \int D[\phi] \prod_x \delta(\phi_n(x) - \phi'(x)) e^{-S[\phi(x)]}
\]

(14)

in this way the field \( \phi \) is constrained to have an average \( \phi_n(x) \) equal to a given configuration of the block variable \( \phi'(x) \). This amounts to performing the path integration in Eq. (14) on the “fast” variables, the \( m \geq n + 1 \) modes. We define the functional generator of the Green’s functions for the blocked field:

\[
Z[J] = \int D[\phi'] e^{-S_n[\phi'(x)\cdot]+\int d\Omega J(x) \phi'(x)} = \int D[\phi] e^{-S[\phi(x)]+\int d\Omega(x)J(x)\phi_n(x)}
\]

(15)

therefore the partition function \( Z \) is invariant under the blocking procedure. This property implies that

\[
e^{-S_n[\phi'(x)\cdot]}
\]

(16)

represents the relative probability distribution for the blocked field \( \phi'(x) \), the average of the original field \( \phi(x) \) to the characteristic volume \( \sim 1/\omega_n^4 \), associated with the \( n \)-eigenmode considered. The mean value of a generic field operator for the blocked system is related with the previous one in the following manner:

\[
\langle O[\phi'] \rangle = Z^{-1} \int D[\phi'] O[\phi'] e^{-S[\phi']}
\]

\[
= Z^{-1} \int D[\phi] D[\phi'] O[\phi'] \prod_x \delta(\phi_n(x) - \phi'(x)) e^{-S[\phi]}
\]

\[
= Z^{-1} \int D[\phi] O[\phi_n] e^{-S[\phi]} = \langle O[\phi_n] \rangle
\]

(17)

This means that \( \langle O[\phi'] \rangle = \langle O[\phi] \rangle \) for all the operators that are functions of the zero-momentum component of the field. Thus the mean value of the field is preserved by the blocking transformation: \( \langle \phi' \rangle = \langle \phi \rangle \). Similarly we see that for the \( p \)-point Green’s functions,

\[
\langle \phi'(x_1)\ldots\phi'(x_p) \rangle \sim \langle \phi(x_1)\ldots\phi(x_p) \rangle
\]

(18)

if \( n \) is sufficiently high.
The blocking procedure generates in (14) all the effective interactions corresponding
to the low-energy \( m \leq n \) in a generic static spacetime. However, in order to perform
explicit calculations we consider the case of almost-homogeneous space sections: If the
disomogeneities of the background manifold are “smooth” within the characteristic regions
in which the original field has been average out (one can always think to the limiting case
of homogeneous space sections) then it is possible to carry out a local derivative expansion
around any given point on the manifold (see [14]):

\[
\phi'^2(x) = \phi'^2(x') + \nabla_\mu \phi'^2(x') (x-x')^\mu + \frac{1}{2} \nabla_\mu \nabla_\nu \phi'^2(x') (x-x')^\mu (x-x')^\nu + ...
\]  

(19)

Therefore we write, according to the Landau-Ginzburg lesson, the following functional form
for the new free-energy:

\[
S_n'[\phi'] = \int d\Omega(x) \sum_i U_n[i] (\nabla_\mu \phi'(x))
\]  

(20)

where \( U_n[i] (\nabla_\mu \phi'(x)) \) is a general polynomial of degree \( i \) in the covariant derivative \( \nabla_\mu \). Note
that the gradient expansion becomes more accurate as we scale in the infrared. To study
the fluctuations around the \( \phi'(x) = \phi_n(x) \) configuration, one rewrites the operator projector
kernel in (14) as

\[
\prod_x \delta(\phi_n(x) - \phi'(x)) = \exp - M^2 \int d\Omega(x) (\phi_n(x) - \phi'(x))^2,
\]  

(21)

where \( M \) is a massive constant that in a more refined analysis could be mode dependent. In
this case it represents a measure for the mean deviation from the \( \phi_n(x) = \phi'(x) \) configuration and it is taken as being far greater than all the relevant masses in the theory at the end of
the computation. The constrained action in (14) reads

\[
e^{-S_n[\phi'(x)]} = \int D[\phi] e^{-S_n[\phi(x),\phi'(x)]}
\]  

(22)

where

\[
S_n[\phi, \phi'] = \int d\Omega(x) \{ \frac{1}{2} (\nabla_\mu \phi(x))^2 + \frac{1}{2} m_B^2 \phi^2(x) + \frac{1}{2} \xi_B R \phi^2(x) + \frac{1}{4!} \lambda_B \phi^4(x) + \\
+ M^2 (\phi_n(x) - \phi'^2(x)) \}
\]  

(23)

To identify the non derivative terms generated in the renormalized action it is enough to
evaluate (20) for a constant field configuration \( \phi'(x) = \Phi \) so that the renormalized action
has the functional form:

\[
S_n'[\Phi] = \Omega U_n(\Phi) \quad U_n(\Phi) = \frac{1}{2} \mu(n) \Phi^2 + \frac{\lambda(n)}{4!} \Phi^4 + ...
\]  

(24)

where \( U_n(\Phi) \) is the average potential. Equation (23) becomes

\[
S_n[\phi, \Phi] = \int d\Omega(x) \{ \frac{1}{2} (\nabla_\mu \phi(x))^2 + \frac{1}{2} m_B^2 \phi^2(x) + \frac{1}{2} \xi_B R \phi^2(x) + \frac{1}{4!} \lambda_B \phi^4(x) + \\
+ M^2 \Phi^2 - 2 M^2 \phi(x)\Phi + M^2 \int d\Omega(x') d\Omega(x') \rho_n(x, x') \phi(x) \phi(x').
\]  

(25)
We evaluate the renormalized action in the loop expansion. In choosing the saddle point in (22) we have to locate the minimum of (25), obtaining the nonlocal motion equation

$$0 = \frac{\delta S_n[\phi, \Phi]}{\delta \phi(x)} = -\nabla_\mu \nabla^\mu \phi(x) + m_B^2 \phi(x) + \xi_B R \phi(x) + \frac{\lambda_B}{3!} \phi(x)^3 + 2M^2 \int d\Omega(x') \rho_n(x', x) \phi(x') - 2M^2 \Phi. \tag{26}$$

Consistently with the hypothesis of weak disomogeneities, we assume that in this approximation \([\nabla_\mu \nabla^\mu, R(x)]\) \(\sim 0\), so that Eq. (26) for a generic constant field configuration \(\phi(x) = \phi_0\) becomes

$$\left(2M^2 + m_B^2 + \xi_B R + \frac{1}{3!} \lambda_B \phi_0^3\right) \phi_0 = 2M^2 \Phi. \tag{27}$$

By substituting in (25) one finds

$$U_n(\Phi) = -\frac{\lambda_B}{4!} \phi_0 + M^2 \Phi(\Phi - \phi_0). \tag{28}$$

Therefore at the tree level in this case the effect of “coarse graining” is not present and we recover the classical potential

$$U_n(\Phi) = \frac{1}{2} m_B^2 \Phi^2 + \frac{1}{2} \xi_B R \Phi^2 + \frac{\lambda_B}{4!} \Phi^4 + O(1/M^2). \tag{29}$$

To include the quantum corrections one writes \(\phi(x) = \phi_0 + \chi(x)\) where \(\chi(x)\) represents the quantum fluctuations. At the one-loop order in the quantum corrections the renormalized potential is given by

$$S_n'[\Phi, \chi] = \Omega V(\Phi) + \Delta S_n[\Phi, \chi], \tag{30}$$

where

$$\Delta S_n[\Phi, \chi] = \frac{1}{2} \int d\Omega(x) d\Omega(x') \left[ \frac{\delta^2 S_n}{\delta \phi(x) \delta \phi(x')} \right]_{\Phi} \chi(x) \chi(x'). \tag{31}$$

with

$$\frac{\delta^2 S_n}{\delta \phi(x) \delta \phi(x')} \bigg|_{\Phi} = (-\nabla_\mu \nabla^\mu + m_B^2 + \xi_B R + \frac{\lambda_B}{2} \Phi^2) \delta(x, x') + 2M^2 \rho_n(x, x'). \tag{32}$$

The contribution of loops with internal quantum numbers \(m \leq n\) is suppressed by a factor \(1/M^2\); thus the renormalized potential at one loop in the loop-expansion can be rewritten

$$U_n(\Phi) = V(\Phi) + \frac{1}{2\Omega} \ln \text{Det} \left[ \frac{\delta^2 S_n}{\delta \phi(x) \delta \phi(x')} \right]_{\Phi}$$

$$= V(\Phi) + \frac{1}{2\Omega} \text{Tr}_{n+1} \ln [(-\nabla_\mu \nabla^\mu + m_B^2 + \xi_B R + \frac{\lambda_B}{2} \Phi^2) \delta(x, x')], \tag{33}$$

where \(\text{Tr}_{n+1}\) means that in the trace operation only modes with \(m \geq n+1\) are retained. We set \(n+1 \rightarrow n\) so that when \(n = 0\) expression (33) matches the familiar effective potential \([3,8]\) defined usually by means of a Legendre transform of the Schwinger function. We observe that the spectrum of \(\delta^2 S_n / \delta \phi(x) \delta \phi(x') \big|_{\Phi}\) is always positive if \(\xi > 0\) and the Ricci scalar is non-negative. This ensures that the path integral is dominated by homogeneous field configurations. In the following we consider \(\xi > 0\).
III. EINSTEIN UNIVERSE

If we work initially at finite temperature, the Einstein universe has topology $S^1 \times S^3$. The high level of symmetry of this spacetime allows one to perform explicit calculations. Let the Euclidean metric be

$$ds^2_E = d\tau^2 + a^2 [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)]$$

where $\chi, \theta, \varphi$ label $S^3$, $a$ is the radius, and $\tau$ spans the euclidean time. In this case the eigenfunctions of the Laplace-Beltrami operator are known: they factorize in the product of plane waves $e^{-2\pi i n \tau / \beta}$ - here $\beta$ is the invariant time - and the hyperspherical functions on $S^3$:

$$Y_{nlm}(\chi, \theta, \varphi) = N_{nm} Y_{lm}(\theta, \varphi) \sin^l \chi C_{n-l}^{l+1}(\cos \chi)$$

where $N_{nm}$ is a normalization factor, $C_l^m(x)$ represents a Gegenbauer polynomial, and is $Y_{lm}(\theta, \varphi)$ the usual spherical harmonic on $S^2$. The quantum numbers $\{n, l, m\}$ are $n = 0, 1, 2, ..., l = 0, 1, 2, ..., n; m = -l, -l+1, ..., l-1, l$. The eigenvalues of the Laplace-Beltrami operator are found to be $\omega^2(n, p) = (2\pi p / \beta)^2 + \lambda_{nm}^2$ where $\lambda_{nm}^2 = a^{-2}((n+1)^2 - 1) + m_B^2 + \xi_B R + \frac{1}{2} \Phi^2$ are the eigenvalues on the spatial sections with degeneration $d = \Sigma_{l=0}^{n} (2l+1) = (n+1)^2$. Here $a^2 = 6/R$ is the radius of $S^3$.

In this manifold a possible realization of the “coarse graining” procedure can be achieved by integrating out the fast fluctuating space modes on $S^3$ and by leaving unconstrained the time evolution of the system. This case is important because it allows us to study the spatial distribution of the fluctuations when the observational characteristic volume shrinks. To this aim we use the following smearing function (see [10] for an analogous $O(3)$ invariant coarse-graining in flat space)

$$\rho_n(P, P') = \sum_{n'=0}^n \sum_{l=0}^{n'} \sum_{m=-l}^{l} Y_{nlm}(P) Y^*_{nlm}(P')$$

and $P, P' \in S^3$. The functional determinant in expression (33) is calculated by means of the generalized zeta-function $\zeta(s)$:

$$U_n(\Phi) = V(\Phi) + \frac{1}{2\Omega} (-\zeta'(s)|_{s=0})$$

where $\Omega = 2\pi^2 / 3 a^3$ is the total volume. In our case, in the low-temperature limit we have

$$\zeta(s) = \frac{\beta \mu^{2s}}{2\pi} \int dp \sum_{n'=0}^{n} \sum_{l=0}^{n'} \sum_{m=-l}^{l} (\omega^2(p, n) + 4 M^2 \pi \delta(p) \Theta_{n,n'})^{-s}$$

with $\Theta_{n,m} = 1$ for $m \leq n$ and zero elsewhere. Explicit evaluation of expression (38) produces a regularized expression for the local potential that is dependent on the undetermined constant $\mu$ (see the appendix for the relevant details). One finds

$$U_n(\Phi) = V(\Phi) - \frac{1}{64 \pi^2 a^4} \{(\check{V}'(\Phi) - 1)^2 [\ln \frac{\mu^2}{R/6} + \frac{3}{2}] + \gamma_n(\Phi)\}$$
where now the range of $n$ starts at $n = 1$. In the above formulas $\tilde{V}''(\Phi) = a^2 V''(\Phi) = a^2 (m^2_B + \xi B R + \lambda B \Phi^2/2)$ and $\gamma_n(\Phi)$ is a function of $\Phi$ and $R$ whose explicit form is given in the Appendix and whose low curvature limit $\tilde{V}''(\Phi) \gg 1$, for $n = 1$ is

$$\frac{R^2}{144 \pi^2} \gamma_1(\Phi) \sim \frac{1}{64 \pi^2} (V''(\Phi) - R/6) \ln \left( \frac{V''(\Phi) - R/6}{R/6} \right)$$

(40)

A subtraction scale is chosen in order to eliminate the $\mu$ dependence. One writes $m^2_B = m^2_R + \delta m^2$, $\xi_B = \xi_R + \delta \xi$, $\lambda_B = \lambda_R + \delta \lambda$ in the bare expression (39), and introduces the following renormalization conditions for the counterterms:

$$m^2_R = \frac{\partial^2 U_{n=1}}{\partial \Phi^2} \bigg|_{\Phi = R = 0}, \quad \xi_R = \frac{\partial^3 U_{n=1}}{\partial \Phi^2 \partial R} \bigg|_{\Phi = R = 0}, \quad \lambda_R = \frac{\partial^4 U_{n=1}}{\partial \Phi^4} \bigg|_{\Phi = R = 0}$$

(41)

where for the sake of simplicity the renormalization point has been chosen in $R = 0$. From expression (40) and from Eq. (41) (see Appendix) we have

$$U_n(\Phi) = \frac{1}{2} m^2_R \left[ 1 + \frac{\xi R}{m^2_R} - \frac{\lambda R}{64 \pi^2} \left( 1 + \frac{3}{2} \frac{3 (\xi_R - 1/6)}{m^2_R} \right) \Phi^2 + \frac{\lambda R}{4!} \left( 1 - \frac{9}{64 \pi^2} \right) \Phi^4 + \frac{R^2 \mathcal{I}_n(\Phi)}{144 \pi^2} \right]$$

$$+ \frac{1}{32 \pi^2} \left( m^2_R + \left( \xi_R - \frac{1}{6} \right) R + \frac{1}{2} \lambda R \Phi^2 \right)^2 \ln \frac{R}{6} + \sqrt{m^2_R + (6 \xi R + n^2 - 1) \frac{R}{6} + \frac{\lambda R}{2} \Phi^2} \left( m^2_R + (6 \xi R + 2 n^2 - 4 n - 1) \frac{R}{6} + \frac{\lambda R}{2} \Phi^2 \right)$$

(42)

where the following counterterms have been introduced

$$\delta m^2 = - \frac{\lambda R}{32 \pi^2} m^2_R (\ln \frac{m^2_R}{\mu^2} - 1), \quad \delta \xi = - \frac{\lambda R}{32 \pi^2} (\xi_R - \frac{1}{6}) \ln \frac{m^2_R}{\mu^2}, \quad \delta \lambda = - \frac{3 \lambda R}{32 \pi^2} \ln \frac{m^2_R}{\mu^2}$$

(43)

and $\mathcal{I}_n(\Phi)$ is an integral function of the field and the renormalized parameters - see Appendix. We note that for $R = 0$ expression (42) reproduces the standard Coleman-Weinberg result [3]. As shown in the appendix, the local potential for $n = 1$ matches the form of the effective potential in the Einstein universe: there is no spontaneous symmetry breaking induced by curvature if $\xi_R > 0$ (note that expression (43) is always real), even though the quantum contribution tends to drive the system in the spontaneously broken phase [15]. However as $n$ increases the new action governs the distribution of the field whose average has been performed within the characteristic space volume $\Omega_n \sim (a/n)^3$. In general the location of the minimum in the blocked action depends on the observation domain $\Omega_n$. It is natural to expect that when the average is performed over regions far larger than the characteristic regions in which the fluctuations of the field are correlated, the details of the original distribution are not yet detectable and statistical equilibrium of the system is reached in $\Phi = 0$: This is encoded in the convex character of $U_n(\Phi)$ for low $n$. But when the “observational” region is small enough, the structure of the original field variables may show up. In fact a numerical
investigation of (42) shows that after some critical value of \( n = n_\sigma \), the blocked potential is no longer convex, and non trivial minima appears – see Fig.1 and Fig.2. Below that scale the convexity of the potential is recovered. In the case where the curvature is not strong, one can gain more insight into this phenomenon by introducing a local momentum space around a generic point in the manifold and examining the Minakshisundaram-Pleijel-DeWitt expansion [19,16] to first order. This is possible because the presence of non trivial minima occurs for high values of \( n \), for small \( R \), and therefore we expect the local-momentum approximation to hold (for example one finds \( n_\sigma \sim 10^3 \) for \( R/m_R^2 \sim 10^{-2} \)). In this case is convenient to use the translation-invariant smearing function

\[
\rho_k(x) = \int_{|p|<k} \frac{d^4p}{(2\pi)^4} e^{-ipx}
\]  

(44)

to define the blocked field (12), and the field is averaged in a region of extent \( \sim 1/k^4 \). By using a cut-off regularization procedure, the one-loop blocked potential is given by

\[
U_k(\Phi) = V(\Phi) + \frac{1}{64\pi^2} \int_k^\Lambda dp^3 \ln \left( 1 + \frac{V''(\Phi) - R/6}{p^2} \right)
\]  

(45)

and Eqs. (41) give the following counterterms

\[
\delta m^2 = -\frac{\lambda R}{32\pi^2} \left( \Lambda^2 + m_R^2 \ln \frac{m_R^2}{\Lambda^2} \right) \quad \delta \xi = -\frac{\lambda R}{32\pi^2} \left( \xi_R - \frac{1}{6} \right) \left( 1 + \ln \frac{m_R^2}{\Lambda^2} \right) \quad \delta \lambda = -\frac{3\lambda_R^2}{32\pi^2} \left( 1 + \ln \frac{m_R^2}{\Lambda^2} \right)
\]  

(46)

The average potential in this approximation reads

\[
U_k(\Phi) = \frac{1}{2} m_R^2 \left[ 1 + \frac{\xi_R R}{m_R^2} - \frac{\lambda_R}{64\pi^2} \left( 1 + \frac{k^2 + 3R(\xi_R - 1/6)}{m_R^2} \right) \right] \Phi^2 + \frac{1}{4!} \lambda_R \left( 1 - \frac{9\lambda_R}{64\pi^2} \right) \Phi^4
\]  

\[
+ \frac{1}{64\pi^2} \left[ \left( m_R^2 + \frac{1}{2} \lambda_R \Phi^2 + \left( \xi_R - \frac{1}{6} \right) R \right)^2 - k^4 \right] \ln \frac{k^2 + m_R^2}{m_R^2} + \frac{1}{2} \lambda_R \Phi^2 + \left( \xi_R - \frac{1}{6} \right) R
\]  

(47)

Let us consider the dimensionless running parameter

\[
\frac{1}{m_R^2} \frac{\partial^2 U_k(\Phi)}{\partial \Phi^2} \bigg|_{\Phi=0} = \mathcal{M}^2(k)
\]  

(48)

In the limit of \( k^2 \gg m_R^2 + \xi_R R \) one finds

\[
\mathcal{M}^2(k) = 1 + \frac{\xi_R R}{m_R^2} - \frac{\lambda_R k^2}{32\pi^2} + \frac{\lambda_R}{32\pi^2} \left[ 1 + \left( \xi - \frac{1}{6} \right) \frac{R}{m_R^2} \right] \ln \frac{k^2}{m_R^2}
\]  

(49)

It is interesting to observe the competition between the two scales \( k \) and \( R^{1/2} \). When the logarithmic term is dominant in (49) \( \mathcal{M}^2(k) \) is positive for \( \xi_R > 0 \), and the only minimum for (47) is at \( \Phi = 0 \). But as the observational scale shrinks, the linear term in \( k^2/m_R^2 \) governs, and the local potential presents two degenerate non symmetric minima at some \( k = k_\sigma \).
Conversely, if the matter field is strongly coupled with gravity at classical level $\xi_R \gtrsim 1$, the linear term in $\xi_R R / m_R^2$ may eventually dominate, and therefore for some $R = R_{\text{cr}}$ it will turn the maximum in $\Phi = 0$ into a minimum. The value of $R_{\text{cr}}$ in this approximation can be read from $\mathcal{M}(k) = 0$

$$R_{\text{cr}}(k) = \frac{-m_R^2 + \frac{\lambda_R}{32\pi^2} \left( k^2 - m_R^2 \ln \frac{k^2}{m_R^2} \right)}{\xi_R - \frac{\lambda_R}{32\pi^2} \left( \xi_R - \frac{1}{6} \right) \ln \frac{k^2}{m_R^2}}$$  \hspace{1cm} (50)

At the leading order, for a conformally coupled field one finds

$$R_{\text{cr}}(k) \approx \frac{3\lambda_R k^2}{16\pi^2} - m_R^2$$  \hspace{1cm} (51)

The expression (50) is independent of the background spacetime but it is reliable only in the small curvature approximation $R \ll k^2$ where the local-momentum analysis works if the coupling strength $\lambda_R$ is weak enough - see relation (51). The new minima in this approximation are located at

$$\Phi^2 = m_R^2 \frac{-1 - \xi_R}{m_R^2} + \frac{\lambda_R}{32\pi^2} \left( \frac{k^2}{m_R^2} - \left( 1 + (\xi_R - 1/6) \frac{R}{m_R^2} \right) \ln \frac{k^2}{m_R^2} \right)$$  \hspace{1cm} \left(1 - \frac{3\lambda_R}{32\pi^2} \right) + \frac{\lambda_R}{64\pi^2} \ln \frac{k^2}{m_R^2} \right) \hspace{1cm} (52)

The presence of degenerate minima in the local potential at a particular value of $n$ and $R$, is a different phenomenon from the spontaneous symmetry breaking, peculiar to the lowest energy mode. As clarified by [9], it can be argued that the field presents a domain structure even in the purely symmetric phase $\langle \phi \rangle_{\text{vac}} = 0$. The path integral is dominated by domains of characteristic size $\iota$ where $\phi(x) \sim \pm \Phi$. Then, if we average the field over regions of linear extension $k^{-1} \gg \iota$ the fluctuations between the domains become uncorrelated and the distribution of the blocked variables is centered around $\Phi \sim 0$. But as the observational scale is comparable with the size $k_{\text{cr}}^{-1} \sim \iota$ of the domains, the fluctuations are governed by a distribution that is strongly peaked around two non-symmetric degenerate values $\pm \Phi$ of the field. In other words, as we move from the ultraviolet towards the infrared domain, a crossover between a double peak and a symmetric distribution occurs. When we average in larger regions the Gaussian limit is approached, according to the central limit theorem, and, since the correlations among domains typically decay as $\sim e^{-m/k}$ where $m$, the mass gap, is the inverse of the correlation length, one concludes that $k_{\text{cr}} \sim m$.

In curved spacetime the other relevant length is the local curvature $R$ that couples quadratically with the field at the classical level, and enters in the propagator as a pure geometric contribution as well. If $R \gg m_R^2$, the local-momentum approximation is questionable. Unless exact expressions for the propagator are available, one has to resort to alternative approaches, by taking into account the global topology of the spacetime, to describing the running of the “effective” correlation length $1/\mathcal{M}(n)m_R$ in the strong curvature regime. In the case of the Einstein universe the large $R$ and large $n$ approximation give, to the leading order,

$$\mathcal{M}^2(n) = 1 + \frac{R}{m_R^2} \left( \xi_R - \frac{\lambda_R n(n-1)}{96\pi^2} \right) = 0$$  \hspace{1cm} (53)
for the critical scale. The crossover region is therefore shifted toward the infrared domain by an increase of curvature as the coefficient of $R/m_R^2$ becomes negative - see Figs.1 and 2. This result is interesting, we believe, because it illustrates how the geometry of the manifold determines the interplay between the two relevant lengths $m_R$ and $R$ in setting the crossover scale. Note that $n$ in this case cannot be approximated by a continuous variable. For instance, $\lambda_R = 0.1$ and $R/m_R^2 \sim 1$, eq.(53) yields $n_{cr} = 138$.

IV. RENORMALIZATION GROUP ANALYSIS

The invariance of the partition function $Z[J]$ under coarse-graining yields the following equation for the local potential

$$U_n(\Phi') - U_{n-\Delta n}(\Phi') = \Delta n \Gamma_n[U_n(\Phi')] \tag{54}$$

where $\Gamma[U_n]$ represents the contribution of the eliminated modes from the original action in the interval $(n, n - \Delta n)$. This ensures [6] that the generator of the Green’s functions $Z[J]$ and its functional derivatives are unchanged when the modes are eliminated in the starting action. Equation (54) governs the scaling of all the new interaction terms generated by the blocking procedure to compensate for the eliminated modes. In a nonflat background the explicit form of $\Gamma_n$ is generally dependent on the geometry of the manifold as we approach the deep infrared region. Since, for a compact manifold, the spectrum of the Laplace-Beltrami operator is discrete, Eq. (54) becomes a partial finite-difference equation for the local potential whose solution close to the low $n$, region determines in principle the relevant operators at the infrared fixed point. At one loop in the loop expansion we see that the contribution of the eliminated modes in the “momentum shell” $(n, n - \Delta n)$ is

$$\Gamma_n[U_n(\Phi)] = \frac{1}{2\Omega} \ln[\omega_n^2 + \partial^2_\Phi U_n(\Phi)] \tag{55}$$

Although a solution of (54) would in principle tell us how the scaling properties of the theory change as we approach the infrared, it should be stressed that the contribution of (55) may not be the leading one in determining the effect of the integrated modes on the evolution of the coupling constants. In general higher order contributions in the loop expansions have to be retained as well. It is however important to study the infrared scaling in the UV region to show in what respect the standard realization of the renormalization group, based on the analysis of the scaling properties of the Green’s function under a geodesic interval [5], and this formulation, obtained by the blocking, are equivalent. To this aim we can work in the local-momentum approximation and in this limit (54) reduces to

$$k \frac{dU_k}{dk} = -\frac{k^4}{16\pi^2} \ln \left(1 + \frac{\partial^2_\Phi U_k}{k^2} - \frac{R/6}{k^2}\right) \tag{56}$$

The blocking procedure generates new effective vertices arising from the coupling of the matter field with gravity. Let us write the potential term in the general form

$$U_k(\Phi, R) = \sum_{n=2}^{\infty} \frac{\lambda_n + \xi_n R}{n!} \Phi^n, \tag{57}$$
and introduce, for the following analysis, the dimensionless variables

\[ R = \bar{R}k^2, \quad \Phi = \bar{\Phi}k, \quad \lambda_n = \bar{\lambda}_n k^{4-n} \quad \xi_n = \bar{\xi}_n k^{2-n} \]  

(58)

The running coupling constants are defined

\[ \xi_n(k) = \left. \frac{\partial^{n+1} U_k(\Phi, R)}{\partial \Phi^n \partial R} \right|_{\Phi = R = 0} \quad \lambda_n(k) = \left. \frac{\partial^n U_k(\Phi, R)}{\partial \Phi^n} \right|_{\Phi = R = 0} \]  

(59)

Equation (56) in terms of those variables reads:

\[ [k \frac{\partial}{\partial k} - \bar{\Phi} \frac{\partial}{\partial \Phi} - 2\bar{R} \frac{\partial}{\partial R} + 4] \bar{U}_k(\Phi, \bar{R}) = -\frac{1}{16\pi^2} \ln [1 + \partial^2 \bar{U}_k(\Phi, \bar{R}) - \bar{R}/6] \]  

(60)

where \( \Gamma = k^4 \bar{\Gamma} \) and \( \bar{U} = k^4 \bar{U} \). We saw that at the tree level \( \Gamma = 0 \) therefore (60) gives the solution \( \bar{\lambda}_n = k^{n-4} C_n \); \( \bar{\xi}_n = k^{n-2} C'_n \); thus for \( n > 4 \) all the \( \bar{\lambda}_n \), and for \( n > 2 \) all the \( \bar{\xi}_n \) couplings are irrelevant. The above procedure could be generalized to describe a potential containing more generic interaction terms of the form \( \chi_{n,m} R^n \Phi^m \), however at the tree level a simple dimensional analysis suffices to classify the interactions. In the deep UV region is \( k^2 \gg \partial^2 \bar{U}_k(\Phi, \bar{R}) - \bar{R}/6 \) and (60) reads in this approximation

\[ [k \frac{\partial}{\partial k} - \bar{\Phi} \frac{\partial}{\partial \Phi} - 2\bar{R} \frac{\partial}{\partial R} + 4] \bar{U}_k(\Phi, \bar{R}) = -\frac{1}{16\pi^2} \partial^2 \bar{U}_k(\Phi, \bar{R}) - \bar{R}/6 - \frac{1}{2} (\partial^2 \bar{U}_k(\Phi, \bar{R}) - \bar{R}/6)^2 + ... \]  

(61)

From this expression one derives the evolution equations for all the coupling constants. This can be done by inserting (57) in (61) and by factoring the coefficient of the generic power of the field. If we arrest the expansion at the relevant and marginal couplings,

\[ (k \frac{\partial}{\partial k} + 2) \bar{\lambda}_2 = -\frac{1}{16\pi^2} \bar{\lambda}_4 (1 - \bar{\lambda}_2) + ... \]

\[ k \frac{\partial \bar{\xi}_2}{\partial k} = \frac{1}{16\pi^2} \bar{\lambda}_4 (\bar{\xi}_2 - \frac{1}{6}) + ... \]

\[ k \frac{\partial \bar{\lambda}_4}{\partial k} = \frac{3}{16\pi^2} \bar{\lambda}_4^2 + ... \]  

(62)

where the ellipses stand for irrelevant couplings. The system (62) reproduces the standard perturbative behavior for interacting scalar field in curved spacetime [5] for the relevant couplings, but this system handles the mixing of all the new local interactions along the renormalized trajectories as we approach the infrared. This result can be extended in the improved scheme where the interaction between modes is taken into account by retaining the effect of the previously integrated modes when the cutoff is lowered. In this way one considers the impact of the UV irrelevant operators during further renormalization group transformations. Let us define the renormalization group coefficient functions (at a fixed value of the cutoff and the bare parameters):
\[
 \frac{k}{m^2(k)} \frac{d m^2(k)}{dk} = \gamma_m \left( \frac{m(k)}{k}, \lambda(k), \xi(k) \right)
\]
\[
 k \frac{d \xi(k)}{dk} = \gamma_\xi \left( \frac{m(k)}{k}, \lambda(k), \xi(k) \right)
\]
\[
 k \frac{d \lambda(k)}{dk} = \beta \left( \frac{m(k)}{k}, \lambda(k), \xi(k) \right)
\]

(63)

From (47) and (59) we find
\[
 \gamma_m \left( \frac{m_R}{k}, \lambda_R, \xi_R \right) = -\frac{\lambda_R}{16\pi^2} \frac{k^4}{m_R^2 (m_R^2 + k^2)}
\]
\[
 \gamma_\xi \left( \frac{m_R}{k}, \lambda_R, \xi_R \right) = \frac{\lambda_R (\xi_R - \frac{1}{6})}{16\pi^2} \frac{k^4}{(k^2 + m_R^2)^2}
\]
\[
 \beta \left( \frac{m_R}{k}, \lambda_R, \xi_R \right) = \frac{3\lambda_R^2}{16\pi^2} \frac{k^4}{(k^2 + m_R^2)^2}
\]

(64)

Therefore the renormalization group improved scale dependence of the relevant coupling constants is given by the equations
\[
 k \frac{\partial m^2(k)}{\partial k} = -\frac{\lambda(k)}{16\pi^2} \frac{k^4}{m^2(k) + k^2}
\]
\[
 k \frac{\partial \xi(k)}{\partial k} = \frac{\lambda(k) (\xi(k) - \frac{1}{6})}{16\pi^2} \frac{k^4}{(k^2 + m^2(k))^2}
\]
\[
 k \frac{\partial \lambda(k)}{\partial k} = \frac{3\lambda^2(k)}{16\pi^2} \frac{k^4}{(k^2 + m^2(k))^2}
\]

(65)

with the initial conditions \( m(0) = m_R \), \( \xi(0) = \xi_R \), \( \lambda(0) = \lambda_R \). It is important to stress that the conformal coupling does not necessarily flow to the conformal value \( \xi_R = 1/6 \) for \( k = 0 \). In fact this ultimate set of equation shows the role played by the mass gap in determining the crossover region. Below the mass scale all the fluctuations are damped and the IR fixed point is present at \( k = 0 \) for arbitrary values of the renormalized coupling constants \( \lambda_R \) and \( \xi_R \). However, in the usual regularization perturbative scheme combined with the minimal subtraction prescription, one introduces counterterms that are mass independent and this fixed point is normally missing for non vanishing coupling strength. But this approximation is accurate only to study the scaling in the deep UV region where all the relevant masses can be neglected.

V. SUMMARY

A realization of the coarse-grained procedure has been presented for a generally coupled scalar field in the Einstein universe, and the non derivative part of the renormalized action has been calculated at the one-loop approximation in the loop expansion. A local renormalized potential governs the spatial distribution of the blocked variables, the average
of the fluctuations in “cubes” of size $\Omega_n \sim (a/n)^3$. This distribution is no longer centered on the symmetric minimum $\Phi = 0$ as $\Omega_n \sim \ell^3$ where $\ell$ is the characteristic linear extent of domains where the field assumes nonzero values. The overall geometry of the manifold determines the crossover scale in the high curvature limit where the local methods may not be accurate. The coarse-graining procedure here presented makes the investigation of the low energy domain possible for a general static manifold. The renormalization group analysis formulated in the Kadanoff-Wilson scheme controls the evolution of the new effective vertices as we scale towards the infrared. In particular, by linearizing the renormalization group transformation close to the UV fixed point, the renormalized field can be explicitly studied in the improved scheme. In the massive model the flow slows down as we move in the infrared direction and the infrared fixed point is recovered below the mass gap for arbitrary values of the conformal coupling $\xi_R$.

These results have been found for a positive $\xi_R R$ term and it would be interesting to address the $\xi_R \leq 0$ case where an increase of curvature can drive the system in the spontaneously broken phase.

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VI. APPENDIX

This appendix contains some relevant details concerning the calculations. The one loop contribution to the local potential has been calculated by means of the “generalized” $\zeta$-function [13]. We have from (38)

$$\zeta_n(s) = \frac{\beta \mu^{2s}}{2\pi} \int dp \sum_{n'=n+1}^{\infty} \sum_{l=0}^{n'} \sum_{m=-l}^{l} (\omega^2(p, n'))^{-s}$$

$$= \frac{\beta(\mu a)^{2s}}{a \pi} \int_0^\infty dp \sum_{m=n+1}^{\infty} \frac{(m+1)^2}{(p^2 + (m+1)^2 + \nu)^{s}}, \quad n \geq 0$$

$$= \frac{\beta(\mu a)^{2s}}{a \sqrt{4\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{m=n}^{\infty} \frac{m^2}{(m^2 + \nu)^{s+1/2}}, \quad n \geq 1$$

(66)

where $\omega^2(m^2 + \frac{1}{2} \lambda \Phi^2 + \xi R) - 1 = \nu > -1$ for $\xi > 0$. In the last line we have set $n+1 \to n$ so that the range of $n$ starts at $n = 1$ and the sum coincides with the $\zeta$-function, extensively studied by several authors [15,17,18,20], whose derivative in $s = 0$ gives the one-loop quantum contribution to the effective potential in the Einstein universe. For general $n$, the sum in the last line can be performed with the Plana summation formula [21,22],

$$g(s) = \sum_{m=n}^{\infty} f(s, m) = \frac{1}{2} f(s, n) + \int_n^\infty f(s, \tau) d\tau + i \int_0^\infty \frac{f(s, n+i\tau) - f(s, n-i\tau)}{e^{2\pi \tau} - 1} d\tau$$

(67)
with

\[ f(s, m) = \frac{m^2}{(m^2 + \nu)^s}. \]  

(68)

The resulting \( g(s) \) has simple poles at \( s - 3/2 = 0, -1, \ldots \). Taking the derivative of (66) in \( s = 0 \) with \( g(s) \) in (67) one obtains for \( s = 0 \) expression (39), where

\[ \Upsilon_n(\Phi) = -\frac{\nu^2}{8} \ln[n + \sqrt{n^2 + \nu}] + \frac{1}{8} n \ln n^2 + \nu(n^2 - 4n) - \mathcal{I}_n(\Phi) \]  

(69)

and

\[ \mathcal{I}_n(\Phi) = \int_0^\infty \frac{4nz e^{-\pi z}}{\sinh \pi z} \frac{2z^2(\nu + 5n^2) - n^2(2\nu + 3n^2) - 3z^4}{\text{Re}[(n + iz)^2 \sqrt{(n + iz)^2 + \nu}]} \]  

(70)

A careful evaluation of the \( \nu \gg 1 \) limit for \( n = 1 \) in expression (69) shows that

\[ \Upsilon_1(\Phi) \sim \frac{\nu^2}{16} \ln \nu + O(1/\nu). \]  

(71)
REFERENCES


