GLOBAL ASPECTS OF ELECTRIC-MAGNETIC DUALITY

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ABSTRACT

We show that the partition function of free Maxwell theory on a generic Euclidean four-manifold transforms in a non-trivial way under electric-magnetic duality. The classical part of the partition sum can be mapped onto the genus-one partition function of a 2d toroidal model, without the oscillator contributions. This map relates electric-magnetic duality to modular invariance of the toroidal model and, conversely, the $O(d,d',Z)$ duality to the invariance of Maxwell theory under the 4d mapping class group. These dualities and the relation between toroidal models and Maxwell theory can be understood by regarding both theories as dimensional reductions of a self-dual 2-form theory in six dimensions. Generalizations to more $U(1)$-gauge fields and reductions from higher dimensions are also discussed. We find indications that the Abelian gauge theories related to 4d string theories with $N = 4$ space-time supersymmetry are exactly duality invariant.

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1. Introduction

The purpose of this paper is to study global aspects of electric-magnetic duality in a very simple model, namely (source-free) Maxwell theory, and to investigate its relation with other known duality symmetries in lower and higher dimensional free field theories. Maxwell theory is the simplest example of a 4d field theory exhibiting a strong-weak coupling duality, and, just as the toroidal conformal models in two dimensions, it serves as a useful ‘toy’-model to illustrate and gain more insight in this phenomenon. Of course, we hope that our results will eventually shed new light on the recent developments in strong-coupling supersymmetric non-abelian gauge theories [1, 2], and the newly discovered dualities in string theory [3, 4]. But, just to keep things simple, we will in this paper restrict our attention to abelian gauge theories, and leave possible generalizations, applications or implications to future work 1.

In the first part of this paper we consider the partition function of Maxwell theory on a euclidean four-manifold without boundary, and exhibit its behaviour as a function of the coupling constants \( g \) and \( \theta \) that appear in the euclidean Maxwell-action

\[
S[A] = \frac{1}{g^2} \int_{M^4} F \wedge *F - i \frac{\theta}{8\pi^2} \int_{M^4} F \wedge F.\tag{1.1}
\]

Here \( A = A_i dx^i \) is the gauge potential written as a one-form, \( F = dA = \frac{1}{2} F_{ij} dx^i \wedge dx^j \) is the 2-form field strength and \( *F = \frac{1}{\sqrt{g}} \epsilon^{ijkl} F_{ij} dx^k \wedge dx^l \) denotes its Hodge-dual. We will study the properties of the partition function under the \( SL(2, \mathbb{Z}) \) duality group [5]

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}\tag{1.2}
\]

where \( \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \) and \( a, b, c \) and \( d \) are integers satisfying \( ad - bc = 1 \). We will find that only for certain four-manifolds the full \( SL(2, \mathbb{Z}) \)-symmetry can be realized. For these manifolds we also find a curious correspondence with 2d toroidal models, that interchanges the role of the duality and mapping class group. This will be discussed and illustrated with examples in section 3. Here we also show that, in analogy with 2d conformal field theory, the correlators of the Wilson-’t Hooft line operators can be obtained as a degenerate limit of the partition function.

The relation between Maxwell theory and the 2d toroidal models is further clarified in section 4. Here we show that both theories correspond to a dimensional reduction of the same 6d theory describing a 2-form with self-dual field strength, and that the duality symmetries arise from the mapping class group of the internal manifold. We also find that the partition function corresponds to a wave-function in a topological theory on which duality acts as a canonical transformation. Finally, in section 4.3 we discuss some string related Abelian gauge theories.

1We became aware that some related work has been done [11], while this paper was being proof-read.
2. Duality of the Partition Function

2.a. The classical partition sum

We consider the partition function of Maxwell theory on a closed four-manifold

\[ Z = \frac{1}{|G|} \int [dA] e^{-S[A]} . \quad (2.1) \]

Here we integrate over all \( U(1) \) gauge-fields \( A \) on the four-manifold \( M^4 \) and we divide as usual by the volume of the gauge group \( |G| \). Applying the standard Faddeev-Popov procedure to factor out this volume one finds that the partition function factorizes into a sum over the classical saddle-points times a product of determinants. Explicitly,

\[ Z = \frac{\det' \Delta_{FP}}{(\det' \Delta_A)^{\frac{n}{2}}} Z_{cl}, \quad (2.2) \]

where \( Z_{cl} \) represents the contribution of the classical saddle-points

\[ Z_{cl} = \sum_{\text{saddle points}} e^{-S[A_{cl}]} . \quad (2.3) \]

Here \( \Delta_{FP} \) and \( \Delta_A \) denote the kinetic operators for the Faddeev-Popov ghosts and the gauge field and, after gauge fixing, are given by the Laplacian acting on functions or one-forms respectively. Both these laplacians can have zero modes, which have to be projected out. In particular, when the four-manifold \( M^4 \) is non-simply-connected, the Laplacian \( \Delta_A \) has zero-modes corresponding to the flat abelian connections on \( M^4 \). In the following we will for simplicity consider simply connected manifolds, so that we do not have to deal with these zero modes.

We will now focus our attention on the sum over classical solutions. When the four-manifold \( M^4 \) has non-trivial homology two-cycles, i.e. closed surfaces that do not correspond to the boundary of a 3-dimensional sub-manifold in \( M^4 \), there exist field configurations with non-zero flux through these surfaces. A generalization of the familiar Dirac quantization condition implies that the flux through the non-trivial homology two-cycles \( \Sigma_I \) must be quantized

\[ \int_{\Sigma_I} F = 2\pi m^I; \quad m^I \in Z \quad (2.4) \]

with \( I = 1, \ldots, \dim H_2(M^4) \). This tells us that in the absence of sources the solutions of the field equations \( d^*F = 0 \) can be decomposed as

\[ F = 2\pi \sum_I m^I \alpha_I \quad (2.5) \]
where \( \alpha \) is an integral basis of harmonic 2-forms, which by definition satisfy \( d\alpha = d^*\alpha = 0 \) and are normalized so that \( \int \alpha \wedge \alpha = \delta \). Thus on a compact four-manifold the classical saddle-points are uniquely labeled by the integer magnetic fluxes \( m^I \). Inserting the expression (2.5) into (1.1) we find that the classical action for this field configuration is

\[
S[m^I] = \frac{4\pi^2}{g^2} m^I G_{IJ} m^J - \frac{i}{2} \theta m^I Q_{IJ} m^J
\]

where

\[
Q_{IJ} = \int_{M^4} \alpha_I \wedge \alpha_J, \quad G_{IJ} = \int_{M^4} \alpha_I \wedge *\alpha_J
\]

represent the intersection form and the metric on the space of harmonic two-forms. In this way we find that the saddle-point contribution to the partition sum is given by

\[
Z_{cl}(g, \theta) = \frac{1}{C} \sum_{m^I} e^{-S[m^I]},
\]

where \( C \) is a normalization constant. Thus, while the full partition function \( Z \) depends on the detailed geometry of \( M^4 \), we find that its classical part \( Z_{cl} \) is completely determined by the matrices \( G_{IJ} \) and \( Q_{IJ} \), and thus requires relatively little information. For any manifold \( G_{IJ} \) is symmetric and positive-definite, and the intersection form \( Q_{IJ} \) has integer entries and determinant equal to one: such matrices are called unimodular. Its inverse \( Q^{IJ} \) counts the number of intersection points of the two surfaces \( \Sigma_I \) and \( \Sigma_J \): \( Q^{IJ} = \#(\Sigma_I, \Sigma_J) \). Further, it follows from \( *(\alpha_I) = \alpha_I \) that

\[
Q^{IK} Q^J_K = \delta^{I,J}
\]

where \( Q^{IJ} \equiv G^{IK} Q_{KJ} \). Thus the eigenvalues of \( Q^{IJ} \) are all +1 or −1 corresponding to the self-dual and anti-self-dual two-forms.

In the following we will take the normalization constant \( C \) to be equal to \( C = g^b \), where \( b = \text{dim} \; H_2(M^4) \), because with this choice we will find that the partition function is (almost) invariant under (a maximal subgroup of) the \( SL(2, Z) \) duality group. To verify that this normalization is correct one could for example use the relation

\[
g^2 \frac{\partial}{\partial g^2} \log Z = \int \langle F \wedge *F \rangle
\]

and calculate the right-hand side using an appropriate regularization procedure. We have not completed this calculation but, by analogy with the 2d Gaussian model, we expect that the regulated one-point function of the marginal operator \( F \wedge *F \) contains metric-dependent terms such as the Euler class. It is likely that this leads to the wanted result for the normalization constant \( C = g^b \).
2. b. Duality properties of the partition function

To derive the behaviour of the full partition function $Z$ under the group of duality transformations (1.2) we only have to consider its action on the classical sum $Z_{cl}$ because this is the only part that depends on the couplings $g$ and angle $\theta$. By repeatedly using the Poisson resummation formula

$$\sum_m f(m) = \sum_n \int dx e^{2\pi i nx} f(x)$$

it is in principle straightforward to compute the action of the $SL(2, \mathbb{Z})$ on the sum $Z_{cl}$. The only ingredients that are used in the calculation are the relation (2.9) and the fact that $Q_{IJ}$ is unimodular.

It turns out that the partition function $Z$ is in general not $SL(2, \mathbb{Z})$-invariant. To describe its transformation properties under duality, let us first introduce the generalized partition sum

$$Z\left[ \frac{\bar{\theta}}{\bar{\phi}} \right] = g^{-\frac{1}{2}} \sum_{m'} e^{-S[m' + \phi'] + 2\pi i (m' + \phi')Q_{IJ}\phi^J}, \quad (2.11)$$

where $S[m]$ is the same quadratic expression given in (2.6) and the ‘characteristics’ $\bar{\theta}$ and $\bar{\phi}$ are half-integers. The physical interpretation of $\bar{\theta}$ and $\bar{\phi}$ is that they represent half-integer shifts in the quantization rule of the magnetic and ‘electric’ fluxes through the homology cycles. Notice that $Z\left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right]$ coincides with the Maxwell partition function, where we dropped the determinants. We find that under $SL(2, \mathbb{Z})$ these partition sums transform as:

$$Z\left[ \frac{\bar{\theta}}{\bar{\phi}} \right] \rightarrow e^{i\varphi} Z\left[ \frac{\bar{\theta}}{\bar{\phi}} \right], \quad (2.12)$$

where

$$\begin{pmatrix} \bar{\theta}' \\ \bar{\phi}' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{\theta} \\ \bar{\phi} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} abQ_i \bar{Q}_j \\ cdQ_i \end{pmatrix}, \quad (2.13)$$

$\varepsilon$ is some eighth root of unity and $\varphi = \frac{1}{2} Q^I \text{arg}(c\tau + d)$. Note that the phase $\varphi$ depends only on the topological data contained in $Q_{IJ}$ and is independent of $G_{IJ}$. The components of the vector $\bar{Q}$ are given by the diagonal elements $Q^{IJ}$ of the (inverse) intersection form. The transformation rule (2.12) is very similar to the modular properties of theta functions associated with 2d Riemann surfaces. In section 4 it will become clear that this similarity is not just a coincidence. When the diagonal elements $Q^{IJ}$ of the intersection form are not all even, the Maxwell partition sum $Z\left[ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right]$ is not invariant under the $SL(2, \mathbb{Z})$ duality group. Only on four-manifolds with an even intersection form does one find that the partition sum is duality invariant up to a phase. This fact and the transformation rules of the partition sum can be naturally understood from the quantum properties of the electric and magnetic fluxes.
2.c. Flux quantization

Let us introduce the ‘electric’ and magnetic flux operators

\[ \Phi_m^\prime = \frac{1}{2\pi} \int_{\Sigma} F, \quad \Phi_e^\prime = \frac{1}{2\pi} \int_{\Sigma} F_D, \quad (2.14) \]

where

\[ F_D = 2\pi i \frac{\delta S}{\delta F} = \frac{4\pi i}{g^2} \ast F + \frac{\theta}{2\pi} F \quad (2.15) \]

is the dual field strength. Here we have chosen the definition of \( \Phi_e \) so that it transforms nicely under duality, but strictly speaking it is a linear combination of electric and magnetic flux. Under \( SL(2, Z) \) the electric and magnetic fluxes transform as

\[ \begin{pmatrix} \Phi_m \\ \Phi_e \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Phi_m \\ \Phi_e \end{pmatrix} \quad (2.16) \]

By a slight modification of the calculation of section 2.1 it can be shown that the expressions (2.11) represent the (conveniently normalized) expectation values

\[ Z \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] = \left\langle \exp 2\pi i [\phi^J Q_{IJ} \Phi_m^J] \right\rangle_{\theta^J} \quad (2.17) \]

where the subscript \( \theta^J \) indicates that the quantization condition for the magnetic fluxes is \( \Phi_m^J \in Z + \theta^J \). Comparing (2.16) with the homogeneous term in the transformation rule (2.13) suggests that \( \phi^J \) must represent a shift in the electric flux quantization. This interpretation of \( \phi^J \) as well as the mysterious inhomogeneous term in (2.13) follow from the fact that, as quantum operators, \( \Phi_e \) and \( \Phi_m \) do not commute when the corresponding surfaces have a non-zero intersection. We find

\[ [\Phi_e^J, \Phi_m^J] = \frac{1}{2\pi i} Q^{IJ}. \quad (2.18) \]

This can be derived, for example, from the short-distance properties of the two-point function \( \langle F_D F \rangle \) by imitating the technique of radial quantization familiar from two-dimensional conformal field theory. The transformation properties of the partition functions can now be understood by interpreting (2.16) as a canonical transformation in the quantum Hilbert space of the flux operators. The result (2.13) precisely describes the unitary transformation of the eigenstates \( |\hat{\theta}, \hat{\phi}\rangle \) of the exponentials \( e^{2\pi i \Phi_m^J} \) and \( e^{2\pi i \Phi_e^J} \) with eigenvalues \( e^{i\theta^J} \) and \( e^{i\phi^J} \). Once this identification is made, it becomes a simple quantum mechanics exercise to derive the transformation rule (2.13): it basically follows from the CBH formula:

\[ e^{2\pi i (a \Phi_e^J + b \Phi_m^J)} = (-1)^{ij} e^{2\pi i a \Phi_e^J} e^{2\pi i b \Phi_m^J}. \]
3. Relations with 2D Toroidal Models

3.a. Self-dual Lorentzian lattices

The classical partition sum of Maxwell theory on a four-manifold has a close similarity to that of a two-dimensional toroidal model. We can make the correspondence almost perfect by noticing that the partition sum can be rewritten as

\[ Z_{cl}(\tau, \bar{\tau}) = \frac{1}{c} \sum_{(p^+, p^-) \in \Gamma_{i+, \bar{i}-}} \exp \left[ i \pi \tau (p^+)^2 - i \pi \bar{\tau} (p^-)^2 \right]. \tag{3.1} \]

where we sum over a self-dual lorentzian lattice with ‘signature’ \((b^+, b^-)\). Here \(b^+ (b^-)\) is the number of (anti-)self dual harmonic two-forms, and coincides with the number of positive (negative) eigenvalues of the intersection form \(Q_{IJ}\). To be more precise, we can represent the lattice in terms of a set of generators

\[ \Gamma_{i+, \bar{i}-} = \bigoplus_i Z (e_i^+ \oplus e_i^-), \tag{3.2} \]

which are related to the \(Q_{IJ}\) and \(G_{IJ}\) via

\[ \frac{1}{2} (G_{IJ} \pm Q_{IJ}) = \sum_{i=1}^{i+} (e_i^+) (e_i^-) \tag{3.3} \]

The expression (3.1) is identical to the partition sum of a 2d toroidal model used for string compactifications, but without the powers of the Dedekind \(\eta\)-function that represent the oscillator modes of the string. We should note also that the lattices that arise for four-manifolds are integral but, unlike those used for toroidal string compactifications, not always even.

The classical partition sum is a function of \(b^+ \times b^-\) moduli parameters that parametrize the shape of the lattice and take values on the coset space

\[ \mathcal{M}_{b^+, b^-} = SO(b^+, b^-) / SO(b^+) \times SO(b^-) / O(b^+, b^-, Z). \]

A well-known example is \(\mathcal{M}_{19,3}\), which represents the moduli space of \(K^3\)-manifolds. The symmetry of the partition sum under the discrete group \(O(b^+, b^-, Z)\) ensures its invariance under the mapping class group of the four-manifold. The fixed points of elements of \(O(b^+, b^-, Z)\) correspond to four-manifolds with accidental discrete symmetries. It is known that the partition sum at these enhanced symmetry points often contains purely \(\tau\)-dependent (or \(\bar{\tau}\)-dependent) lattice sums. These ‘characters’ represent the contributions of abelian instantons (= purely self-dual (or anti-self-dual) solutions of Maxwell’s equations).

\footnote{The results of this and the preceding section have been reported at the Strings '95 conference at USC and at the Spring School in Trieste, and appear to have some overlap with [11].}
3.b. Some examples

As an illustration let us discuss a few simple examples. The basic example of a manifold
with an odd-intersection form is $CP^2$ for which $b^+ = 1$ and $b^- = 0$. The classical Maxwell
partition sum on this manifold is given by a Jacobi theta-function

$$Z_{cl}(\tau)_{CP^2} = \frac{1}{C} \theta_3(\tau) = \frac{1}{C} \sum_m e^{i\pi m^2}.$$ 

This partition-function is invariant only under $\tau \rightarrow \tau + 2$, and up to a phase under $\tau \rightarrow -1/\tau$, provided we choose the normalization constant to be $1/C = \sqrt{\frac{1}{2\pi}}$.

The simplest non-trivial example of an ‘even’ manifold is $S^2 \times S^2$ which has intersection
form $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. When we specialize our result to this case we find the Maxwell partition
sum for electro-magnetism on $S^2 \times S^2$ is identical to the momentum sum for the $c = 1$
gaussian model

$$Z_{cl}(\tau, \pi, R)_{S^2 \times S^2} = \frac{1}{C} \sum_{m,n} e^{\frac{1}{C} \pi (m^2 + R^2)} e^{-\frac{1}{C} \pi (m^2 - R^2)},$$

where the ‘radius’ $R$ equals the ratio of the size of the two spheres. Note that the familiar
$R \rightarrow 1/R$-symmetry is just a consequence of the invariance under exchange of the two
spheres. Hence, in this context, $R \rightarrow 1/R$-duality corresponds to a kind of 4d modular
invariance’, or, more precisely, invariance under the mapping class group $Z_2$ of $S^2 \times S^2$.

These observations can be generalized to manifolds with higher dimensional second co-
homology, such as the connected sum of $N$ copies of the $S^2 \times S^2$-manifold. On this manifold,
which we simply denote as $N(S^2 \times S^2)$, we can choose a canonical basis of $A$- and $B$-cycles
$\Sigma_i$ and $\Sigma_j$ such that the corresponding two-forms $\alpha_i$ and $\beta_j$ have intersection

$$\int_{M^4} \alpha_i \wedge \beta_j = \delta_{ij}$$

with $i, j = 1, \ldots, N$. All other components of the intersection form $Q$ vanish. By making
use of the relation (2.9) we find that the metric on these two-forms takes the form

$$\int_{M^4} \beta_j \wedge \star \beta_i = G^{ij}, \quad \int_{M^4} \alpha_i \wedge \star \beta_i = B_i^j$$

and

$$\int_{M^4} \alpha_i \wedge \star \alpha_j = G_{ij} - B_{ik} G^{kl} B_{lj},$$

where $G_{ij}$ is symmetric and positive definite and the matrix $B_{ij} \equiv B_i^k G_{kj}$ is antisymmetric.

The classical action for the saddle-point configurations may thus be written as

$$S[m, n] = \frac{8\pi^2}{g^2} \left[ m_i G_{ij} m_j + (n_i - B_{ik} m^k) G^{ij} (n_j - B_{jk} m^l) \right] - i\theta m^i n_i.$$

This expression exactly coincides with the spectrum of vertex operators for a toroidal model with constant metric $G_{ij}$ and anti-symmetric tensor field $B_{ij}$. The properties of the resulting partition sum have been well studied in this string context (see e.g. [6]), and we gratefully make use of this.

For $N(S^2 \times S^2)$, or any other four-manifold with the same intersection form, we can rewrite the partition in a manifestly duality-invariant way by performing a Poisson resummation over the fluxes through the $B$-cycles $n^i$. This gives

$$Z_{c,i}(G, B, \tau, \bar{\tau}) = \sum_{m,n} e^{-2\pi E[m,n]}$$

(3.8)

where

$$E[m,n] = \frac{1}{Im\tau}(n^i + \tau m^i)(G_{ij} + B_{ij})(n^j + \tau m^j).$$

(3.9)

The integers $n^i$ and $m^i$ now represent the electric and magnetic flux through the $A$-cycles $\Sigma_i$. To get to this $SL(2, Z)$-invariant representation of the partition sum, we had to choose a canonical decomposition of the two-cycles. Of course, we would get the same partition sum if we had relabelled the basis of two-spheres without changing the intersection form. This fact gives rise to the familiar $O(N, N, Z)$ symmetry.

3.c. Correlators through factorization.

It is also interesting to study the partition function on certain degenerate four-manifolds. This should give information on the spectrum of states of the theory, and the correlation function of observables. It turns out that the relevant type of degenerations are those in which a two-cycle shrinks to size zero. When an $A$-cycle $\Sigma_i$ is pinched the corresponding metric-element $G_{ii}$ blows up and goes to infinity. We deduce from (3.8) that for $G_{ii} \to \infty$ the term $Z[m,n]$ in the partition sum labelled by $m^i$ and $n^i$ is suppressed by an exponential factor $e^{-2\pi G_{ii} \Delta_{mn}}$ with

$$\Delta_{mn} = |n + m\tau|^2/Im\tau.$$  

In analogy with 2d conformal field theory we would like to interpret $\Delta_{mn}$ as the ‘scaling dimension’ of the operators in the theory. Indeed, when we ‘pinch’ all ‘$A$’-cycles the partition function can be seen to go over in the correlation function of the abelian versions of the Wilson–t’Hooft line operators,

$$Z[m,n] \to \prod_i e^{-2\pi G_{ii} \Delta_{mn} i} \langle \prod_i W_{m_in_i}(C_i) \rangle.$$  

(3.10)

The observables $W_{mn}$ are represented by

$$W_{mn}(C) = \exp i (n \oint_C A + m \oint_C A_D),$$  

(3.11)
where $A_D$ is the dual gauge field whose field strength $F_D = dA_D$ is given in (2.15). The integers $n$ and $m$ are the electric and magnetic charge resp. Assuming that the loop $C$ is contractible we can rewrite the line integrals over $A$ and $A_D$ as a surface integral of the $F$ and $F_D$ over a disk $D$ with boundary $\partial D = C$. Then, because the functional integral is just a gaussian, we can formally express the correlation functions of the observables $W_{m,n}$ in the two-point function of the field strength $F$. In this way one finds

$$
\left\langle \prod_i W_{n_i,m_i}(C_i) \right\rangle = \prod_{i \neq j} \exp \frac{2\pi}{Im \tau} (n_i + \tau m_i) L(C_i, C_j)(n_j + \tau m_j)
$$

where

$$
L(C_i, C_j) = g^{-2} \int_{D_i} \int_{D_j} \langle F^+ F^- \rangle,
$$

with $F^\pm = \frac{1}{2} (F \pm *F)$ and $\partial D_i = C_i$. Notice that this has indeed a form identical to the term in the partition function labelled by $m_i$ and $n_i$, as would be expected from the factorization equation (3.10).

**4. Duality from Dimensional Reduction**

**4.a. Self-dual 2-form theory in $d = 6$ and its reductions.**

In this section we further clarify the relation between the duality and modular symmetries of Maxwell theory and the two-dimensional toroidal models by showing that both these theories can be regarded as dimensional reductions of the same theory, namely of a six-dimensional theory describing a 2-form field $C$ with self-dual field strength $H = dC$: Maxwell theory is obtained by compactifying the 6d self-dual theory on a torus, while the toroidal models arise through compactification on $M^4$.

First let us explain how to describe the self-dual 2-form theory in $d = 6$. As a starting point, let us consider the following first order action

$$
S_{\text{6d}} = \frac{1}{2\pi i} \int dC \wedge H + \frac{1}{4\pi} \int H \wedge *H.
$$

where $H$ is a three-form field and the field $C$ is a two-form satisfying the flux quantization condition $\int_{\Xi} dC \in 2\pi Z$ for all three-cycles $\Xi$. Integrating out $H$ implies that $H = i *dC$ and gives the standard free action for a $C$. Integrating out $C$ implies $H = dC_D$ and gives the

\footnote{For definiteness, we restrict our attention to the 6d theory, but the generalization to self-dual 2p-forms in 4p + 2 dimensions should be obvious.}
dual action for the two-form field $C_D$. In (4.1) we have chosen the coupling constant at its self-dual value so that the action for $C$ and its dual $C_D$ are identical.

To construct an action for the self-dual 2-form field we now take this first order action and identify half of the components of $H$ with those of $dC$. The only problem with this procedure is that one has to give up manifest covariance of the theory. For example, one way to do this is by choosing a global vector field $V$ (=‘time’-direction) and to equate all the ‘spatial’ components of $H$ with those of $dC$, or in a more invariant notation

$$i_V(H - dC) = 0. \quad (4.2)$$

This leads to the non-covariant action for self-dual 2-forms of Henneaux and Teitelboim [7]. The physical content of the theory should not depend on the choice of the vector field $V$, and hence there must be a ‘canonical transformation’ that relates different choices for $V$. This is somewhat analogous to the ‘choice of polarization’ in quantum mechanics, and suggests a possible reinterpretation of this procedure in terms of geometric quantization.

We now describe how the above procedure leads to the Maxwell action when one reduces to four dimensions by compactifying on a torus. Let us choose complex coordinates $z$ and $\bar{z}$ on the torus $T^2$ with modular parameter $\tau$. The dimensional reduction is performed by taking the following ansatz for the fields $H$ and $C$

\begin{align*}
H &= \frac{1}{Im \tau} \left[ (F_D - \tau F) dz + (F_D - \tau F) d\bar{z} \right], \quad (4.3) \\
C &= \frac{1}{Im \tau} \left[ (A_D - \tau A) dz + (A_D - \tau A) d\bar{z} \right]
\end{align*}

where $F$ and $F_D$ are four-dimensional two-forms which for the moment are unrelated to the four-dimensional gauge fields $A$ and $A_D$. Inserting this ansatz into the action (4.1) and performing the integrations over $z$ and $\bar{z}$ gives

$$S_{4d} = \frac{1}{2\pi i} \int (dA_D \wedge F + dA \wedge F_D) + \frac{1}{4\pi Im \tau} \int (F_D - F\tau) \wedge \tau (F_D - F\tau). \quad (4.4)$$

To proceed we now choose our vector field $V$ to be along the $b$-cycle of the torus. The condition (4.2) then gives $F = dA$. The next step is to integrate out the field $F_D$ from the action(4.4). One easily checks that this gives the Maxwell action (1.1). Notice that by the field equations $F_D$ becomes identified with the dual field strength $F_D = iIm \tau^* F + Re \tau F$ introduced in (2.15). And with this one can verify that the field $H$ in (4.3) is indeed self-dual.

In this construction the $SL(2, Z)$-duality symmetry coincides with the modular group of the internal torus, which is a remnant of $6d$-covariance. The fact that the duality symmetry is not manifest is because our choice of the vector field $V$ breaks the modular symmetry. Different choices for $V$ are related by modular transformations: for example, if we choose this vector in the direction of the $a$-cycle we would have obtained the dual action with
coupling $-1/\tau$. An other choice would be to take the vector in a preferred direction in the 4d space(time). This would lead to the duality invariant but non-covariant action [8]. Our results suggest that it is indeed impossible to have manifest duality and covariance at the same time: otherwise all partition functions would have been invariant under duality. The deviations of duality invariance are thus directly related to the global gravitational anomalies of the 6 dimensional theory.

Let us now explain how one obtains the toroidal model by reducing the self-dual 6d theory to $d = 2$ on a simply-connected internal manifold $M^4$. For definiteness we assume that $M^4$ has an even intersection form and $b^+ = b^-$. The reduction of the three-form field strength to 2d is performed by imposing $d_4^* H = 0$, where $d_4$ denotes the exterior derivative on $M^4$. Thus we can write

$$H = \sum_i \alpha_i dX^i + \beta \Pi_i, \quad (4.5)$$

where $d$ is now the exterior derivative in the two un-compactified dimensions and $\alpha_i$ and $\beta^i$ are the same harmonic two-forms that we introduced in section 3. We now also assume that there exists a vector field $V$ with $i_V \beta^i = 0$. With this choice of $V$ the condition (4.2) implies that the field $X^i$ is identified with the periods of the two-form field: $X^i = f_{2i} C_i$. Inserting this ansatz for $H$ and $C$ into the action (4.4) and performing the integrations over $M^4$ gives the two-dimensional action

$$S_{2d} = \frac{1}{2\pi i} \int \Pi_i \wedge dX^i + \frac{1}{4\pi} \int \left[ dX^i G^*_{ij} dX^j + (\Pi_i - B_{ik} dX^k) \wedge G^*_{ij} (\Pi_j - B_{jl} dX^l) \right]. \quad (4.6)$$

Self-duality of the field strength $H$ implies $\Pi_i = iG^*_{ij} dX^j + B_{ij} dX^i$ which is indeed one of the field equations that follow from $S_{2d}$. Finally, integrating out $\Pi_i$ leads to the action of the 2d toroidal model

$$S_{2d} = \frac{1}{2\pi i} \int dX^i \wedge (iG^*_{ij} dX^j + B_{ij} dX^i). \quad (4.7)$$

Again the hidden duality symmetries of the dimensionally reduced theory are directly related to the symmetries of the internal compactification manifold. We further note that if we reduce on a four-manifold with $b^\pm \neq b^-$, we will find a theory with unequal left- and right-moving bosons: for example for $K^3$ we get 19 left-movers and 3 right-movers.

The fact that the toroidal model and Maxwell theory have the same classical partition sum can now be understood as follows: the ansätze that we used for $H$ to reduce to $d = 4$ and to $d = 2$ are compatible with the classical solutions of the full 6d theory. Thus the classical solutions of Maxwell theory and those of the toroidal models can be extended to the same self-dual three forms on $M^4 \times T^2$, and thus are in one-to-one correspondence with the classical solutions of the 6d theory. This makes clear that the classical partition sum for all these three theories are indeed identical.
4.b. A simple topological theory in $d = 7$ and its dimensional reductions.

Consider the following topological theory in 7 dimensions

$$S_{7d} = \frac{1}{2\pi} \int H \wedge dH$$

where $H$ is an (unconstrained) three-form. The action (4.8) is identical to the level $k = 1$ $U(1)$ Chern-Simons theory, except that we have replaced the abelian gauge-field with a three-form. It is known that the Hilbert space of the $U(1)_k$ CS-theory may be identified with the characters ($=\text{chiral partition functions}$) of the chiral boson at $k$ times the self-dual radius. In a similar way one can show that the Hilbert space corresponding to $S_{7d}$ is related to the self-dual 2-form theory in $d = 6$. The only subtle point concerns the gauge symmetry: in order to achieve the correspondence with the 2-form theory at its self-dual coupling the abelian gauge symmetry $H \rightarrow H + dC$ must be ‘compact’: this means that $C$ is allowed to be multi-valued, as long as it has integral periods $\int_{\Xi} dC \in 2\pi \mathbb{Z}$ for all three-cycles $\Xi$.

Let us now consider this topological theory on a seven-manifold of the type $T^2 \times M^4 \times R$, where we interpret $R$ as time. Now, in a similar way as in subsection 4.1 we can dimensionally reduce the theory in various ways. First, the reduction to $d = 3$ gives a familiar theory: by using the ansatz $H = \sum \alpha_I A^I$, where $\alpha_I$ are again the integral harmonic two-forms on $M^4$ we reduce the theory to 3d abelian Chern-Simons theory on $T^2 \times R$

$$S_{3d} = \frac{1}{2\pi} \int A^I \wedge Q_{IJ} dA^J$$

with integral coupling constants $Q_{IJ}$ given by the intersection form of the four-manifold. It is well known that the Hilbert space of this theory is related to the 2d toroidal conformal models [9]. In a completely analogous way it is shown that the 5-dimensional topological theory

$$S_{5d} = \frac{1}{2\pi} \int F_D \wedge dF,$$

which is obtained from $S_{7d}$ by inserting the ansatz (4.3), is related to 4d Maxwell theory. The fields $F$ and $F_D$ in $S_{5d}$ are independent and unconstrained two-forms, but, by the field equations and after quantization, $F_D$ and $F$ become identified with the Maxwell field strength and its dual.

A curious fact about these topological dimensional reductions is that they do not reduce the number of physical degrees of freedom: all these theories have a finite number of degrees of freedom living on a ‘small phase space’ parametrized by two sets of angles $\theta^I \in [0, 2\pi]$ and $\phi^I \in [0, 2\pi]$. In the 5d topological model on $M^4 \times R$ the classical solutions for the fields $F$ and $F_D$ are of the form $F = \sum \theta^I \alpha_I$, $F_D = \sum \phi^I \alpha_I$. In the abelian Chern-Simons theory on $T^2 \times R$ these same angles $\theta^I$ and $\phi^I$ represent the $U(1)$ holonomies around the $a$- and $b$-cycles. So, without losing any physical degrees of freedom we can even reduce the theories.
to a simple quantum mechanical model in $d = 1$ that contains all the relevant topological information. The action is

$$S_1 = \frac{1}{2\pi} \int \theta^i Q_{1,j} d\phi^j. \quad (4.11)$$

After quantization the angle-variables satisfy the canonical commutation relations $[\theta^i, \phi^j] = 2\pi i Q_{i,j}$, which for is just the algebra (2.18) of electric and magnetic fluxes. As we explained in section 2.3, the $SL(2, \mathbb{Z})$ duality group correspond to the canonical linear transformations of $\theta^i$ and $\phi^j$. Now let $|\bar{\theta}, \bar{\phi}\rangle$ be the simultaneous eigenstate of the exponentials $e^{i\theta^i}$ and $e^{i\phi^j}$ (notice that they commute), and let us define the state $|0, \tau\rangle$ by $(\theta^i + \tau \phi^j)|0, \tau\rangle = 0$ Then we have

$$\langle \bar{\theta}, \bar{\phi}|0, \tau\rangle = Z \left[ \begin{array}{c} \bar{\theta} \\ \bar{\phi} \end{array} \right]. \quad (4.12)$$

This same overlap can be computed in the ‘big phase space’ in the various topological field theories. This yields a functional integral representation that, depending on which topological theory we consider, is identical to the partition function of the corresponding free field theories in terms of $H = dC$, $F = dA$ or $A^I = dX^I$. The result should be of course the same for all these different cases.

4.c. Generalizations and string-related models.

An obvious way to generalize our results is to consider higher dimensional theories with (self-dual) forms. For example, we can take a $2(p+q)$-form theory in $d = 4(p+q) + 2$ with self-dual $2(p+q)+1$-form field strength $H = dC$, and dimensionally reduce this theory down to $d = 4q$ on a internal compactification manifold $X$ with dimension $4p+2$. By using an ansatz of the form $H = \sum \alpha_A F^A + \beta^A F^D_A$ and following the same procedure as in section 4.1 we find a $2q-1$-form theory in $d = 4q$ with action

$$S = \frac{1}{2\pi i} \int \left( F^A_+ \Omega_{AB} F^B_+ - F^A_- \Omega_{AB} F^B_- \right), \quad (4.13)$$

where $F^A$ are 2$q$-form field strengths and $\Omega_{AB}$ is the period matrix of the internal manifold $X$. This theory possesses a duality symmetry that is inherited from the mapping class group of $X$, and that acts on the matrix $\Omega_{AB}$ as a $Sp(2b, \mathbb{Z})$ fractional linear transformation where $2b = \text{dim} H_{2p+1}(X)$. The physical observables of this theory are labelled by electric and magnetic quantum numbers $n_A$ and $m_A$ and have $Sp(2b, \mathbb{Z})$-invariant ‘scaling dimensions’

$$\Delta_{m,n} = (m_A + \Omega_{AC} n_C)(\Omega^{-1} m_A)_{AB}(m_B + \Omega_{BC} n_C).$$

A particularly relevant case for string theory is $p = q = 1$. Namely, the type IIB superstring has in its 10d effective action a 4-form field with self-dual 5-form field strength.

\footnote{For a related discussion in supersymmetric theories see [10]}

14
When we compactify this field down to \(d = 4\) on a 6d internal manifold of the form \(K^3 \times T^2\) following the outlined procedure we get

\[
S_{4d} = \frac{1}{g^2} \int_{M^4} F^I \wedge G_{IJ} F^J - i \frac{\theta}{8\pi^2} \int_{M^4} F^I Q_{IJ} F^J. \tag{4.14}
\]

where \(G_{IJ}\) and \(Q_{IJ}\) are the ‘metric’ and intersection form of the internal \(K^3\)-manifold, and the couplings \(g\) and \(\theta\) come from the internal \(T^2\). It is interesting to note that in this case the partition sum of this Abelian gauge theory is \(SL(2, \mathbb{Z})\)-invariant up to a phase on any four-manifold; for example, the partition sum on \(CP^2\) is given by a sum over the self-dual Lorentzian lattice \(\Gamma_{19,3}\). The deviation of exact duality is related to the global gravitational anomaly of the self-dual form in \(d=10\). In the full string theory these anomalies cancel, and thus we know that the phase that arises in the duality transformations in the 4d theory must be cancelled by the other fields in the low energy action.

The same appears to hold for the kind of 4d abelian gauge theories that arise in toroidal compactifications of the heterotic string from \(d = 10\) to \(d = 4\): also in this case the couplings and the theta-angles of the various \(U(1)\)-gauge fields are precisely right to have duality invariance (up to a phase) of the resulting 4d effective theory\(^5\). Our analysis suggest that the ‘duality-anomaly’ is related or proportional to the global gravitational anomaly in 6d or 10d. It will be interesting to verify this explicitly and to check that all obstructions (including phase factors) to exact \(s\)-duality cancel in the complete (effective) string theory on all possible (= differentiable four-dimensional spin-)manifolds. This is left for future work.

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