Quantum states on supersymmetric minisuperspace with a cosmological constant

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Abstract

Spatially homogeneous models in quantum supergravity with a nonvanishing cosmological constant are studied. A class of exact nontrivial solutions of the supersymmetry and Lorentz constraints is obtained in terms of the Chern-Simons action on the spatially homogeneous 3-manifold, both in Ashketar variables where the solution is explicit up to reality conditions, and, more concretely, in the tetrad-representation, where the solutions are

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given as integral representations differing only by the contours of integration. In the limit of a vanishing cosmological constant earlier exact solutions for Bianchi type IX models in the tetrad-representation are recovered and additional asymmetric solutions are found.

In the present-day search for a consistent theory comprising both general relativity and the quantum theory in appropriate limits one of the still promising lines of approach is the nonperturbative quantization of gravity or supergravity. In order to explore many of the recent new ideas developed in this approach [1] spatially homogeneous 'minisuperspace' models of the full theory have proved to be a very valuable tool, in particular in gravity [2], less so, but for no fundamental reasons, in supergravity [3]-[15]. In pure gravity spatially homogeneous minisuperspace models without or with anisotropy, cosmological constant, or matter-coupling have been successfully explored. In supergravity, on the other hand, until recently only minisuperspace models of the isotropic Friedmann type without or with cosmological constant or scalar field coupling [4] have been successfully quantized. The reason is that the correct treatment of anisotropy in the fermionic components of such supersymmetric models was only given very recently in [17, 18]. There anisotropic spatially homogeneous supersymmetric minisuperspace models restricted to the case without cosmological constant or matter coupling were treated. An important simplifying feature of this restriction is the fact that the fermion number defined by the Rarita-Schwinger field then is a good quantum number and each sector with a fixed fermion number may be treated separately.
If a cosmological constant (or any form of matter) is allowed for [4], [12]-[14] all fermion sectors become coupled and have to be considered simultaneously. Previous work on anisotropic supersymmetric Bianchi class A models with a cosmological constant [12]-[14] attempted to study the coupled fermion sectors but concluded that a nontrivial quantum state solving all constraints of supergravity does not exist, which is the same as stating that a consistent quantization of such models is not possible. However, in the light of the recent results in [17, 18] it is clear that the treatments of anisotropy in these papers need revision and the problem must be considered open.

In the present paper we therefore study the quantization of supersymmetric Bianchi class A models with a cosmological term. We shall, in fact, find an analytic expression for a family of nontrivial quantum states with components in all fermion sectors, satisfying all constraints. In the limit of vanishing cosmological constant these states give back explicit exact solutions obtained in [18] for Bianchi type IX.

As shown in [11] the study of minisuperspace models in supergravity can be greatly simplified by the use of Ashketar-variables, which we shall therefore employ to do most calculations. However, instead of applying reality conditions to the Ashketar variables, we shall at the end transform our results to the tetrad representation, which is more easy to interpret, and which was used in all previous work [12-14]. The starting point is the Hamiltonian formulation of $N = 1$-supergravity with a cosmological term in Ashketar variables which was given by
Jacobson [19]. For simplicity we shall adopt the notational conventions made there.

In the tetrad representation the independent variables are the spatial real components \( e_p^{AA'} \) of the tetrad \((p \in 1, 2, 3; A, A' \in 1, 2)\) and the Grassmannian components \( \psi_p^A \) of the Rarita-Schwinger field. The \( e_p^{AA'} \) form the metric tensor \( h_{pq} = -e_p^{AA'} \epsilon_q^{AA'} \) on the space-like homogeneity 3-surfaces in the symmetric basis of 1-forms \( \omega^p \), satisfying, in Bianchi class-A models,

\[
d\omega^p = \frac{1}{2} m^{pq} \varepsilon_{qrs} \omega^r \wedge \omega^s
\]

where \( \varepsilon_{qrs} \) is the spatial Levi-Civita tensor density. We shall denote \( h = det(h_{pq}) \).

The constant symmetric matrix \( m^{pq} \) is defined by the particular Bianchi type chosen within the class A. The volume of the homogeneous 3-surfaces (compactified, if necessary) is denoted by \( V = \int \omega^1 \wedge \omega^2 \wedge \omega^3 \). Due to the choice of a symmetric (non-coordinate) basis the \( e_p^{AA'} \) and \( \psi_p^A \) are independent of spatial coordinates.

In Ashketermin variables, on the other hand [19], one first adds a suitable complex term to the Lagrangean, which does not change the equations of motion, and then uses the complexified spin-connection \( A_{pAB} = A_{p(AB)} \) and the tensor density \( \bar{\sigma}^{pAB} \)

\[
\sqrt{2} \bar{\sigma}^{pAB} = -\varepsilon^{pqr} \epsilon_q^{AA'} \epsilon_r^{BB'} = \sqrt{2} \bar{\sigma}^{p(AB)}
\]

as a canonically conjugate pair of (complex) coordinates \( A_{pAB} \) and momenta \( \bar{\sigma}^{pAB} \). \( \varepsilon^{pqr} \) numerically equals \( \varepsilon_{pqr} \). In view of the reality of \( e_p^{AA'} \) the \( \bar{\sigma}^{pAB} \) must be Hermitian with respect to some Hermitian matrix \( n^{AA'} \) [19]

\[
(\bar{\sigma}^{pAB})^\dagger = n^{A}_{\ A'} n^{B}_{\ B'} \sigma^{pA'B'} = \bar{\sigma}^{AB}
\]
where $n^{AA'}$ satisfies
\[ n_{AA'} n^{AA'} = 2, \quad n_{AA'} \epsilon^{AA'}_p = 0 \] (3)
and is thereby determined (up to a sign) as a function of the $\epsilon^{AA'}_p$, or the $\tilde{\sigma}^{pAB}$, if the latter is more convenient. In the context of supergravity this choice of variables has the additional advantage that there are no second-class constraints and there is therefore no need to introduce Dirac brackets. Thus the diffeomorphism-, Hamiltonian-, Lorentz-, and supersymmetry constraints of the theory can be obtained very directly [19], and are also easily reduced to our present, spatially homogeneous case [11, 14]. In the following we shall only need the Lorentz constraints and the supersymmetry constraints which imply all others via the algebra of the symmetry generators. After canonical quantization in the $(A^{AB}_p, \psi^A_p)$-representation where $\tilde{\sigma}^{AB}_p = -\frac{1}{\sqrt{2}} \partial / \partial A^{AB}_p$ they take the form
\[ J_{AB} \Psi = \frac{1}{\sqrt{2}} \left( -2 A^{C}_p [A \frac{\partial}{\partial A_{p} B C}] + \psi^A_p (A \frac{\partial}{\partial \psi^B_p}) \right) \Psi = 0 \] (4)
\[ S^A \Psi = \frac{1}{\sqrt{2}} \left( A^{AB}_p \frac{\partial}{\partial \psi^B_p} - 2 \sqrt{2} V^2 m i \frac{\partial}{\partial A^{AB}_p} \psi^B_p \right) \Psi = 0 \] (5)
\[ S^{\dagger A} = \frac{1}{4} \varepsilon_{pqr} \frac{\partial}{\partial A^{AB}_p} \frac{\partial}{\partial A^{C}_q} \left( V m \varepsilon^{str} A^D_{sC} \psi^D_{tD} - 2 \sqrt{2} m i \frac{\partial}{\partial \psi^C_p} \right) \Psi \neq 0 \]

Here, the scalar $2\sqrt{2} m$ denotes the cosmological constant in the notation of [19].

The operator ordering chosen in (6) is motivated by the fact that polynomiality of this constraint in $\tilde{\sigma}^{AB}_p$ was only achieved after supplying an additional (non-vanishing) factor $h^{1/2} n_{AA'} [19]$, which can only be supplied from the left. The notation $S^{\dagger A}$ is not meant to imply that $S^{\dagger A}$ is the adjoint of $S^A$, because we leave open the problem of defining a scalar product.
The physical wave-functions satisfying these constraints are to be holomorphic in the complex variables \( A_p^{AB} \). The transformation to the tetrad-representation is achieved in two steps: (i) The change from the \( A_p^{AB} \)-representation to the \( e_p^{AA'} \)-representation is achieved by the generalized Fourier-transform

\[
\Psi'(e_p^{AA'}) = \int \left[ \prod_p \prod_{\{A \leq B\}} dA_p^{AB} \right] e^{-A_p^{AB} \epsilon_{AA'} \epsilon_{AB} \epsilon_{p} e^{p} e^{p'}} \Psi(A_p^{AB})
\]

along a suitable 9-dimensional contour in the complex manifold spanned by the \( A_p^{AB} \) chosen in order to achieve convergence, and permitting partial integration without boundary terms. Apart from this condition the contour may still be chosen arbitrarily, and, indeed there are different possible choices corresponding to different linearly independent solutions [20].

(ii) In order to undo the initial non-canonical complex transformation of the Lagrangean an additional similarity transformation has to be performed [22] which takes the form

\[
\Psi(e_p^{AA'}) = e^{-\phi(e_p^{AA'})} \Psi'(e_p^{AA'})
\]

with

\[
\phi = -\frac{V}{2} m^{pq} e_p^{AA'} e_{p'AA'} = \frac{V}{2} m^{pq} \hbar_{pq}.
\]

Indeed, applying these transformations on the generators (4)-(6), and contracting the generator (6) between the step (i) and (ii) with \( h^{-1/2} r_{AA'} \) from the left [19] we obtain the Lorentz generator \( J_{AB} \) and the supersymmetry generators \( S_A, \tilde{S}_A' \) for the class A Bianchi models in the tetrad representation [17, 18], with a cosmological term. The operator ordering obtained is that chosen in [18]. As long as \( h \neq 0 \),
each step in these transformations can be inverted, i.e. after specifying the integration contour in (7) the Ashketar representation and the tetrad-representation are mathematically if not physically equivalent for our models (see [21] for a similar discussion in the context of full gravity).

Let us now turn to a solution of the constraints (4)-(6). We start by noting that (6) may be rewritten in terms of the function $F(A_p^{AB}, \psi_p^A)$ defined by

$$F = -\frac{1}{4\sqrt{2}m} \left( V m^{pq} \psi_p^A \psi_q^A + \epsilon^{pqr} \psi_p^A A_q^B \psi_r^B \right)$$  \hspace{1cm} (10)

as

$$\epsilon_{pqr} \frac{\partial}{\partial A_{pAB}} \frac{\partial}{\partial A_{qBC}} \left( i \frac{\partial}{\partial \psi_r^C} + \frac{\partial F}{\partial \psi_r^C} \right) \Psi = 0.$$  \hspace{1cm} (11)

A special class of solutions (and we shall restrict ourselves to this class in the following, even though, undoubtedly, more general solutions do exist) is therefore

$$\Psi = \text{const exp} \left[ i \left( F(A_p^{AB}, \psi_p^A) + G(A_p^{AB}) \right) \right]$$  \hspace{1cm} (12)

where $G$ is independent of the $\psi_p^A$ but otherwise arbitrary. Choosing this function appropriately we can next satisfy the constraint (5). This yields

$$G = \frac{i}{(4mV)^2} \left( V m^{pq} A_p^{AB} A_q^{AB} + \frac{2}{3} \epsilon^{pqr} A_p^{AB} A_q^B C A_r^C \right).$$  \hspace{1cm} (13)

One can finally check that the Lorentz constraint (4) is fulfilled, which is obvious because $F + G$ is a manifest Lorentz-scalar. The function $G$ can be expressed by the Chern-Simons functional integrated over the spatially homogeneous 3-manifolds. It is already known that an exponential of the Chern-Simons func-
tional is a formal solution of canonically quantized pure gravity with a cosmological term in Ashketar variables [22]. Furthermore, exponentials of supersymmetric extensions of the Chern-Simons functional have also previously been obtained as semi-classical WKB solutions of quantum supergravity [23, 24]. Here we find such a wave-function for our spatially homogeneous models as an exact solution of all constraints. It seems a safe conjecture that even in full supergravity with a cosmological constant an exact formal solution of this form exists. We note, however, that the solution (12) is not yet fully specified, as ‘reality conditions’ for $A_p^{AB}$ still need to be imposed. Instead of doing this we prefer to transform back to the physically more transparent tetrad representation.

The transformation of the wave-function (12) with (10), (13) from the Ashketar-representation to the metric representation (7), (8), (9) can be performed in the two steps described above. The $A_p^{AB}$ integrals required in the first step need a prior specification of the integration contour. Fortunately, not all of these integrals need to be done, because only three of the nine degrees of freedom of $A_p^{AB}$ are physical, while six correspond to gauge freedoms (three from basis changes of the $\omega^p$, three from Lorentz frame rotations) which can be fixed by a choice of gauge and are not integrated over in that gauge. However, even the remaining three integrals cannot all be performed analytically. In the limit of vanishing cosmological constant $m \to 0$ a stationary-phase approximation becomes possible. As a preparation for performing this approximation we expand the wave-function
in the fermions

\[ \Psi = \text{const} \sum_{n=1}^{3} \frac{(iF(A_p^{AB}, \psi_p^A))^n}{n!} e^{iG(A_p^{AB})}. \]  

(14)

One stationary phase point is at \( A_p^{AB} = 0 \) for all \( p = 1, 2, 3 \) and all \( A, B \in 1, 2 \). The first solution is therefore defined by choosing a suitable contour passing through this point. To discuss its limit for \( m \to 0 \) we need to keep only the dominant fermion term for \( m \to 0 \) (which, because of the appearance of \( m \) in the denominator of (10), has \( 6 \psi_p^A \)-factors and therefore fermion number 6). Then we obtain from the stationary phase at \( A_p^{AB} = 0 \)

\[ \Psi(\epsilon_p^{AA'}) = \text{const} \left( \prod_{p=1}^{3} \prod_{A=1}^{2} \psi_p^A \right) \exp \left( -\phi(\epsilon_p^{AA'}) \right) \]  

(15)

where \( \phi \) is defined in eq.(9). This is the well-known 'worm-hole state' in the 6-fermion sector [18]. In the constant prefactor we have also absorbed the divergent factor \( m^{-3} \).

Other stationary phase points are at \( A_p^{AB} \neq 0 \) and further solutions are obtained by choosing integration contours through any of them [20]. To be specific we shall discuss this for the case of Bianchi-type IX, but the other Bianchi types in class A can be treated similarly, replacing the SO(3) group by the respective Lie groups. For Bianchi type IX we have

\[ m^{pq} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Then there is a stationary phase at

\[ A_p^{AB} = V \frac{1}{2} \tau_p^{AB} \]  

(16)
where the $\tau^{pAB} = m^{pq} \tau_q^{AB}$ are a basis of 2-dimensional symmetric matrices satisfying $\tau^{pAB} = \frac{1}{2} m^{pq} \varepsilon_{grs} \tau^{rAC} \tau^{sB} C$, which determines the $\tau^{pAB}$ up to an orthogonal rotation of the basis. We choose the explicit representation

$$
\tau^{1AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{AB}, \quad \tau^{2AB} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^{AB}, \quad \tau^{3AB} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}^{AB}
$$

and adopt the convention

$$
\varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{AB}, \quad \varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_A^B.
$$

Let us now parametrize $A_p^{AB}$ by the two rotation matrices $P_{pq}$ (for coordinate rotations) and $Q_{pq}$ (for Lorentz frame rotations), each depending on three different Euler angles, which are gauge freedoms, and by the remaining three (still complex) elements $A_q$, which, after imposing some reality conditions, are three physical degrees of freedom (one combination of them playing the role of time):

$$
A_p^{AB} = \sum_{q,r} P_{pq} Q_{qr} \tau^{rAB}.
$$

Here, and until further notice we drop the summation convention. As expected $\Psi$ does not depend on $P$, $Q$. The canonical momenta of the physical variables $A_q = \frac{1}{2} \sum_{p,r} P_{pq} Q_{qr} A_p^{AB} \tau^{rAB}$ are $\sigma^{q} = -\frac{\partial}{\partial q} = \sqrt{2} \sum_{p,r} P_{pq} Q_{qr} \sigma^{p}_{AB} \tau^{rAB}$.

Let us now fix the gauges by the choice $P = 1, Q = 1$, i.e. by the condition $A_{pq} = 0$, $p \neq q$. The remaining Fourier integral over the $A_q$ then introduces the $\sigma^q$-representation for the same gauge, and we define the variables $b_1, b_2, b_3$ by $\sigma^1 = b_2 b_3$ and cyclic in 1, 2, 3. Considering the whole family of gauges for fixed
but arbitrary $C$-number rotation matrices $P$, $Q$, we may define

$$
\tilde{\sigma}_{AB}^p(P, Q) = \frac{1}{2\sqrt{2}} \sum_{qr} P_{pq} \sigma^q Q_{ qr} \tau_{ rAB} .
$$

(20)

which coincides with the full operator $\tilde{\sigma}_{AB}^p$ up to terms proportional to derivatives of the Euler angles contained in $P$ and $Q$, i.e. to generators of the gauge group. The latter are zero when acting on physical states, i.e. in Dirac’s sense [25] we have the weak equality $\tilde{\sigma}_{AB}^p \approx \tilde{\sigma}_{AB}^p(P, Q)$. For the tetrad matrix $e_p^{AB} \equiv i n^{BA'} e_p^{A'}$ and the spatial metric this gives us the further weak equalities

$$
e_p^{AB} \approx \frac{1}{\sqrt{2}} \sum_{qr} P_{pq} b_q Q_{ qr} \tau_{ rAB}$$

$$
h_{pq} \approx \sum_r P_{pq} b_r^2 P_{ qr}
$$

(21)

which explains the significance of the $b_q$. Returning to the gauge $P = 1$, $Q = 1$ and also returning to the summation convention where possible) we may now write $\Psi'$ of eq. (7) in the form

$$
\Psi'(b_1, b_2, b_3) = \text{const} \sum_{n} \frac{i^n}{n!} F^n \int_{(C)} dA_1 dA_2 dA_3 (A_1 A_2 A_3)^2 \exp \left[ \sum_y A_y \sigma^y + i G(A_y) \right].
$$

(22)

The prefactor $(A_1 A_2 A_3)^2$ comes from the Jacobian of the gauge fixing. We have introduced the abbreviations

$$
F = - \frac{1}{4\sqrt{2m}} \left( \epsilon^{mnq} \psi_p^A \psi^q_A - \sum_q \epsilon^{mqr} \psi_p^A \tau^q_{rAB} \psi_r^B \frac{\partial}{\partial \sigma^r} \right)
$$

$$
G(A_y) = \frac{1}{2m^2 V^2} \left( \frac{V}{4} (A_1^2 + A_2^2 + A_3^2) - A_1 A_2 A_3 \right).
$$

(23)

The 3-dimensional contour $C$ in the complex $A_1$, $A_2$, $A_3$ manifold now remains
to be chosen. Points of stationary phase, for \( m \to 0 \), satisfy the equations

\[
2A_1 A_2 = VA_3 \quad \text{and cyclic}
\]

with solutions

\[
\begin{align*}
(i) \quad &A_1 = A_2 = A_3 = 0, \quad (ii) A_1 = A_2 = A_3 = \frac{V}{2}; \\
(iii) \quad &A_1 = A_2 = -\frac{V}{2}, \quad A_3 = \frac{V}{2}; \\
(iv) \quad &A_2 = A_3 = -\frac{V}{2}, \quad A_2 = \frac{V}{2}; \\
v) \quad &A_3 = A_1 = -\frac{V}{2}, \quad A_2 = \frac{V}{2};
\end{align*}
\]

Choosing the contour \( C \) to run through any of these stationary points we generate a family of five linearly independent solutions \[20\]. We discuss them briefly in the asymptotic limit \( m \to 0 \) where we now also use eq. \( 8 \). The first state, arising from the choice \((i)\), has already been discussed above. In the stationary point the Jacobian in \(20\) is singular and it is better to use \((7)\) in this case. The second state with choice \((ii)\) yields

\[
\psi(b_1, b_2, b_3) \simeq \text{const} \ e^{-\frac{V}{2}(b_1^2 + b_2^2 + b_3^2) + V(b_1b_2 + b_2b_3 + b_3b_1)} F_0^2
\]

with

\[
F_0 = -\frac{V}{4\sqrt{2} m} \left( m^{pq} \psi_p^A \psi_q^A - \frac{1}{2} \sum_q \varepsilon^{pqr} \psi_p^A \tau^q_{AB} \psi_r^B \right).
\]

After some algebra it turns out that the term \( F_0^3 \) vanishes and does not appear in eq. \( 26\), even though, if nonzero, it would be the dominant term. The state \( 26\) is the Hartle-Hawking state in the 4-fermion sector, first found in \[19\] by a completely different approach.
The states arising from the choices (iii) to (v) are asymmetric but related by cyclic permutations of the coordinate directions 1, 2, 3. It is therefore sufficient to consider choice (iii) only, where we obtain asymptotically

\[ \psi(b_1, b_2, b_3) \simeq \text{const} \ e^{-\frac{V}{2}(b_1^2 + b_2^2 + b_3^2) + V(b_1 b_2 - b_2 b_3 - b_3 b_1) F_1^2} \] (28)

with

\[ F_1 = -\frac{V}{4 \sqrt{2} m} \left( m^p \psi_p^A \psi_{\bar{A}} - \frac{1}{2} \sum_q \varepsilon^{pq} \psi_p^A \left( -1 \gamma^1 + 1 \right) \tau^2_{AB} \psi^B_{\bar{r}} \right). \] (29)

(28) is also a state in the 4-fermion sector. This state could have, but in fact has not been discussed before.

In summary, we have obtained a special family of solutions of the quantized constraints of the supersymmetric Bianchi type models in class A with a nonvanishing cosmological constant. In Ashketar variables these solutions are all given by the exponential of a supersymmetric extension of the complex Chern-Simons functional, restricted to the homogeneous spatial 3-manifold under study. The different solutions of the family arise by transforming back to the tetrad variables after fixing a diffeomorphism and Lorentz gauge and using different integration contours C. In stationary phase approximation for \( m \to 0 \) we recover our earlier results, obtained for \( m \to 0 \), and find three additional asymmetric solutions in the 4-fermion sector. It is obvious that in addition to the special family more general solutions of the quantized constraints also exist. Our result is in contrast to earlier work [12]-[14] which, on the basis of an overly restrictive ansatz
for the wave-function, concluded that for these anisotropic models the quantized constraints have only the trivial solution.

A lot remains to be done, even within the restrictions of these spatially homogeneous anisotropic models: What is the significance of the appearance of the Chern-Simons functional? Can one find more general analytical solutions? Is it possible to place a scalar product and a Hilbert space structure on the space of solutions? Can one find solutions for anisotropic homogeneous models including matter? To these and related questions we hope to return in future work.

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