Generalizing the Coleman Hill Theorem

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ABSTRACT

In terms of the effective action, we show to one-loop order the Coleman-Hill theorem can be generalized to systems with spontaneous symmetry breaking. Although the correction to the parity-odd part of the vacuum polarization looks complicated in the Higgs phase, it turns out that the correction to the Chern-Simons term is identical to that in the symmetric phase, with the difference coming only from the contribution of the would be Chern-Simons term. We also discuss the implication of our result to nonabelian systems.
It is known that the Chern-Simons theories can give rise to particle excitations with fractional spin and statistics, and are thus relevant to the fractional quantum Hall effect [1, 2, 3]. Further studies show that this property is also enjoyed by the topological vortices in the Higgs phase of these systems [4]. Since the inverse of the Chern-Simons coefficient plays the role of statistical parameter, the knowledge of its quantum correction is important for a complete understanding of the quantum physics in these systems.

In the absence of massless charged particles and spontaneous symmetry breaking, Coleman and Hill have shown that the only correction to the Chern-Simons coefficient comes from the fermion one-loop contribution [5]. When the two conditions are not satisfied, higher-loop effect is generally non-vanishing and the corrections are complicated functions of couplings and particle masses [6, 7]. By analyzing the one-loop correction for a system without fermion, Khare et al show that in terms of the effective action the Chern-Simons term does not get renormalized even in the Higgs phase [8]. This suggests that the above theorem can be generalized to systems with spontaneous symmetry breaking if recast in terms of the effective action.

When the gauge symmetry is nonabelian, the coefficient must be quantized for the system to be quantum-mechanically consistent. In the symmetric phase, this has been explicitly verified to one-loop [9]. In the Higgs phase, the situation is more subtle. If the gauge symmetry is completely broken, since there is no well-defined symmetry generator, we do not expect the Chern-Simons coefficient to be quantized [10]. On the other hand, if there is remaining symmetry in the Higgs phase, we do believe and it has been shown that the corresponding Chern-Simons coefficient satisfy the quantization condition [11].

In this letter, we extend the result in [8] to systems containing also a fermion. Using the background field method, we calculate the coefficient of the would be Chern-Simons term to one loop. In terms of the effective action, we show that the one-loop correction to the Chern-Simons term in the Higgs phase is the same as that in the symmetric phase, with the difference coming solely from the contribution of the would be Chern-Simons term. We speculate that similar situation happens in the nonabelian case so that all the corrections to the Chern-Simons coefficients
are identical, if we subtract out the contribution from the would be Chern-Simons

term.

Let us consider a model with a gauge field \( A_\mu \), a complex Higgs field \( \phi \), and a
Dirac fermion field \( \psi \). The most general gauge-invariant renormalizable Lagrangian
is given as

\[
\mathcal{L} = -\frac{1}{4\epsilon^2} F^2_{\mu\nu} + \frac{\kappa}{2} e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + |D_\mu \phi|^2 + i \overline{\psi} \gamma_\mu D_\mu \psi
- (m^2 |\phi|^2 + \frac{4\lambda}{4!} |\phi|^4 + \frac{8\tau}{6!} |\phi|^6) - (M + 2g_1 |\phi|^2) \overline{\psi} \psi - g_2 [\phi^2 \overline{\psi} \psi^* + \phi^* 2 \overline{\psi} \psi].
\]

(1)

where \( D_\mu = \partial_\mu + i A_\mu \) and all coupling constants are real [12]. The metric and the
gamma matrices are chosen to be \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \), and \( \gamma^\mu = (\sigma^2, i\sigma^3, i\sigma^1) \)
so that the gamma matrices satisfy \( \gamma^\mu \gamma^\nu = \eta^{\mu\nu} - i e^{\mu\nu\rho} \gamma_\rho \) with \( \epsilon_{012} = \epsilon_{012} = 1 \).

For later convenience, we express the scalar and spinor fields in terms of real and
Majorana fields: \( \phi = (\phi_1 + i\phi_2)/\sqrt{2} \) and \( \psi = (\psi_1 + i\psi_2)/\sqrt{2} \). To proceed,
we separate \((\phi_a, A_\mu)\) into the background part \((\hat{\phi}_a, \hat{A}_\mu)\) and the quantum part
\((\phi_a, A_\mu)\). The background Lagrangian is then given by

\[
\mathcal{L}_B = \mathcal{L}(\hat{A} + \hat{A}, \hat{\phi} + \phi, \psi) - \mathcal{L}(\hat{A}, \hat{\phi}, 0) - \phi_a \frac{\partial \mathcal{L}(\hat{A}, \hat{\phi}, 0)}{\partial \hat{\phi}_a} - A_\mu \frac{\partial \mathcal{L}(\hat{A}, \hat{\phi}, 0)}{\partial \hat{A}_\mu}
\]

(2)

In the background \( R_\xi \) gauge,

\[
\mathcal{L}_{gf} = -\frac{1}{2\xi} (\partial^\mu A_\mu + \xi \hat{\phi} \times \phi)^2,
\]

(3)

which gives rise to the Fadeev-Popov ghost Lagrangian

\[
\mathcal{L}_{FP} = \bar{\eta} \left\{ -\partial_\mu^2 - \xi (\hat{\phi}^2 - \hat{\phi} \cdot \phi) \right\} \eta.
\]

(4)

Here,

\[
\hat{\phi} \times \phi \equiv \epsilon_{abc} \hat{\phi}_a \phi_b
\]

\[
\hat{\phi} \cdot \phi \equiv \hat{\phi}_a \phi_a.
\]

(5)

Combining the Lagrangian (2) and the gauge fixing terms (3) and (4), we see
to quadratic term

\[ \mathcal{L}_B = \frac{1}{2} A^\mu U^\nu (\hat{\phi}) A^\nu + A^\mu V_{\mu a} (\hat{\phi}, \hat{A}) \phi_a + \frac{1}{2} \phi_a W_{ab} (\hat{\phi}, \hat{A}) \phi_b \\
+ \frac{1}{2} \bar{\psi}_a T_{ab} (\hat{\phi}, \hat{A}) \psi_b + \bar{\eta} R(\hat{\phi}) \eta, \]  

(6)

with

\begin{align*}
U_{\mu \nu}(\hat{\phi}) & = \left[ -P^2/\epsilon^2 + \hat{\phi}^2 \right] \eta_{\mu \nu} + \left[ -1/\xi + 1/\epsilon^2 \right] P_\mu P_\nu + i \kappa \epsilon_{\mu \nu \rho} P^\rho \\
V_{\mu a}(\hat{\phi}, \hat{A}) & = -2 \epsilon_{\mu b} (\partial_{\mu} \hat{\phi}_b) - 2 \hat{A}_\mu \hat{\phi}_a \\
W_{ab}(\hat{\phi}, \hat{A}) & = \left[ P^2 \delta_{ab} - m_1^2(\hat{\phi}) \right] \left( \hat{\phi}_a \hat{\phi}_b \right) + \left[ P^2 - m_2^2(\hat{\phi}) - \xi \hat{\phi}^2 \right] \left( \delta_{ab} - \frac{\hat{\phi}_a \hat{\phi}_b}{\hat{\phi}^2} \right) \\
T_{ab}(\hat{\phi}, \hat{A}) & = \left[ \gamma \cdot P - M_1(\hat{\phi}) \right] \otimes I - M_2(\hat{\phi}) \otimes \sigma_3 - M_3(\hat{\phi}) \otimes \sigma_1 + \gamma \cdot A \otimes \sigma_2 \\
R(\hat{\phi}) & = \left[ P^2 - \xi \hat{\phi}^2 \right].
\end{align*}

(7)

Here, \( P_\mu \) is the momentum operator, and\n
\begin{align*}
m_1^2(\hat{\phi}) & = m^2 + \frac{\lambda}{6} \hat{\phi}^2 + \frac{\tau}{120} \hat{\phi}^4 \\
m_2^2(\hat{\phi}) & = m^2 + \frac{\lambda}{2} \hat{\phi}^2 + \frac{\tau}{24} \hat{\phi}^4 \\
M_1(\hat{\phi}) & = M + g_1 \hat{\phi}^2 \\
M_2(\hat{\phi}) & = g_2 (\hat{\phi}_1^2 - \hat{\phi}_2^2) \\
M_3(\hat{\phi}) & = 2g_2 (\hat{\phi}_1 \hat{\phi}_2).
\end{align*}

(8)

To one-loop order, the effective action is given by [13]

\[ \Gamma[\hat{\phi}, \hat{A}] = \frac{i}{2} \text{Tr} \{ \log W \} + \frac{i}{2} \text{Tr} \left\{ \log (U - VW^{-1}V^\dagger) \right\} + \frac{i}{2} \text{Tr} \{ \log T \} + i \text{Tr} \{ \log R \}. \]

(9)

Note that the 1/2 factor in the third term comes from the fact that \( \psi_a \) is a Majorana spinor.
On the other hand, there could exist in the effective action the following
parity-odd term [8]

\[
\Gamma^{\text{odd}}[\hat{\phi}(x), \hat{A}(x)] = \int d^3x \, e^{\mu\nu\rho} \left\{ \frac{\delta\kappa}{2} \hat{A}_\nu \partial_\rho \hat{A}_\rho + C(\hat{\phi}^2) \epsilon_{ab} \hat{\phi}_a D_\mu \hat{\phi}_b \partial_\nu \hat{A}_\rho + \ldots \right\}. \tag{10}
\]

Let \( \hat{\phi}_a = \varphi_a + \tilde{\varphi}_a(x) \). Expanding \( \Gamma^{\text{odd}}[\hat{\phi}(x), \hat{A}(x)] \) around \( \varphi \) to linear order in \( \tilde{\varphi}(x) \), we have

\[
\Gamma^{\text{odd}}[\hat{\phi}(x), \hat{A}(x)] \approx \int d^3x \, e^{\mu\nu\rho} \left\{ \left[ \frac{\delta\kappa}{2} + \varphi^2 C(\varphi^2) \right] \hat{A}_\nu \partial_\rho \hat{A}_\rho \\
+ \frac{\partial}{\partial \varphi_2} \left[ \varphi^2 C(\varphi^2) \right] \left[ 2(\varphi \cdot \tilde{\varphi}) \hat{A}_\mu \partial_\nu \hat{A}_\rho \right] \right\} + C(\varphi^2) \left[ \epsilon_{ab} \varphi_a D_\mu \tilde{\varphi}_b \partial_\nu \hat{A}_\rho \right]. \tag{11}
\]

Therefore, to determine \( \delta\kappa \), we must find out the coefficients of the first two terms in the above equation. Note that although the coefficient of the third term is much easier to calculate, it is unfortunately a total derivative term, and thus its coefficient cannot be uniquely determined.

Since we are interested in parity-odd part of the effective action, only the second and the third terms in (9) contribute. Define

\[
X = U - VW^{-1}V^i. \tag{12}
\]

For the purpose of power counting, it is convenient to expand \( X, U, V, W \) with respect to \( \hat{A} \)

\[
X = X_0 + X_1 + X_2 + \ldots \\
= U_0 - (V_0 + V_1)(W_0 + W_1 + W_2)^{-1}(V_1^i + V_1^i), \tag{13}
\]

where the subscripts denotes the powers of \( \hat{A} \).
Focusing on the relevant terms, we have

\[
\Gamma^{\text{odd}}[\phi(x), \hat{A}(x)] = \frac{i}{2} \text{Tr} \left\{ [X^{-1}_0(\phi)]^{\text{odd}} X_2(\phi) \right\} - \frac{i}{2} \text{Tr} \left\{ [X^{-1}_0(\phi)]^{\text{odd}} X_1(\phi) [X^{-1}_0(\phi)]^{\text{even}} X_1(\phi) \right\} + \frac{i}{4} \text{Tr} \left\{ T^{-1}_0(\phi) T_1(\phi) T^{-1}_0(\phi) T_1(\phi) \right\} \ldots
\]

(14)

To calculate the one-loop coefficient of the Chern-Simons terms, we set \( \hat{\phi}_a = \varphi_a \) in (14). Since \( V_0 = 0 \), we see \( X_0 = U_0 \) and the second term vanishes. Consequently,

\[
\Gamma^{\text{odd}}[\varphi, \hat{A}(x)] = -\frac{i}{2} \text{Tr} \left\{ [U^{-1}_0(\varphi)]^{\text{odd}} V_1(\varphi) W^{-1}_0(\varphi) V_1(\varphi) \right\} + \frac{i}{4} \text{Tr} \left\{ T^{-1}_0(\varphi) T_1(\varphi) T^{-1}_0(\varphi) T_1(\varphi) \right\},
\]

(15)

to one-loop order. Here, in the Landau gauge

\[
[U^{-1}_0(\varphi)]_{\mu\nu} = -\frac{e^2(p^2 - e^2\varphi^2)}{(p^2 - e^2\varphi^2)^2 - \kappa^2 e^4 p^2} \left( \eta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2} \right) - i\kappa e^4 \epsilon_{\mu\nu\rho\sigma} p^\rho,
\]

\[
[W^{-1}_0(\varphi)]_{ab} = \frac{1}{p^2 - m^2_1(\varphi)} (\varphi_a \varphi_b) + \frac{1}{p^2 - m^2_2(\varphi)} (\delta_{ab} - \varphi_a \varphi_b),
\]

(16)

\[
[T^{-1}_0(\varphi)] = \frac{\{ [\gamma \cdot p - M_1(\varphi)] \otimes I + M_2(\varphi) \otimes \sigma_3 + M_3(\varphi) \otimes \sigma_1 \}}{[\gamma \cdot p - M_+ (\varphi)] [\gamma \cdot p - M_- (\varphi)]},
\]

with

\[
M_{\pm}(\varphi) = M \pm (g_1 \pm |g_2|)\varphi^2.
\]

After some algebra, we have

\[
\Gamma^{\text{odd}}[\varphi, \hat{A}(x)] = -\frac{i}{2} \text{Tr} \left\{ \frac{-i\kappa e^4 \epsilon_{\mu\nu\rho\sigma} p^\rho}{(p^2 - e^2\varphi^2)^2 - \kappa^2 e^4 p^2} \left[ A^\mu \right] \left[ \frac{4\varphi^2}{p^2 - m^2_1(\varphi)} \right] \left[ A^\mu \right] \right\} + \frac{i}{2} \text{Tr} \left\{ \frac{1}{\gamma \cdot p - M_+ (\varphi)} [\gamma \cdot \hat{A}] \left[ \frac{1}{\gamma \cdot p - M_- (\varphi)} \right] [\gamma \cdot \hat{A}] \right\}.
\]

(17)

Employing the technique of derivative expansion and Wick rotation [14], one can
see the bosonic part contributes

\[
\frac{4}{3} \int \frac{d^3p}{(2\pi)^3} \frac{\kappa e^4 \varphi^2 p^2}{[(p^2 + \epsilon^2 \varphi^2)^2 + \kappa^2 e^4 p^2] \left[p^2 + m^2(\varphi)\right]^2}
\]

(18)
to the coefficient of the Chern-Simons term. In the pure Chern-Simons limit

\[ \epsilon \to \infty, \]

it gives

\[
\frac{m(2|m| + |m_1|)}{6\pi(|m| + |m_1|)^2},
\]

(19)
where \( m = \varphi^2/\kappa \). When \( m_1 = m \), the above result reduces to \( m/(8\pi|m|) \).

Similarly, we see the fermionic part contributes

\[
-\frac{2}{3} \int \frac{d^3p}{(2\pi)^3} \frac{[M_+(\varphi) + M_-(\varphi)] \left[p^2 + M_+(\varphi)M_-(\varphi)\right]}{\left[(p^2 + M_+^2(\varphi))^2 \left[p^2 + M_-^2(\varphi)\right]^2\right]},
\]

(20)
After integration, it leads to the following expression

\[
-\frac{(M_+ + 2M_-)|M_+| - (2M_+ + M_-)|M_-|}{12\pi(|M_+| + |M_-|)^2}.
\]

(21)
Note that the results in Eqs. (19) and (21) are identical to those obtained in [12].

For the special case \( M_+ = M_- \), we have a Dirac fermion with mass \( M_+ \) and the above result simplifies to \(-M_+/(8\pi|M_+|)\). In contrast when \( M_+ = -M_- \), we have two Majprana fermions with opposite spin and their contributions exactly cancel out.

From (11), it is easy to see

\[
\frac{\delta \kappa}{2} + \varphi^2 C(\varphi^2) = \frac{4}{3} \int_B(\varphi^2) - \frac{2}{3} \int_F(\varphi^2),
\]

(22)
where \( \int_B(\varphi^2) \) and \( \int_F(\varphi^2) \) are the integrations in Eq.(18) and (20).
To further determine $C(\varphi^2)$, we must evaluate (14) to linear order in $\dot{\varphi}$. It is easy to see that the second term still has no contribution and

$$\Gamma^{odd}[\varphi + \varphi(x), \dot{A}(x)] = \frac{i}{2} \text{Tr} \left\{ \left[ U^{-1}_{0}(\varphi) \frac{\partial U_{0}(\varphi)}{\partial \varphi} U^{-1}_{0}(\varphi) \right]^{odd} V_{1}(\varphi) W^{-1}_{0}(\varphi) V_{1}^{\dagger}(\varphi) \right\}$$

$$+ \frac{i}{2} \text{Tr} \left\{ \left[ U^{-1}_{0}(\varphi) \frac{\partial W_{0}(\varphi)}{\partial \varphi} W^{-1}_{0}(\varphi) \right] V_{1}^{\dagger}(\varphi) \right\}$$

$$- \frac{i}{2} \text{Tr} \left\{ T_{0}^{-1}(\varphi) T_{1}(\varphi) \left[ T_{0}^{-1}(\varphi) \frac{\partial T_{0}(\varphi)}{\partial \varphi} T_{0}^{-1}(\varphi) \right] T_{1}(\varphi) \right\} + \ldots$$

(23)

After tedious but straightforward calculation, we obtain the coefficient for the second term in (11):

$$\frac{\partial}{\partial \varphi^2} \left\{ \frac{4}{3} I_{B}(\varphi^2) - \frac{2}{3} I_{F}(\varphi^2) \right\}.$$  

(24)

Imposing the boundary condition that $[\varphi^2 C(\varphi^2)]_{\varphi=0} = 0$, we have

$$\varphi^2 C(\varphi^2) = \frac{4}{3} I_{B}(\varphi^2) - \frac{2}{3} I_{F}(\varphi^2) + \frac{M}{8\pi|M|}.$$  

(25)

From Eq.(22), it is easy to see that

$$\delta \kappa = -\frac{M}{4\pi|M|},$$  

(26)

In other words, the one-loop corrections to $\kappa$ in the symmetric and asymmetric phases are the same, if we subtract out the contribution from the would be Chern-Simons term.

Let us now apply the above results to the $N = 2$ and $N = 3$ self-dual Chern-Simons Higgs systems [15, 16]. For the $N = 3$ case, there are two Dirac fermions $(\psi, \chi)$ with mass $m/2$ and $-m/2$ in the symmetric phase. In the Higgs phase, the gauge boson and the Higgs field have the same mass $m$. Moreover, the mass of $\psi$ flips sign while $\chi$ splits into two Majorana fermions with mass of opposite sign.
Hence, we have

\[
\delta \kappa \bigg|_{N=3} = 0 \\
\left[ \varphi^2 C (\varphi^2) \right]_{N=3} = \frac{1}{2\pi} \frac{\kappa}{|\kappa|}. \tag{27}
\]

The \( N = 2 \) result can be obtained by discarding the contribution from \( \chi \) and

\[
\delta \kappa \bigg|_{N=2} = \frac{-1}{2\pi} \frac{\kappa}{|\kappa|} \\
\left[ \varphi^2 C (\varphi^2) \right]_{N=2} = \frac{3}{4\pi} \frac{\kappa}{|\kappa|}. \tag{28}
\]

In view of (10), we see the quantization of the corrections to the Chern-Simons coefficient in these systems is a reflection of the quantization of the anomalous magnetic moment of the charged scalars.

In this letter, we have shown that the one-loop correction to the Chern-Simons coefficient in the Higgs phase is identical to that in the symmetric phase and therefore originates only from the fermionic part, if we properly remove the contribution from the would be Chern-Simons term. An interesting question is whether the Coleman Hill theorem restated in terms of the effective action holds to all loops. As for the nonabelian case, our result suggests that after we subtract out the contribution from the would be Chern-Simons term, the correction to the Chern-Simons coefficient obtained from evaluating the vacuum polarizations of the broken and unbroken gauge bosons are identical and quantized [11]. This should also apply to the \( SU(2) \) case, even though it is not expected there as the gauge symmetry is completely broken. Finally, since the would be Chern-Simons term is invariant even under the large gauge transformation, these theorists are quantum-mechanically consistent.

Acknowledgement

The author thanks K. Lee for helpful comments and criticisms. This work is supported National Science Council, Taiwan (Grant No. NSC84-2112-M-001-022).
REFERENCES


