HECKE ALGEBRAS, TYPE III FACTORS AND PHASE TRANSITIONS WITH SPONTANEOUS SYMMETRY BREAKING IN NUMBER THEORY

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In this paper we shall discuss an example of a quantum statistical mechanical system, arising from the statistical theory of prime numbers, which exhibits a phase transition with spontaneous symmetry breaking (cf. [Bos-C]). The original motivation for these results comes from the work of B. Julia [J] (cf. also [Spe]).

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1. Description of the system and its phase transition

Let us first recall the general discussion of quantum statistical mechanics and phase transitions. A quantum statistical system is given by:

$\alpha$) The $C^*$-algebra of observables: $A$.

$\beta$) The time evolution $(\sigma_t)_{t \in \mathbb{R}}$, which is a one parameter group of automorphisms of $A$.

An equilibrium, or KMS state, at inverse temperature $\beta$ on $(A, \sigma_t)$ is a state $\varphi$ on $A$ which fulfills the KMS$_\beta$ condition with respect to $\sigma_t$, namely for any $x, y \in A$, there exists a
bounded holomorphic function, (continuous on the closed strip), $F_{x,y}(z), 0 \leq \text{Im} \, z \leq \beta$ such that:

$$F_{x,y}(t) = \phi(x \sigma_t(y)) \quad \forall t \in \mathbb{R}$$
$$F_{x,y}(t + i\beta) = \phi(\sigma_t(y)x) \quad \forall t \in \mathbb{R}.$$ 

In the simplest case where $A = M_N(\mathbb{C})$ is the algebra of $N \times N$ matrices, any one parameter group of automorphisms $(\sigma_t)_{t \in \mathbb{R}}$ of $A$ is of the form:

$$\sigma_t(x) = e^{itH} x e^{-itH} \quad \forall x \in A, \ t \in \mathbb{R}$$

for some selfadjoint element $H = H^* \in A$. Then for any $\beta \in [0, \infty[$, one has a unique KMS$_\beta$ state for $\sigma_t$, and it is given by the formula:

$$\varphi_\beta(x) = \frac{\text{Trace}(e^{-\beta H} x)}{\text{Trace}(e^{-\beta H})} \quad \forall x \in A.$$ 

Note here that $H$ is only defined up to an additive constant by $\sigma_t$, so that the normalization factor: $\text{Trace}(e^{-\beta H})$, cannot be recovered from $\sigma_t$. However the following formula holds:

$$\log \text{Trace}(e^{-\beta H}) = \sup_{\varphi} (S(\varphi) - \beta \varphi(H))$$

where $\varphi$ varies over all states on $A$ and $S(\varphi)$ is the entropy of the state:

$$S(\varphi) = -\text{Trace}(\rho \log \rho) \quad \text{for} \quad \varphi = \text{Trace}(\rho \cdot).$$

In a slightly more involved situation, that of systems without interaction it is still true that for any $\beta \in [0, \infty[$ there exists a unique KMS$_\beta$ state. More precisely one has the following immediate:

**Proposition 1.** Let $A = \bigotimes_{\nu \in I} A_{\nu}$ be an infinite tensor product of matrix algebras $A_{\nu} = M_{n_{\nu}}(\mathbb{C})$, and $\sigma_t = \bigotimes_{\nu \in I} \sigma_t^\nu$ a product time evolution. Then for any $\beta \geq 0$, there exists a unique KMS$_\beta$ state $\varphi_\beta$ for $(A, \sigma_t)$, and one has $\varphi_\beta = \bigotimes_{\nu} \varphi_{\beta,\nu}$, where $\varphi_{\beta,\nu}$ is the unique KMS$_\beta$ state for $(A_{\nu}, \sigma_t^\nu)$.

For interesting systems with interaction, one expects in general that for large temperature, i.e. small $\beta$, the disorder will be predominant so that there will exist only one KMS$_\beta$ state. For small enough temperature some order should set in and allow for the existence of various thermodynamical phases, i.e. of various KMS$_\beta$ states. It is a very important general fact of the $C^*$-algebraic formulation of quantum statistical mechanics that for a
given \( \beta \) every KMS\(_\beta \) state decomposes uniquely as a statistical superposition of extreme KMS\(_\beta \) states:

**Proposition 2.** ([Br-R] [H]) Let \((A, \sigma_t)\) be a \(C^*\)-dynamical system and \( \beta \in [0, \infty[ \). Then the space of KMS\(_\beta \) states is a compact convex Choquet simplex.

For a careful discussion of the link between extreme KMS\(_\beta \) states and thermodynamical phases we refer the reader to [H].

As a simple (classical) example illustrating the coexistence of phases at small temperature one can think of the phase diagram for water and vapor or better for the ferromagnet. In the latter example when the temperature \( T \) is larger than the critical temperature \( T_c \), of the order of \( 10^3K \), the disorder dominates, while for \( T < T_c \) the individual magnets tend to align with each other, which in the classical 3 dimensional set-up yields a set of thermodynamical homogeneous phases parametrized by the 2 dimensional sphere of directions in 3 space.

This example serves to illustrate the phenomenon of spontaneous symmetry breaking: The group \(SO(3)\) of rotations in \( \mathbb{R}^3\) is a symmetry group of the dynamical system one starts with, and for large \( T, T > T_c \), the equilibrium state is unique and hence invariant under rotation. For small \( T \) however, \( T < T_c \), the group \(SO(3)\) acts non trivially on the set of thermodynamical phases and the choice of an equilibrium state breaks the symmetry.

The \(C^*\)-algebraic formulation of this is straightforward. One has a (compact) group \( G \) of automorphisms of the \(C^*\)-algebra \( A \) which commutes with the time evolution:

\[
\alpha_g \in \text{Aut} \, A \quad \forall g \in G \quad \alpha_g \sigma_t \alpha_g^* = \sigma_t \forall t \in \mathbb{R}.
\]

Such a group obviously acts on the compact convex space of KMS\(_\beta \) states and hence on its extreme points.

We shall now describe (the precise motivation will be explained below) a \(C^*\)-dynamical system intimately related to the distribution of prime numbers and exhibiting the above behaviour of spontaneous symmetry breaking.

The \(C^*\)-algebra \( A \) is a Hecke algebra, which contains the algebra of usual Hecke operators of number theory, i.e. those related to Hecke correspondences for lattices in \( \mathbb{C} \) ([Sh], [Ser1]). The latter algebra is commutative and is essentially the algebra of composition of double cosets:

\[
\gamma \in GL(2, \mathbb{Z}) \backslash GL(2, \mathbb{Q}) / GL(2, \mathbb{Z})
\]

More generally, given a discrete group \( \Gamma \) and a subgroup \( \Gamma_0 \) which is almost normal, namely which satisfies the condition
"The orbits of $\Gamma_0$ acting on the left on $\Gamma/\Gamma_0$ are finite.".

one defines the Hecke algebra $\mathcal{H}(\Gamma, \Gamma_0)$ as the convolution algebra of ($\mathbb{C}$-valued for our purposes) functions with finite support on $\Gamma_0 \backslash \Gamma/\Gamma_0$. More specifically given two such functions $f, f' \in \mathcal{H}(\Gamma, \Gamma_0)$, their convolution is

$$(f * f')(\gamma) = \sum_{\gamma \gamma_1^{-1} \in \Gamma_0 \backslash \Gamma} f(\gamma \gamma_1^{-1}) f'(\gamma_1) \quad \forall \gamma \in \Gamma.$$ 

In this formula $f$ and $f'$ are viewed as $\Gamma_0$ biinvariant functions on $\Gamma$ with finite support in $\Gamma_0 \backslash \Gamma/\Gamma_0$.

To complete $\mathcal{H}$ to a $C^*$-algebra we just close it in norm in the following regular representation of $\mathcal{H}$ in $\ell^2(\Gamma_0 \backslash \Gamma)$ (cf. [Bi]).

**Proposition 3.** Let $\Gamma_0 \subset \Gamma$ be an almost normal subgroup of the discrete group $\Gamma$. Then the following formula defines an (involutive) representation $\lambda$ of $\mathcal{H}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0 \backslash \Gamma)$:

$$(\lambda(f)\xi)(\gamma) = \sum_{\gamma \in \Gamma} f(\gamma \gamma_1^{-1}) \xi(\gamma_1) \quad \forall \gamma \in \Gamma_0 \backslash \Gamma, \forall f \in \mathcal{H}.$$ 

One checks that $\lambda(f)$ is bounded for any $f \in \mathcal{H}(\Gamma, \Gamma_0)$. The involution on $\mathcal{H}$ such that:

$$\lambda(f^*) = \lambda(f)^* \quad \forall f \in \mathcal{H}$$

is given by the following equality:

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \quad \forall \gamma \in \Gamma_0 \backslash \Gamma/\Gamma_0.$$ 

Thus we let $A$ be the $C^*$-algebra norm closure of $\mathcal{H}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0 \backslash \Gamma)$. A good notation for it, compatible with the discrete group case is:

$$\overline{\mathcal{H}}(\Gamma, \Gamma_0) = C^*_r(\Gamma, \Gamma_0).$$

Let us now define the one parameter group of automorphisms $\sigma_t \in \text{Aut} \ A$. We first need to introduce a notation. Since each $\Gamma_0$ orbit on $\Gamma/\Gamma_0$ is finite we shall let, for $\gamma \in \Gamma$

$$L(\gamma) = \text{cardinality of the image of } \Gamma_0 \gamma \Gamma_0 \text{ in } \Gamma/\Gamma_0$$

$$R(\gamma) = \text{cardinality of the image of } \Gamma_0 \gamma \Gamma_0 \text{ in } \Gamma_0 \backslash \Gamma.$$ 

Thus by construction $L(\gamma) \in \mathbb{N}^*$, $R(\gamma) \in \mathbb{N}^*$. $R(\gamma) = L(\gamma^{-1})$, $L$ and $R$ are both $\Gamma_0$ biinvariant functions.
Proposition 4. Let \( \Gamma_0 \subset \Gamma \) be an almost normal subgroup of the discrete group \( \Gamma \). There exists a unique one parameter group of automorphisms \( \sigma_t \in \text{Aut}(C^*_r(\Gamma, \Gamma_0)) \) such that

\[
(\sigma_t(f))(\gamma) = \left( \frac{L(\gamma)}{R(\gamma)} \right)^{-it} f(\gamma) \quad \forall \gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0.
\]

In fact, as we shall see later, \( \sigma_{-i} \) is the restriction of the modular automorphism group \( \sigma_t^\rho \) for the state on \( M = \lambda(\mathcal{H})'' \) given by the unit vector corresponding to the coset \( \gamma_0 \in \Gamma_0 \backslash \Gamma \).

Let us now consider the Hecke algebra \( \mathcal{H} \) for the groups:

\[
\Gamma = P^+_Q, \quad \Gamma_0 = P^+_Z
\]

where \( P \) is the group of \( 2 \times 2 \) matrices \( P = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} : aa^{-1} = a^{-1}a = 1 \right\} \) and the + indicates that we restrict to \( a > 0 \).

One checks that \( P^+_Z \) is almost normal in \( P^+_Q \) (cf. Lemma 13).

We shall now describe the phase transition with spontaneous symmetry breaking for the dynamical system corresponding to \( \Gamma = P^+_Q, \quad \Gamma_0 = P^+_Z \).

Let us denote by \( \psi_\beta \) the following function on the group \( Q/Z \). Given \( n = a/b \in Q/Z \), with \( a, b \in Z \), with \( a \) relatively prime to \( b > 0 \), one lets

\[
b = \prod_{p \in P} p^{k_p}
\]

be the prime factor decomposition of \( b \) and one sets:

\[
\psi_\beta(n) = \prod_{p \in P, k_p \neq 0} p^{-k_p} \cdot (1 - p^{-2})(1 - p^{-1})^{-1}.
\]

The inclusion of the unipotent subgroup

\[
\left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in Q \right\} \subset P^+_Q
\]

defines an imbedding \( Q/Z \subset \Gamma_0 \backslash \Gamma / \Gamma_0 \) and injective morphisms of involutive algebras

\[
C[Q/Z] \subset \mathcal{H}(\Gamma, \Gamma_0)
\]

and

\[
(\cdot^* Q/Z) \subset C^*_r(\Gamma, \Gamma_0).
\]

The main result of this paper is:
Theorem 5. Let \((A, \sigma_t)\) be the \(C^*\)-dynamical system associated to the almost normal subgroup \(P^+_2\) of \(P^+_Q\). Then:

a) For \(0 < \beta \leq 1\) there exists a unique KMS\(_\beta\) state \(\varphi_{\beta}\) on \((A, \sigma_t)\). Its restriction to \(C^*(\mathbb{Q}/\mathbb{Z}) \subset C^*_r(\Gamma, \Gamma_0)\) is given by the above function of positive type \(\psi_{\beta}\) on \(\mathbb{Q}/\mathbb{Z}\). Each \(\varphi_{\beta}\) is a factor state and the associated factor is the hyperfinite factor of type III\(_1\), \(R_\infty\).

b) For \(\beta > 1\) the KMS\(_\beta\) states on \((A, \sigma_t)\) form a simplex whose extreme points \(\varphi_{\beta, \lambda}\) are parametrized by the complex imbeddings \(\lambda: \mathbb{Q}^{\text{cycl}} \rightarrow \mathbb{C}\) of the subfield \(\mathbb{Q}^{\text{cycl}}\) of \(\mathbb{C}\) generated by the roots of unity and whose restrictions to \(C^*(\mathbb{Q}/\mathbb{Z})\) are given by the following formula:

\[
\varphi_{\beta, \lambda}(\gamma) = \zeta(\beta)^{-1} \sum_{n=1}^{\infty} n^{-\beta} \lambda(\gamma)^n.
\]

These states are type \(I_\infty\) factor states.

The normalization factor is the inverse of the Riemann \(\zeta\) function evaluated at \(\beta\).

In other words the critical temperature here is \(T_c = 1\) and at low temperature (\(\beta > 1\)) the phases of the system are parametrized by all possible embeddings of \(K = \mathbb{Q}^{\text{cycl}}\) in the field of complex numbers.

As we shall see below the Galois group \(G = \text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})\) does act naturally as a group of automorphisms of \(C^*_r(\Gamma, \Gamma_0)\) commuting with the time evolution \((\sigma_t)_{t \in \mathbb{R}}\), and the spontaneous symmetry breaking occurs for \(\beta > 1\).

Before we begin the proof of Theorem 5, we shall explain how the above \(C^*\)-dynamical system is related to the distribution of prime numbers.

2. Bosonic second quantization and prime numbers as a subset of \(\mathbb{R}\)

It is a saying of E. Nelson that first quantization is a mystery while second quantization is a functor. In the bosonic case this functor \(S\) from the category of Hilbert spaces to itself, assigns to every Hilbert space \(\mathcal{H}\) the new Hilbert space \(S\mathcal{H}\) given by:

\[
S\mathcal{H} = \oplus_{n=0}^{\infty} S^n\mathcal{H}
\]

where \(S^n\mathcal{H}\) is the \(n\)th symmetric power of \(\mathcal{H}\) endowed with the following inner product:

\[
\langle \xi_1 \cdots \xi_n, \eta_1 \cdots \eta_n \rangle = \sum_{\sigma} \prod_{i=1}^{n} \langle \xi_i, \eta_{\sigma(i)} \rangle \quad \forall \xi, \eta \in \mathcal{H}
\]
(see for instance [G]). Given an operator $T$ in $\mathcal{H}$ (more generally $T : \mathcal{H}_1 \to \mathcal{H}_2$), the operator $ST$ in $\mathcal{SH}$ is given by:

$$(ST)(\xi_1 \cdots \xi_n) = (T\xi_1)(T\xi_2) \cdots (T\xi_n) \quad \forall \xi_i \in \mathcal{H}.$$ 

Even if $T$ is bounded, $ST$ is not bounded in general but if $T$ is selfadjoint (unbounded) so is $ST$. Thus we shall work with such operators. One has the following formula:

$$\text{Trace}(ST) = \frac{1}{\det(1 - T)} \quad (*)$$

which makes good sense if $\| T \| < 1$ and $T \in \mathcal{L}^1(\mathcal{H})$.

The problem we shall now consider is the following: to give a simple characterization of selfadjoint operators $T$ in $\mathcal{H}$ whose spectrum is the subset $\mathcal{P} \subset \mathbb{R}$ formed of all prime numbers, each with multiplicity one:

$$\mathcal{P} = \{2, 3, 5, 7, 11, 13, 17, \ldots\} \subset \mathbb{R}.$$ 

The corresponding problem for the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of natural numbers, or $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, is easier, and was solved by Dirac’s paper [Dir] which inaugurated quantum field theory. In that case the solution is simply that there exists an operator $a$ such that:

$$aa^* - a^*a = 1, \quad a^*a = T.$$ 

(For $\mathbb{N}^*$ one requires that $aa^*$ be equal to $T$.)

Let us now state the result for the subset $\mathcal{P} \subset \mathbb{R}$:

**Lemma 6.** Let $T$ be a selfadjoint operator in a Hilbert space $\mathcal{H}$ then (counting multiplicities):

$$\text{Spectrum } T = \mathcal{P} \iff \text{Spectrum } ST = \mathbb{N}^*.$$ 

**Proof.** Let us first assume that Spectrum $ST = \mathbb{N}^*$. Then, as quite generally $\text{Spec} T \subset \text{Spec} ST$, (using the inclusion $\mathcal{H} \subset \mathcal{SH}$) we see that $\Sigma = \text{Spec}(T) \subset \mathbb{N}^*$. Let us show that $\mathcal{P} \subset \Sigma$. Indeed let $p \in \mathcal{P}$ not be in $\Sigma$. Then because $\Sigma \subset \mathbb{N}^*$ one has $p \notin \Sigma^n$ for any $n$ (with $\Sigma^n = \{k_1k_2\cdots k_n : k_j \in \Sigma\}$). This shows that $p \notin \text{Spec}(ST) = \cup \Sigma^n$ whence a contradiction. Thus $\mathcal{P} \subset \Sigma$. If $k \in \Sigma \setminus \mathcal{P}$ then as $k \in \mathcal{P}^n$ for some $n > 1$ this would mean that $k$ is not a simple eigenvalue for $ST$. Thus $\mathcal{P} = \Sigma$. The converse is obvious from Euclid's unique factorization theorem, but we shall fix the corresponding notations: we let $\mathcal{H}_1 = \ell^2(\mathcal{P})$ be the Hilbert space with basis $(\varepsilon_p)_{p \in \mathcal{P}}$, and we identify it to the one particle
subspace of $\mathcal{S}\mathcal{H}_1 = \ell^2(\mathbb{N}^*)$, the Hilbert space of square integrable sequences of complex numbers, with canonical basis the $\varepsilon_n$, $n \in \mathbb{N}^*$. We shall denote by $T$ the operator:

$$T : \ell^2(\mathcal{P}) \rightarrow \ell^2(\mathcal{P}) ; \ T\varepsilon_p = p\varepsilon_p \quad \forall p \in \mathcal{P}$$

and by $ST$ the corresponding operator:

$$ST : \ell^2(\mathbb{N}^*) \rightarrow \ell^2(\mathbb{N}^*) ; \ (ST)\varepsilon_n = n\varepsilon_n \quad \forall n \in \mathbb{N}^* .$$

We shall let $H = \log(ST)$. It is the generator of a one parameter unitary group $U_t = \exp(itH) = T^it$, whose role is made clear by the following special case of formula (⋆) which is the Euler product formula for the Riemann $\zeta$ function:

$$\text{For } \Re s > 1; \ \zeta(s) = \text{Trace } (ST)^s = \frac{1}{\zeta(1 - is)} .$$

The meaning of Lemma 6 is that the subset $\mathcal{P} \subset \mathbb{R}$ has a neat definition provided one is ready to use the formalism of bosonic quantum field theory. That formalism includes the algebra of creation and annihilation operators, respectively $a^*(\xi)$ and $a(\eta)$, for $\xi, \eta \in \mathcal{H}$, given by:

$$a^*(\xi)\xi_1 \cdots \xi_n = \xi\xi_1 \cdots \xi_n \quad \forall \xi_j \in \mathcal{H}$$

$$a(\eta) = (a^*(\eta))^* .$$

It also includes the time evolution, in Heisenberg’s picture, given by:

$$\sigma_t(x) = U_t \ x \ U_t^* = e^{itH} \ x \ e^{-itH} \quad \forall t \in \mathbb{R} .$$

In our case the corresponding $C^*$-algebra in $\mathcal{S}\mathcal{H} = \ell^2(\mathbb{N}^*)$ and time evolution are given by the following:

**Proposition 7.** a) For every $p \in \mathcal{P}$ let $\mu_p$ be the isometry in $\ell^2(\mathbb{N}^*)$ given by the polar decomposition of the creation operator associated to the unit vector $\varepsilon_p \in \mathcal{H}$. The $C^*$-algebra $C^*(\mathbb{N}^*)$ generated by the $\mu_p$'s is the same as that generated by the isometries $\mu_n$, $n \in \mathbb{N}^*$, defined by:

$$\mu_n \varepsilon_k = \varepsilon_{kn} \quad \forall k \in \mathbb{N}^* .$$

b) This $C^*$-algebra is the infinite tensor product:

$$C^*(\mathbb{N}^*) = \bigotimes_{p \in \mathcal{P}} \tau_p$$

where each $\tau_p$ is the $C^*$-algebra generated by $\mu_p$ and is the Toeplitz $C^*$-algebra.
c) The equality $\sigma_t(x) = e^{itH}x e^{-itH}$, $\forall x \in C^*(\mathbb{N}^*)$, $t \in \mathbb{R}$ where $H = \log(ST)$, defines a one parameter group of automorphisms of $C^*(\mathbb{N}^*)$ given as

$$\sigma_t = \bigotimes_{p \in \mathbb{P}} \sigma_{t,p} ; \sigma_{t,p}(\mu_p) = \mu_p^{it} \mu_p \quad \forall t \in \mathbb{R}.$$ 

**Proof.** a) By construction $\mu_p$ is the one sided shift in the Hilbert space $S \subseteq \varepsilon_p$ tensored by the identity in each of the Hilbert spaces $S \subseteq \varepsilon_q$, $q \neq p$ in the decomposition

$$S\mathcal{H} = \bigotimes_{q \in \mathbb{P}} (S(\mathbb{C} \varepsilon_q)).$$

In terms of the basis $(\varepsilon_n)$ of $S\mathcal{H}$ one thus has:

$$\mu_p \varepsilon_n = \varepsilon_{pn} \quad \forall n \in \mathbb{N}$$

so that a) follows.

b) We recall that the Toeplitz $C^*$-algebra $\tau$ is the $C^*$-algebra defined by a unique generator $u$ satisfying the relation $u^*u = 1$. If $u$ is any non-unitary isometry in a (separable) Hilbert space the smallest $C^*$-algebra containing $u$ is isomorphic to $\tau$. This $C^*$-algebra is nuclear so that the finite tensor products $\bigotimes_{p \leq n} \tau_p$ are unambiguously defined. The $C^*$-algebra $\bigotimes_{p \in \mathbb{P}} \tau_p$ is their inductive limit. Now for each $p$ the isometry $\mu_p$ generates $\tau_p$ in $S \subseteq \varepsilon_p$ and since the finite tensor products $\bigotimes_{p \leq n} \tau_p$ are faithfully represented in $\mathcal{H}$ we get b).

c) follows from a direct computation.

The $C^*$-dynamical system thus obtained is not very interesting because it is without interaction (see Proposition 8. a)). Nevertheless the unique associated KMS$_\beta$ states will be useful later and are given by the following corollary of Proposition 1 and of the Araki-Woods classification of ITPF1.

**Proposition 8.** a) For every $\beta > 0$, there exists a unique KMS$_\beta$ state on $(C^*(\mathbb{N}^*), \sigma_t)$.

It is the infinite tensor product:

$$\varphi_\beta = \bigotimes_{p \in \mathbb{P}} \varphi_{\beta,p}$$

where $\varphi_{\beta,p}$ is the unique KMS$_\beta$ state on $(\tau_p, \sigma_{t,p})$ for $\sigma_{t,p}$. The eigenvalue list of $\varphi_{\beta,p}$ is:

$$\{(1 - p^{-n\beta})p^{-n\beta} : n \in \mathbb{N}\}.$$
b) For $\beta > 1$, the state $\varphi_\beta$ is of type $I_\infty$ and is given by:

$$\varphi_\beta(x) = \zeta(\beta)^{-1} \text{Trace}(e^{-\beta x}) \quad \forall x \in C^*(\mathbb{N}^*) .$$

c) For $\beta = 1$, the state $\varphi_\beta$ is a factor state of type $III_1$ given by:

$$\varphi_\beta(x) = \text{Trace}_\omega(e^{-x}) \quad \forall x \in C^*(\mathbb{N}^*)$$

where $\text{Trace}_\omega$ is the Dixmier trace.

d) For $0 < \beta \leq 1$, $\varphi_\beta$ is a factor state of type $III_1$ and the associated factor is the factor $R_\infty$ of Araki-Woods.

Statement d) for $\beta = 1$ is due to B. Blackadar ([Bl]). We refer to [Co] IV.2 for the definition of the Dixmier trace, whose general properties make it clear that the equality c) defines a KMS$_1$ state.

**Proof.** a) Let us first show that there exists a unique KMS$_\beta$ state on $\tau_p$ for the group $\sigma_{t,p}$. We let $\tau_p$ be the $C^*$-algebra in $\ell^2(\mathbb{N})$ generated by the one sided shift $S$, while $\sigma_{t,p}(S) = p^{it}S$ is the one parameter group of automorphisms. First let $\varphi_{\beta,p}$ be the restriction to $\tau_p$ of the state on $\mathcal{L}(\ell^2(\mathbb{N}))$ given by

$$\varphi(T) = \sum_{n \in \mathbb{N}} A_{n,n} \nu(n)$$

where $\nu(n) = p^{-n\beta} \left( \sum_{m=0}^\infty p^{-m\beta} \right)^{-1}$.

One checks that $\varphi_{\beta,p}$ is a KMS$_\beta$ state on $\tau_p$. Conversely, let $\varphi$ be a KMS$_\beta$ state on $\tau_p$. Then the KMS$_\beta$ condition shows that $\varphi$ vanishes on any eigenvector $A \in \tau_p$,

$$\sigma_{t,p}(A) = \lambda^{it} A \quad \forall t \in \mathbb{R} \quad \text{provided } \lambda \neq 1 .$$

It also shows that $\varphi(SS^*) = \varphi(S^* \sigma_{-t\beta}(S)) = p^{-\beta} \varphi(S^*S) = p^{-\beta}$ so that $\varphi(1 - SS^*) = 1 - p^{-\beta}$. More generally, one has for any $k, \ell \in \mathbb{N}$:

$$\varphi(S^k S^* \ell) =
\begin{cases}
0 & \text{if } k \neq \ell \\
p^{-k\beta} & \text{if } k = \ell .
\end{cases}$$

This shows the uniqueness of $\varphi = \varphi_{\beta,p}$ on the ideal $\mathcal{K}$ of compact operators, $\mathcal{K} \subset \tau_p$.

Thus the difference $\psi = \varphi - \varphi_{\beta,p}$ vanishes on $\mathcal{K}$ and is a continuous linear form on the quotient $C^*$-algebra $\tau_p/\mathcal{K} = C(\mathbb{S}^1)$. We saw that $\varphi(S^n) = \varphi_{\beta,p}(S^n) = 0$ for any $n > 0$ and similarly for $(S^*)^n$ so that $\psi = 0$ and $\varphi = \varphi_{\beta,p}$.
The uniqueness of $\varphi_\beta$ then follows by a general argument for tensor products: Let $(A, \sigma_t^A)$ and $(B, \sigma_t^B)$ be $C^*$-dynamical systems and $\varphi$ a KMS$_\beta$ state on $(A \otimes B, \sigma_t^A \otimes \sigma_t^B)$. Then for any $b \in B$ the functional on $A$:

$$\varphi_b(a) = \varphi(a \otimes b)$$

is KMS$_\beta$ on $A$. Indeed for any $x, y$ in $A$ one has:

$$\varphi(\sigma_t^A(x) y \otimes b) = \varphi(\sigma_t^{A \otimes B}(x \otimes 1)(y \otimes b))$$

$$\varphi(y \sigma_t^A(x) \otimes b) = \varphi((y \otimes b) \sigma_t^{A \otimes B}(x \otimes 1)) .$$

b) One uses the finiteness of $\text{Trace}(e^{-\beta H})$ for $\beta > 1$.

c) By construction one has an infinite tensor product of type I factors with eigenvalue list given by

$$\lambda_{p,n} = p^{-n\beta}(1 - p^{-\beta})$$

thus the assertion follows from [A-W].

d) One directly checks the KMS$_1$ condition, and the same argument as for c) shows that $\varphi_1$ is of type III$_1$.

3. Products of trees and the non-commutative Hecke algebra

In this section we shall relate the $C^*$-dynamical system $(C^*(\mathbb{N}^*), \sigma_t)$, of section $\beta$) with basic number theory notions [We$_2$] and get the Hecke dynamical system of Theorem 5.

Let $P$ be the $ax + b$ group, i.e. the group of triangular $2 \times 2$ matrices of the form

$$\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} ,$$

with $a$ invertible. We view it as an algebraic group over $\mathbb{Z}$, i.e. as a functor $A \rightarrow P_A$ from commutative rings to groups. It plays an important role in the elementary classification of locally compact (commutative and non discrete) fields (cf. [We$_2$]). Indeed given such a field $K$ then the group $G = P_K$ is a locally compact group, and as such it has a module

$$\delta : G \rightarrow \mathbb{R}_+^* ,$$

defined by the lack of invariance of a left Haar measure $dg$ on $G$ under right translations:

$$d(gk) = \delta(k)dg \quad \forall k \in G .$$

(1) (Or equivalently $d(g^{-1}) = \delta(g)^{-1}dg$ as measures on $G$.)
This module $\delta : P_K \to \mathbb{R}_+^*$ is 1 on the additive group and its restriction to the multiplicative group (extended by 0 on $K \setminus K^*$) yields a proper continuous multiplicative map:

$$\mod_K : K \to \mathbb{R}_+.$$  

In fact, if $dx$ denotes the Haar measure on the (locally compact) additive group $K$, we have

$$d(ax) = \mod_K(a) \, dx \quad \forall a \in K^*.$$

Moreover the open sets \( \{ k \in K : \mod_K(k) < \varepsilon \} \) form a basis of neighborhoods of 0. The image of $\delta$ is a closed subgroup of $\mathbb{R}_+^*$ and except for the case of the archimedian fields $\mathbb{R}$ or $\mathbb{C}$, this closed subgroup is discrete equal to $\lambda^\mathbb{Z}$ for some $\lambda \in ]0, 1[ \) whose inverse $q = \lambda^{-1}$ is called the module of $K$. The function on $K \times K$ $d(x, y) := \mod_K(x - y)$ is then a ultrametric distance giving back the topology of $K$ ([We2]). In other words, we have:

**Proposition 9.** (cf. [We2]) Let $K$ be a non-discrete commutative locally compact field, $K \neq \mathbb{R}$ or $\mathbb{C}$. Then there exists a prime $p$ such that $\mod_K(p) < 1$. Call $R$, $R^*$ and $P$ the subsets of $K$ respectively given by

$$R = \{ x \in K : \mod_K(x) \leq 1 \}, \quad R^* = \{ x \in K : \mod_K(x) = 1 \},$$

$$J = \{ x \in K : \mod_K(x) < 1 \}.$$

Then $R$ is the unique maximal compact subring of $K$; $R^*$ is the group of invertible elements of $R$; $J$ is the unique maximal ideal of $R$, and there is $\pi \in J$ such that $J = \pi R = R\pi$. The topology on the topological group $K$ is the unique (ultrametric) topology such that the ideals $\pi^n R$ form a basis of neighborhood of 0. Moreover, the residue field $k = R/J$ is a finite field of characteristic $p$: if $q$ is the number of its elements, the image of $K^*$ in $\mathbb{R}_+^*$ under $\mod_K$ is the subgroup of $\mathbb{R}_+^*$ generated by $q$.

Given $x \in K$ the integer $v(x)$ such that $\mod_K(x) = q^{-v(x)}$ is called the valuation of $x$.

As a basic example the field $\mathbb{Q}_p$ of $p$-adic numbers is defined (given any prime number $p$), as the completion of the field $\mathbb{Q}$ of rational numbers for the distance function:

$$d(x, y) = |x - y|_p$$

where for $x \in \mathbb{Q}$, $x = p^n \frac{a}{b}$ (with $u, a, b$ integers and $a, b$ relatively prime to $p$) one sets:

$$|x|_p = p^{-u}.$$
The maximal subring $R$ of $K = \mathbb{Q}_p$ is the ring $\mathbb{Z}_p$ of $p$-adic integers and the residual field $k = R/J$ is the finite field $\mathbb{F}_p$.

One obtains in this way, together with the inclusion $\mathbb{Q} \subset \mathbb{R}$, all the inclusions $\mathbb{Q} \subset K$ of the field of rational numbers as a dense subfield of a local field $K$. Such inclusions (or rather in general, equivalence classes of completions) are called places and to distinguish the real place $\mathbb{Q} \subset \mathbb{R}$ from the others, the latter ones are called finite places.

Putting together the inclusions of $\mathbb{Q}$ in its completions $\mathbb{Q}_p = K$ parametrized by the places of $\mathbb{Q}$ one gets a single inclusion of $\mathbb{Q}$ in the locally compact commutative ring of adeles which is the restricted product of the fields $\mathbb{Q}_p$. More specifically this ring is the product $\mathbb{R} \times \mathcal{A}$ where the ring $\mathcal{A}$ of finite adeles is obtained as follows:

a) Elements $x$ of $\mathcal{A}$ are arbitrary families $(x_p)_{p \in \mathcal{P}}$, with $x_p \in \mathbb{Q}_p$, such that $x_p \in \mathbb{Z}_p$ for all $p$ but a finite number.

b) $(x + y)_p = x_p + y_p$; $(xy)_p = x_p y_p$ define the addition and product in $\mathcal{A}$.

c) Finally $\mathcal{A}$ has the unique topology of locally compact ring such that the subring

$$\mathcal{R} = \prod_{p \in \mathcal{P}} \mathbb{Z}_p$$

is open and closed and inherits its compact product topology.

We shall now relate the $C^*$-dynamical system $(C^*(\mathbb{N}^*), \sigma_t)$ of section $\beta$ with the locally compact ring $\mathcal{A}$ of finite adeles.

We just need to recall that given any locally compact group $G$ with modular function $\delta : G \rightarrow \mathbb{R}_+^*$, one has a natural one parameter group of automorphisms $\sigma_t$ of $C^*(G)$ given by the following formula valid say on $L^1(G)$:

$$(\sigma_t(f))(g) = \delta(g)^{-it} f(g) \quad \forall g \in G, \ t \in \mathbb{R}.$$  

This automorphism group defines also an automorphism group of the reduced $C^*$-algebra $C^*_r(G)$, and the group $\sigma_{-t}$ is the modular automorphism group of the Plancherel weight on $C^*(G)$.

**Proposition 10.** Let $\mathcal{A}$ be the ring of finite adeles over $\mathbb{Q}$, and $\mathcal{R}$ its maximal (open) compact subring. Let $G$ be the locally compact group $G = P_{\mathcal{A}}$, and $\varepsilon \in C^*(G)$ the characteristic function of the open and compact subgroup $P_{\mathcal{R}} \subset P_{\mathcal{A}}$. Then:

1) One has $\varepsilon = \varepsilon^* = \varepsilon^2$, and the reduced $C^*$-algebra $C^*(G)_\varepsilon = \{ x \in C^*(G) \ ; \ \varepsilon x = xe = x \}$ is canonically isomorphic to the $C^*$-algebra $C^*(\mathbb{N}^*)$ of section $\beta$.

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2) One has \( \sigma_t(e) = e \ \forall t \in \mathbb{R} \), and the restriction of \( \sigma_t \) to the reduced \( C^* \)-algebra \( C^*(\mathbb{N}^*) \) is the time evolution of section (3).

We think of the characteristic function of \( P_K \) as an element of \( L^1(G, dg) \subset C^*(G) \), with \( dg \) the unique left Haar measure which gives measure 1 to \( P_K \). The group \( G \) is solvable and hence amenable so that there is no distinction between \( C^*(G) \) and the reduced \( C^* \)-algebra \( C^*_r(G) \).

The proof of Proposition 10 reduces immediately to a local statement, namely, if \( K = \mathbb{Q}_p \) and \( R \subset K \) is the maximal compact subring, the \( C^* \)-dynamical system \( (C^*(P_K), \sigma_t) \) given by (5), reduced by the projection \( \epsilon_p \) defined as the characteristic function of \( P_K \), is isomorphic to the Toeplitz \( C^* \)-algebra \( \tau_p \), with the time evolution \( \sigma_{1,p} \) of Proposition 7, 3).

To check this one uses the isomorphisms

\[ C^*(P_K) \cong C^*(K) \cong C_0(K) \cong K^* \]

given by the identification of the additive group \( K \) with its Pontrjagin dual (cf. [We2]; we use in the sequel the same normalizations for the Haar measure and the Fourier transform on \( K \) as in \textit{loc.cit.}, namely the additive Haar measure of \( R = \mathbb{Z}_p \) is 1, and the identification of \( K \) with its Pontrjagin dual is such that \( R^\perp = R \)). Then to \( \epsilon_p \) corresponds the element of the crossed product given by \( \int_{R^*} 1_R U_{\sigma} \ d\sigma \). The reduced \( C^* \)-algebra \( C^*(P_K)_{\epsilon_p} \) is then generated by the isometry \( \mu_p, \mu_p \in C^*(P_{\mathbb{Q}_p})_{\epsilon_p} \), given by the following \( L^1 \) function:

\[
(6) \quad \mu_p \left( \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \right) = 1 \quad \text{if} \quad a \in R, \ \text{val}(a) = 1, \ \text{and} \ \text{equal} \ \text{to} \ 0 \ \text{otherwise}.
\]

Let us check that \( \mu_p \) is an isometry, i.e. that \( \mu_p^* \mu_p = \epsilon_p \). The left Haar measure on \( P_K \) is given by \( d \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} = db \ d^*a \) where \( d^*a \) is the multiplicative Haar measure normalized so that \( \int_{R^*} d^*a = 1 \). The module \( \delta \) of the locally compact group \( P_K \) is given by \( \delta \left( \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \right) = |a| \). The adjoint \( \mu_p^* \) of \( \mu_p \) is given by the function

\[
(7) \quad \mu_p^*(g) = \frac{1}{\mu_p(g^{-1})} \delta(g^{-1}) \cdot
\]

Thus the convolution \( \mu_p^* \mu_p \) is given by the integral

\[
(8) \quad (\mu_p^* \mu_p)(g) = \int_{P_K} \frac{\mu_p(g_1)}{\mu_p(g_1 g)} \mu_p(g_1 g) \ dg_1.
\]

This vanishes unless \( g \in P_R \) as can be seen using \( g = g_1^{-1} \ g_2 \) for \( \mu_p(g_i) = 1 \). With

\[
g_i = \begin{bmatrix} 1 & b_i \\ 0 & a_i \end{bmatrix}, \ \text{one gets} \ g_1^{-1} g_2 = \begin{bmatrix} 1 & -a_1^{-1}b_1 \\ 0 & a_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & b_2 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & b_2 - a_2a_1^{-1}b_1 \\ 0 & a_1^{-1} \end{bmatrix} \in P_R.
\]

Moreover the integral \( \int_{P_K} \mu_p(g) \ dg \) is equal to 1 so that one gets \( \mu_p^* \mu_p = \epsilon_p \).
One has $\sigma_t(\mu_p)(g) = 0$ unless $g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$, val$(a) = 1$. $b \in R$, and for such $g$'s one has $\sigma_t(\mu_p)(g) = \delta(g)^{-it} \mu_p(g) = |a|^{-it} \mu_p(g) = p^{it} \mu_p(g)$. Thus $\sigma_t(\mu_p) = p^{it} \mu_p$.

We may also write the state $\varphi_{\beta, p}$ on $C^*(P_K)_{e_{\beta}}$ in terms of biinvariant functions. One obtains the following identity for any $f \in C^*(P_K)_{e_{\beta}}$:

$$\varphi_{\beta, p}(f) = f \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) + \left( \sum_{k \geq 0} p^{k(1-\beta)} f \begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix} \right) (1 - p^{\beta-1}).$$

(Observe that the elements of $C^*(P_K)$ may be identified with some $P_R$-biinvariant functions in $L^2(P_K)$, and therefore to some locally constant functions on $P_K$; in particular they have well defined values on points of $P_K$.) According to Proposition 8 a) and its proof, to prove (9) we only need to check that the linear functional on $C_c(P_K)_{e_{\beta}}$ defined by the right hand side of (9) satisfies the KMS$_\beta$ condition: this follows from a straightforward computation.

The $C^*$-dynamical system $(C^*(P_A), \sigma_t)$ of proposition 10 is without interaction, exactly as is $(C^*(N^*), \sigma_t)$ (Proposition 8), and there is an exact analogue of Proposition 8, which states the existence and uniqueness (up to scale) of KMS$_\beta$ weights on the above system. One needs to use weights because one is dealing with non-unital $C^*$-algebras. At the technical side such weights have to be semicontinuous and semifinite (for the norm topology) (cf. [Com]). It is however instructive to work out the explicit formula for those KMS$_\beta$ weights. Using the natural isomorphism of the Pontrjagin dual $\hat{A}$ of the additive group $A$ with itself (cf. [We2]), one gets an isomorphism:

$$C^*(P_A) = C_0(A) \bowtie A^*$$

where the multiplicative group $A^*$ of finite ideles acts by homotheties in the locally compact space $A$. The KMS$_\beta$ weight on $C^*(P_A)$, $\sigma_t$ is then the dual weight of the following measure $\mu_\beta$ on $A$:

$$\mu_\beta(f) = \zeta(\beta)^{-1} \int_{A^*} |j|^\beta f(j) d^*j.$$

Here $d^*j$ is the Haar measure on the multiplicative group $A^*$, $j \rightarrow |j|$ is the module, and the formula makes sense as such for $\beta > 1$, and by analytic continuation for $0 < \beta < 1$ (cf. [T], [We1], [We2]).

It is clear that to obtain a $C^*$-dynamical system with interaction we need to use not only the locally compact ring $A$ but also the fundamental inclusion:

$$Q \subset A.$$

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We shall use the corresponding inclusion \( P_{\mathbb{Q}} \subseteq P_{\mathbb{A}} = G \) and the action of \( P_{\mathbb{A}} \) on the \( C^* \)-module \( \mathcal{E} = C^*(G)e \) over \( C^*(\mathbb{N}^*) \) given by the isomorphism of proposition 10: \( C^*(G)e = C^*(\mathbb{N}^*) \). Indeed, given any \( C^* \)-algebra \( B \) and (self adjoint) projection \( e \in B \), the space \( \mathcal{E} = B e = \{ x \in B ; \; xe = x \} \) is in a natural way a (right) \( C^* \)-module over the reduced \( C^* \)-algebra \( B_e = \{ x \in B ; \; ex = xe = x \} \). Thus we let:

\[
\langle \xi, \eta \rangle = \xi^* \eta \in B_e, \quad \forall \xi, \eta \in \mathcal{E} = B e
\]

\[
\xi a \in \mathcal{E} \quad \forall \xi \in \mathcal{E}, \; a \in B_e.
\]

This \( C^* \)-module has moreover a natural left \( B \)-module structure, given by \( (b, \xi) \to b \xi \in \mathcal{E}, \forall b \in B, \xi \in \mathcal{E} \).

In our case \( \mathcal{E} = C^*(G)e \) is a space of functions on \( G \) which are invariant by right multiplications by elements of \( P_{\mathbb{R}} \subseteq G \), or in other words it is a space of functions on the homogeneous space:

\[
\Delta = \frac{G}{P_{\mathbb{R}}}.
\]

This space \( \Delta \) is by construction the restricted product of the spaces:

\[
\Delta_p = \frac{P_{\mathbb{Q}_p}}{P_{\mathbb{Z}_p}}
\]

relative to the base point given by \( P_{\mathbb{Z}_p} \).

**Proposition 11.** The homogeneous space \( \Delta_p = \frac{P_{\mathbb{Q}_p}}{P_{\mathbb{Z}_p}} \) over the group \( P_{\mathbb{Q}_p} \subseteq GL(2, \mathbb{Q}_p) \) is naturally isomorphic to the (set of vertices of the) tree of \( SL(2, \mathbb{Q}_p) \). The group \( P_{\mathbb{Q}_p} \) acts by isometries of \( \Delta_p \) and preserves a point at \( \infty \).

Let us recall (cf. [Ser2]) that the tree of \( SL(2, K) \), where \( K \) is a local field, is defined in terms of equivalence classes of lattices in a two dimensional vector space \( V \) over \( K \). With the notations of proposition 9, a lattice \( L \subset V \) is a sub \( R \)-module of \( V \) which is of finite type and generates \( V \) as a vector space. The multiplicative group \( K^* \) operates on the set of lattices by \( (L, x) \to xL \) for \( x \in K^* \), and one lets \( T \) be the set of orbits of this action of \( K^* \). Given a lattice \( L \subset V \) and a class \( \Lambda' \in T \) there exists a unique representative \( L' \in \Lambda' \) such that \( L' \subset L \) and \( L' \not\subset \pi L \) (with \( \pi \) given by proposition 9). Then \( L/L' = R/\pi^n R \) and the integer \( n \), which only depends upon the classes of \( L \) and \( L' \), defines a distance \( d \) on \( T \), by the equality:

\[
d(\text{class of } L, \text{ class of } L') = n.
\]

Using the set of pairs with mutual distance equal to 1 to define a 1 dimensional simplicial complex, one gets a tree, the tree of \( SL(2, K) \), and the above distance is the length of the
unique shortest injective path joining two elements of this tree (cf. [Ser2]). The group \( GL(V) \) of automorphisms of the vector space \( V \) acts on the set of lattices by

\[
(L, g) \rightarrow gL \quad \forall g \in GL(V) .
\]

and since this action commutes with that of \( K^* \) it gives an action, by isometries, of \( GL(V) \) on the tree \( T \). Let us identify \( V \) with \( K^2 \), \( GL(V) \) with \( GL(2, K) \), and consider \( P_K \) as a subgroup of \( GL(2, K) \) : \( P_K = \begin{Bmatrix} [\begin{array}{cc} 1 & b \\ 0 & a \end{array}] : a \in K^* . b \in K \end{Bmatrix} \). Let \( L_0 \) be the lattice \( R^2 \subset K^2 \). Then one checks that \( P_K \) acts transitively on \( T \) and that the stabilizer of the class of \( L_0 \) is \( P_R \). We thus get a canonical identification \( T = P_K/P_R \). Taking \( K = \mathbb{Q}_p \) yields the conclusion.

**Proposition 12.** 1) The homogeneous space \( G/P_R = \Delta \) equipped with the base point \( P_R \) is canonically isomorphic to the restricted product of the trees \( T_p \) equipped with the base points \( P_{z_p} \), and the action of \( G \) on \( \Delta \) is simplicial.

2) the subgroup \( P^+_Q \subset P_A = G \) acts transitively on \( \Delta \), and the isotropy subgroup of the base point \( * \) is \( P^+_z \).

**Proof.** 1) Since both \( P_A \) and \( P_R \) are restricted products the proof of 1) is straightforward.

2) First \( \mathbb{Q} \) is dense in the additive group \( A \) of adeles (with the \( \infty \) place removed), cf. [Ser1].

The subgroup \( \mathbb{Q}_+^* \) of \( A^* \) is discrete and one has \( A^* = \mathbb{Q}_+^* \cdot \mathcal{R}^* \) where \( \mathcal{R}^* = \{ (x_p) ; \text{val}(x_p) = 0 \} \forall p \) is a compact subgroup of \( A^* \) (cf [We]).

Consider the exact sequence of groups:

\[
0 \rightarrow A \rightarrow P_A \overset{\rho}{\rightarrow} A^* \rightarrow 1 .
\]

The closure of \( P^+_Q \) in \( P_A \) is given by \( \rho^{-1}(\mathbb{Q}_+^*) \). Thus \( \overline{P^+_Q} = A \rtimes \mathbb{Q}_+^* \).

Given \( g = \begin{bmatrix} 1 & n \\ 0 & h \end{bmatrix} \in P_A \) one can write \( h = rs \) with \( r \in \mathbb{Q}_+^* \), \( s \in \mathcal{R}^* \), then:

\[
g = \begin{bmatrix} 1 & n \\ 0 & h \end{bmatrix} = \begin{bmatrix} 1 & ns^{-1} \\ 0 & r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \in \overline{P^+_Q} \cdot P_R .
\]

Thus \( g \in \overline{P^+_Q} \cdot P_R \) and since \( P_R \) is open in \( P_A \) we get \( g \in P^+_Q \cdot P_R \). Thus \( P_A = P^+_Q \cdot P_R \) and \( P^+_Q \) acts transitively on \( \Delta \).
Finally the isotropy subgroup of the base point \( * = \mathcal{P}_R \) is given by \( \mathcal{P}_R^+ \cap \mathcal{P}_R = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} = \mathcal{P}_R^+ \).

We can thus identify \( \Delta \) with \( \mathcal{P}_Q^+ / \mathcal{P}_R^+ \), and we shall now get the Hecke algebra of theorem 5 from the commutant of the action of \( \mathcal{P}_R^+ \) in the space of functions on \( \Delta \). We need for that purpose to consider the Hilbert space \( \ell^2(\Delta) = \ell^2(\mathcal{P}_Q^+ / \mathcal{P}_R^+) \). Let \( \Gamma = \mathcal{P}_Q^+ \), \( \Gamma_0 = \mathcal{P}_R^+ \subset \mathcal{P}_Q^+ \).

Let us check first

**Lemma 13.** The action of \( \Gamma_0 = \mathcal{P}_R^+ \) on \( \Gamma / \Gamma_0 \) only has finite orbits.

**Proof.** Let \( g = \begin{bmatrix} 1 & a \\ 0 & k \end{bmatrix} \in \mathcal{P}_Q^+ \). Then \( g \Gamma_0 = \)

\[
\left\{ g \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} = \left\{ \begin{bmatrix} 1 & n+a \\ 0 & k \end{bmatrix} : n \in \mathbb{Z} \right\}.
\]

One has

\[
\begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix} g \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n + a + n_1 k \\ 0 & k \end{bmatrix}.
\]

When \( n_1 \) varies the \( n_1 k \) only take finitely values modulo \( \mathbb{Z} \), their number depending upon the denominator of \( k \). Thus we see that we get indeed a finite orbit of cardinality the denominator of \( k \).

**Lemma 14.** a) Any element \( T \) of the commutant of \( \Gamma \) acting in \( \ell^2(\Gamma / \Gamma_0) \) is characterized by the \( \Gamma_0 \)-biinvariant function \( f_T(g) = \langle T \varepsilon, g^{-1} \varepsilon \rangle \).

b) One has \( \sum_{\Gamma_0 \setminus \Gamma} |f_T(g)|^2 < \infty \).

c) One has \( f_{T^*}(g) = \overline{f_T}(g^{-1}) \).

d) The \( \Gamma_0 \)-biinvariant function \( f_{T_1 T_2} \) associated to the product of two elements \( T_1 \) and \( T_2 \) in this commutant is given by:

\[
f_{T_1 T_2}(g) = \sum_{\Gamma / \Gamma_0} f_{T_1}(gg_1) f_{T_2}(g_1^{-1}).
\]

**Proof.** a) Each element of \( \Gamma \) acts by permutation of the basis \( \varepsilon_x, x \in \Gamma / \Gamma_0 \) of \( \mathcal{H} = \ell^2(\Gamma / \Gamma_0) \). Since \( T \) commutes with \( \Gamma \) it is characterized by \( T \varepsilon \), which is determined by \( f_T \). One has \( f_T(g) = \langle Tg \varepsilon, \varepsilon \rangle \) so that \( f_T \) is \( \Gamma_0 \) biinvariant.

b) Obvious.
c) \( f_{T^*}(g) = \langle T^* \varepsilon_e, g^{-1} \varepsilon_e \rangle = \langle g \varepsilon_e, T \varepsilon_e \rangle = \langle T \varepsilon_e, g \varepsilon_e \rangle \)

d) \( \langle T g_1 \varepsilon_e, g_2 \varepsilon_e \rangle = f_T(g_2^{-1} g_1) \), \( \langle T_1 T_2 \varepsilon_e, g^{-1} \varepsilon_e \rangle = \)

\[ = \langle T_2 \varepsilon_e, g^{-1} T_1^* \varepsilon_e \rangle = \sum \langle T_2 \varepsilon_e, g_1 \varepsilon_e \rangle \langle g_1 \varepsilon_e, g^{-1} T_1^* \varepsilon_e \rangle = \langle T_1 \varepsilon_e, g_1^{-1} g^{-1} \varepsilon_2 \rangle \]

\[ = \sum_{G/G_0} f_{T_1}(g g_1) f_{T_2}(g_1^{-1}) . \]

One can write d) as

\[ f_{T_1 T_2}(g) = \sum_{G_0 / G} f_{T_1}(g g_1^{-1}) f_{T_2}(g_1) . \]

Condition b) shows that \( f_T(g) = 0 \) unless there exists a finite set \( F \subset \Gamma \) with:

\[ \Gamma_0 \ g \ G_0 \subset F \ G_0 \]

i.e. if \( g \ G_0 \) belongs to a finite orbit of \( G_0 \) on \( \Gamma / G_0 \). By Lemma 13 we can take a basis of \( G_0 \) biinvariant functions, \( \epsilon_X \), where \( X \) runs through the double classes \( X \in \Gamma_0 \backslash \Gamma / G_0 \), \( \epsilon_X(g) = 0 \) if \( g \notin X \), \( \epsilon_X(g) = 1 \) if \( g \in X \). For each such class one has two associated finite integers:

\[ L(X) = \text{cardinality of the image of } X \text{ in } \Gamma / G_0 \]

\[ (19) \]

\[ R(X) = \text{cardinality of the image of } X \text{ in } G_0 \backslash \Gamma . \]

In fact to each double class \( X \) corresponds the positive rational number given by \( k \) for any

\[ \begin{pmatrix} 1 & a \\ 0 & k \end{pmatrix} \in X, \text{ and } L(X) \text{ is the denominator of } k \text{ while } R(X) \text{ is the numerator of } k. \]

**Proposition 15.** a) Let \( f \) be a \( G_0 \) biinvariant function on \( \Gamma \) with finite support in \( G_0 \backslash \Gamma / G_0 \). Then there exists a unique element \( r(f) \) of the commutant of \( \Gamma \) in \( \ell^2(\Gamma / G_0) \) such that

\[ f(g) = \langle r(f) \varepsilon_e, g^{-1} \varepsilon_e \rangle \quad \forall g \in \Gamma. \]

b) The map \( r \) extends to an isomorphism of \( C^*_r(\Gamma, G_0) \) with the \( C^* \)-algebra \( C = C_q \) generated by the \( r(f) \) in \( \ell^2(\Delta) \).

**Proof.** a) Let \( X \) be a double class, \( X \in \Gamma_0 \backslash \Gamma / G_0 \) and \( \epsilon_X \) the corresponding \( G_0 \) biinvariant function. Let \( T \) be the matrix given by

\[ \langle T g_1 \varepsilon_e, g_2 \varepsilon_e \rangle = \epsilon_X(g_2^{-1} g_1) . \]
One has, by Lemma 13, the bounds

$$\sup_{\alpha} \sum_{\beta} |T_{\alpha,\beta}| \leq L(X) \quad \sup_{\beta} \sum_{\alpha} |T_{\alpha,\beta}| \leq R(X)$$

which shows that $T$ defines a bounded operator in $\ell^1(\Delta)$ and in $\ell^\infty(\Delta)$, hence in $\ell^2(\Delta)$. The uniqueness is clear.

b) follows from Lemma 14 d) and the definitions of $H(\Gamma, \Gamma_0)$ and $C^*_\tau(\Gamma, \Gamma_0)$ (cf. Proposition 3).

Next we let $\varphi$ be the state on $C_\mathfrak{a}$ defined by

$$\varphi(T) = \langle T \varepsilon, \varepsilon \rangle .$$

**Lemma 16.** $\varphi$ is a KMS$_1$ state on $C$ with respect to the one parameter group $\sigma_t$,  

$$\sigma_t(\epsilon_X) = k^{-t} \epsilon_X, \quad k = \frac{R(X)}{L(X)} .$$

**Proof.** The product of $\Gamma_0$-biinvariant functions is given by:

$$(f_1 * f_2)(g) = \sum_{\Gamma/\Gamma_0} f_1(g_1) f_2(g_1^{-1} g) .$$

One has

$$\varphi(f_1 * f_2) = (f_1 * f_2)(\epsilon) = \sum_{\Gamma/\Gamma_0} f_1(g_1) f_2(g_1^{-1})$$

$$\varphi(\epsilon_X * f) = \sum_{\Gamma/\Gamma_0} \epsilon_X(g_1) f(g_1^{-1})$$

$$\varphi(f * \epsilon_X) = \sum_{\Gamma/\Gamma_0} f(g_2) \epsilon_X(g_2^{-1}) .$$

Let $g \in \Gamma$ with $X = \Gamma_0 g \Gamma_0$ a fixed double class; then as $f$ is $\Gamma_0$-biinvariant, the two above expressions are multiples of $f(g^{-1})$:

$$\varphi(\epsilon_X * f) = L(X) f(g^{-1}) . \quad \varphi(f * \epsilon_X) = R(X) f(g^{-1}) .$$

Thus $\varphi(\epsilon_X * f) = \frac{L(X)}{R(X)} \varphi(f * \epsilon_X)$. This shows that $\epsilon_X$ is an eigenvector for the modular automorphism group of the faithful normal state

$$T \rightarrow \langle T \varepsilon, \varepsilon \rangle$$
on the weak closure $C''$ of $C$ acting in $\mathcal{H} = \ell^2(\Gamma/\Gamma_0)$. It follows thus that $\varphi$ is a KMS$_1$ state for the group $\sigma_t = \tau_{\Delta t}$. The vector $\varepsilon_e \in \ell^2(\Delta)$ is still separating for the weak closure of $C_\mathbb{Q}$ because it is cyclic for $P_\mathbb{Q}^+$ by Proposition 12.2). The subspace $\mathcal{H}_r$ of $\ell^2(\Delta)$ given by:

$$\mathcal{H}_r = \overline{C_\mathbb{Q} \varepsilon_e}$$

is in the bicommutant of the action of $P_\mathbb{Q}^+$ on $\Delta$ and we shall compute it.

**Lemma 17.** Let $\xi \in \ell^2(\Delta)$; then $\xi \in \mathcal{H}_r$ iff $\xi$ is fixed by $\Gamma_0 \subset P_\mathbb{Q}^+$ acting on the left on $\Delta = P_\mathbb{Q}^+/\Gamma_0$.

**Proof.** Let $\xi \in \mathcal{H}$ be fixed under $\Gamma_0$. Then the function $f(g) = (\xi, \varepsilon_g)$, $g \in \Gamma/\Gamma_0$ is $\Gamma_0$ biinvariant. To show that $\xi \in \mathcal{H}_r$ we can assume, using an orthogonal decomposition, that $f$ is the characteristic function of a double class, $f = \varepsilon_X$. We get as above that $e_X^* \varepsilon_e = \sum_{g \in F} g \varepsilon_e$ where $X = FT \Gamma_0$ so that all the characteristic functions of double classes belong to $\mathcal{H}_r$. Conversely if $\xi \in \mathcal{H}_r$ and $\xi = T \varepsilon_e$ for some operator $T$ commuting with $P_\mathbb{Q}^+$ then as $T$ commutes with $\Gamma_0 \subset P_\mathbb{Q}^+$ and $g \varepsilon_e = \varepsilon_e$ $\forall g \in \Gamma_0$ one gets that $\xi$ is fixed by $\Gamma_0$.

4. **Presentation of the $C^*$-algebra $C_\mathbb{Q} = C^*(P_\mathbb{Q}^+, P_\mathbb{Z}^+)$**

Let us first consider the Hecke algebra $\mathcal{H} = \mathcal{H}(\Gamma, \Gamma_0)$ where $\Gamma = P_\mathbb{Q}^+$, $\Gamma_0 = P_\mathbb{Z}^+$ are as above. It is an involutive algebra over $\mathbb{C}$ with a linear basis $\varepsilon_X$, indexed by the double cosets $X \in \Gamma_0 \backslash \Gamma/\Gamma_0$. We shall use the following notations:

**a)** For $n \in \mathbb{N}^*$, $\mu_n = n^{-1/2} \varepsilon_{X_n}$ where $X_n$ is the double class of the element $\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$ of $P_\mathbb{Q}^+$.

**b)** For $\gamma \in \mathbb{Q}/\mathbb{Z}$, $\varepsilon(\gamma) = \varepsilon_{X^\gamma}$ where $X^\gamma$ is the double class of the element $\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ of $P_\mathbb{Q}^+/P_\mathbb{Z}^+$.

**Proposition 18.** a) The elements $\mu_n$, $\varepsilon(\gamma)$, $n \in \mathbb{N}^*$, $\gamma \in \mathbb{Q}/\mathbb{Z}$ generate the involutive algebra $\mathcal{H}$ and the following relations give a presentation of $\mathcal{H}$.

(a) $\mu_n^* \mu_n = 1$ $\forall n$

(b) $\mu_{nm} = \mu_n \mu_m$ $\forall n, m$

(c) $\mu_n \mu_m^* = \mu_m^* \mu_n$ if $(n, m) = 1$

(d) $\varepsilon(\gamma)^* = \varepsilon(-\gamma)$, $\varepsilon(\gamma_1 + \gamma_2) = \varepsilon(\gamma_1) \varepsilon(\gamma_2)$ $\forall \gamma, \gamma_1, \gamma_2$

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(e) \( e(\gamma) \mu_n = \mu_n e(n\gamma) \quad \forall n, \forall \gamma \)

(f) \( \mu_n e(\gamma) \mu_n^* = \frac{1}{n} \sum_{n\delta = \gamma} e(\delta) \quad \forall n, \forall \gamma. \)

**Proof.** We first have to check that the relations (a)\ldots(f) are fulfilled using the definitions of the product and involution in \( \mathcal{H} \), namely:

\begin{equation}
(1) \quad f_1 * f_2(g) = \sum_{g_1 \in \Gamma_0 / \Gamma_0} f_1(g_1) f_2(g_1^{-1} g) \end{equation}

\begin{equation}
(2) \quad f^*(g) = \overline{f(g^{-1})}. \end{equation}

For \( n \in \mathbb{N}^* \), the right class \( \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \Gamma_0 \) is already a double class: \( X_n \). This shows that for any \( \Gamma_0 \) biinvariant function \( f \in \mathcal{H} \):

\begin{equation}
(3) \quad n^{1/2}(\mu_n * f)(g) = f \left( \begin{bmatrix} 1 & 0 \\ 0 & n^{-1} \end{bmatrix} g \right) \quad \forall g \in \Gamma. \end{equation}

Similarly the right class \( \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \Gamma_0, \gamma \in \mathbb{Q}/\mathbb{Z} \) is already a double class \( X^\gamma \) so that:

\begin{equation}
(4) \quad (e(\gamma) * f)(g) = f \left( \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix} g \right) \quad \forall g \in \Gamma \end{equation}

and using the equality \( \Gamma_0 \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} = X^\gamma. \)

\begin{equation}
(5) \quad (f * e(\gamma))(g) = f \left( g \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix} \right) \quad \forall g \in \Gamma. \end{equation}

Using (3) (4) (5) one proves directly the relations b) – d) and e). The left multiplication by \( \mu_n^* \) is given by

\begin{equation}
(6) \quad n^{1/2}(\mu_n^* * f)(g) = \sum_{k=0}^{n-1} f \left( \begin{bmatrix} 1 & k \\ 0 & n \end{bmatrix} g \right) \quad \forall g \in \Gamma \end{equation}

where the biinvariance of \( f \) shows that \( f \left( \begin{bmatrix} 1 & k \\ 0 & n \end{bmatrix} g \right) \) only depends on \( k \) modulo \( n \), since:

\[
\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & n \end{bmatrix} = \begin{bmatrix} 1 & k + nb \\ 0 & n \end{bmatrix}. \]
The equality a) follows directly from (6). Let \( n, m \) be integers such that \( (n, m) = 1 \), using (3) one gets that the biinvariant function \( n^{1/2}m^{1/2} \mu_n \ast \mu_m^* \) is the characteristic function of the \( \Gamma_0 \) double coset: 
\[
\begin{bmatrix}
1 & Z/m \\
0 & n/m
\end{bmatrix}
\]. Using (6) one gets that:
\[
m^{1/2} \mu_m^* \ast \mu_n(g) = \sum_{k=0}^{m-1} \mu_n \left( \begin{bmatrix} 1 & k \\ 0 & m \end{bmatrix} g \right).
\]

Let \( g = \begin{bmatrix} 1 & \beta \\ 0 & \alpha \end{bmatrix} \in \Gamma \). Then the \((k + 1)\)-st term on the right hand side vanishes unless 
\[
\begin{bmatrix} 1 & k \\ 0 & m \end{bmatrix} g \in \begin{bmatrix} 1 & \mathbb{Z} \\ 0 & n \end{bmatrix},
\]
i.e. unless \( \alpha = n/m, \beta + \frac{nk}{m} \in \mathbb{Z} \). Thus the left hand side vanishes unless \( \beta \in \mathbb{Z}/m \) and is equal to \( n^{-1/2} \) if \( \beta \in \mathbb{Z}/m \) since, as \( (n, m) = 1 \) only one value of \( k \) will contribute to the sum. This proves the relation c). To prove f) one combines (3) and (4) which gives
\[
n^{1/2}(\mu_n \ast \epsilon(\gamma) \ast f)(g) = f \left( \begin{bmatrix} 1 & -\gamma n^{-1} \\ 0 & n^{-1} \end{bmatrix} g \right) \forall g \in \Gamma
\]
which one applies to \( f = \mu_n^* \). One has \( f(g) = n^{-1/2} \) if \( g \in \begin{bmatrix} 1 & \mathbb{Z}/n \\ 0 & 1/n \end{bmatrix} \) and \( f(g) = 0 \) otherwise. This shows that \( (\mu_n \ast \epsilon(\gamma) \ast \mu_n^*)(g) \) vanishes unless \( g = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \) with
\[
\begin{bmatrix} 1 & -\gamma n^{-1} \\ 0 & n^{-1} \end{bmatrix} \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \in \begin{bmatrix} 1 & \mathbb{Z}/n \\ 0 & n^{-1} \end{bmatrix}
\]
i.e., \( n\beta = \gamma \) modulo \( n \). Since this equation has \( n \) solutions \( \beta \in \mathbb{Q}/\mathbb{Z} \) one gets f).

We have thus proved the relations (a)...(f). Conversely let \( \mathcal{A} \) be an involutive algebra generated by elements \( (\mu_n), (\epsilon_\gamma) \) satisfying (a)...(f). We shall show that the monomials of the form:
\[
t_{n, m, \gamma} = \mu_n \epsilon(\gamma) \mu_m^* \quad n, m \in \mathbb{N}^*, (n, m) = 1, \gamma \in \mathbb{Q}/\mathbb{Z}
\]
form a set of generators of the vector space \( \mathcal{A} \). It is enough for this to express the adjoint and the products of such monomials as elements of their linear span \( \mathcal{L} \). First, if we continue to denote by \( t_{n, m, \gamma} \) the expression \( \mu_n \epsilon(\gamma) \mu_m^* \) when \( n, m \) are not relatively prime but have \( (n, m) = q > 1 \), we can write:
\[
t_{n, m, \gamma} = \mu_{n/q} \mu_q \epsilon(\gamma) \mu_{m/q}^*.
\]

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and use (f) to express it as an element of $\mathcal{L}$. It is clear that $(t_{n,m,\gamma})^* = t_{m,n,-\gamma}$ so that $\mathcal{L} = \mathcal{L}^*$. Let us now compute the product: $t_{n_1,m_1,\gamma_1} t_{n_2,m_2,\gamma_2}$. Let $q = (m_1,n_2)$ then 

$$\mu_{m_1}^* \mu_{n_2} = \mu_{m_1/q}^* \mu_{n_2/q} \mu_{m_1/q} \mu_{n_2/q} = \mu_{m_2/q} \mu_{n_1/q} \rho_{m_1/q} \rho_{n_2/q},$$

using (a), (b), (c).

Thus $t_{n_1,m_1,\gamma_1} t_{n_2,m_2,\gamma_2} = \mu_n e(\gamma_1) \mu_{m_2/q} \rho_{m_1/q} e(\gamma_2) \mu_m^*$. Using (e) and its adjoint one thus obtains:

$$t_{n_1,m_1,\gamma_1} t_{n_2,m_2,\gamma_2} = \mu_{n_1,n_2/q} e \left( \frac{n_2}{q} \gamma_1 + \frac{m_1}{q} \gamma_2 \right) \mu_{m_1,m_2/q}^* \mu_{m_1,m_2/q},$$

which is a $t_{n,m,\gamma}$ with $(n,m)$ not necessarily 1 and can be expressed as above as an element of $\mathcal{L}$.

We have shown that the linear span $\mathcal{L}$ of the $t_{n,m,\gamma}$ is an involutive algebra and hence that $\mathcal{L} = \mathcal{A}$. In the algebra $\mathcal{H}$ the $t_{n,m,\gamma}$ are given by

$$(7) \quad t_{n,m,\gamma} = (nm)^{-1/2} e_X, \quad X = \text{double class of } \begin{bmatrix} 1 & \gamma \\ 0 & n/m \end{bmatrix}.$$  

Thus they are linearly independent in $\mathcal{H}$ and this is enough to conclude that (a) . . . (f) is a presentation of $\mathcal{H}$.

It is crucial that the above presentation of $\mathcal{H}$ is defined over the field $\mathbb{Q}$ of rational numbers, since this will allow for a natural action of the Galois group $G$ of the cyclotomic extension $\mathbb{Q}^{\text{cycl}}$ of $\mathbb{Q}$ on certain representations of $\mathcal{H}$ which we shall construct later in section 6.

A result similar to Proposition 18 holds for the $C^*$-algebra $C_{\mathbb{Q}} = C^*_r(\Gamma, \Gamma_0)$ and due to the amenability of $\Gamma = P_{\mathbb{Q}}$ the nuance between $C^*_r(\Gamma, \Gamma_0)$, the universal $C^*$-algebra generated by the $(\mu_n, e_\gamma)$ with the above relations, and $C^*_r(\Gamma, \Gamma_0)$ does not arise. One has $C^*_r(\Gamma, \Gamma_0) = C^*_r(\Gamma, \Gamma_0)$.

**Proposition 19.** Let $\pi$ be an involutive representation of $\mathcal{H}$ as operators in a Hilbert space $\mathcal{H}$. Then $\pi$ extends uniquely by continuity to a representation of $C^*_r(\Gamma, \Gamma_0) = C_{\mathbb{Q}}$.

**Proof.** The relations (a) and (d) show that $\pi(\mu_n)$ is an isometry and $\pi(e(\gamma))$ a unitary. Thus one has:

$$\|\pi(\mu_n e(\gamma) \mu_{m}^*)\| \leq 1 \quad \forall n, m, \gamma .$$

This shows that $\pi(f)$ is bounded for any $f \in \mathcal{H}$, with

$$\|\pi(f)\| \leq \|f\|_1$$

where

$$\|f\|_1 = \sum_{\gamma \in \Gamma / \Gamma_0} \delta(\gamma)^{-1/2} |f(\gamma)|, \quad \delta(\gamma) = \frac{L(\gamma)}{R(\gamma)} .$$

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It follows then that the following equality defines a norm on $\mathcal{H}$ whose completion is a $C^*$-algebra:

\begin{equation}
\|f\|_{\text{max}} = \text{Sup}\{\|\pi(f)\| : \pi \in \text{Rep} \mathcal{H}\}
\end{equation}

where $\text{Rep} \mathcal{H}$ is the class of all involutive representations of $\mathcal{H}$ in a fixed separable Hilbert space.

Let us now show that:

\begin{equation}
\|f\|_{\text{max}} = \|f\|_{C^*_r(\Gamma, \Gamma_0)} \quad \forall f \in \mathcal{H},
\end{equation}

which is a statement of amenability of the pair $(\Gamma, \Gamma_0)$. Let us first prove (9) for elements of the group ring $f \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$, i.e. linear combinations of the $\epsilon_{\gamma}$, $\gamma \in \mathbb{Q}/\mathbb{Z}$. Using the representation of $\mathcal{H}$ in $\ell^2(\Gamma_0 \backslash \Gamma)$ one has:

\begin{equation}
\|f\|_{\text{max}} \geq \|f\|_{C^*_r} \quad \forall f \in \mathcal{H}.
\end{equation}

Thus we only need to prove the other inequality. The amenability of the group $\mathbb{Q}/\mathbb{Z}$ shows that

\begin{equation}
\|\pi(f)\| \leq \|f\|_{C^*_r(\mathbb{Q}/\mathbb{Z})} \quad \forall f \in \mathbb{C}[\mathbb{Q}/\mathbb{Z}]
\end{equation}

for any unitary representation $\pi$ of $\mathbb{Q}/\mathbb{Z}$.

Observe that in the representation of $\mathcal{H}$ in $\ell^2(\Gamma_0 \backslash \Gamma)$ the restriction to $\mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ defines a faithful representation of $C^*_r(\mathbb{Q}/\mathbb{Z}) = C^*(\mathbb{Q}/\mathbb{Z})$. Indeed the restriction of the action of $\mathbb{Q}/\mathbb{Z}$ to the orbit of $\epsilon_0 = \Gamma_0$, is isomorphic to the regular representation of $\mathbb{Q}/\mathbb{Z}$, so the answer follows. This proves (9) for elements of $\mathbb{C}(\mathbb{Q}/\mathbb{Z})$ and allows to view $C^*(\mathbb{Q}/\mathbb{Z})$ as a $C^*$-subalgebra of the $C^*$ completion $C^*(\Gamma, \Gamma_0)$ of $\mathcal{H}$ for the norm (8). Let us identify the dual of the group $\mathbb{Q}/\mathbb{Z}$ with the additive group $\mathcal{R} \subset \mathcal{A}$ using the equality $\mathbb{Q}/\mathbb{Z} = \mathcal{A}/\mathcal{R}$ and the identification of the additive group $\mathcal{A}$ with its Pontrjagin dual. Consider now the locally compact groupoid $\mathcal{G}$ defined as follows. As a locally compact space $\mathcal{G}$ is the following subset of $\mathcal{R} \times \mathbb{Q}_+^*$,

\begin{equation}
\mathcal{G} = \{(b, a) \in \mathcal{R} \times \mathbb{Q}_+^* : ab \in \mathcal{R}\}
\end{equation}

which may be identified with a countable union of open and closed subsets of $\mathcal{R}$. One has $\mathcal{G}^{(0)} = \mathcal{R} \times 1 = \mathcal{R}$ and the range and source maps are

\begin{equation}
r(b, a) = ab, \quad s(b, a) = b
\end{equation}

while the composition is given by

\[(b_1, a_1) \circ (b_2, a_2) = (b_2, a_1 a_2).\]

By construction the fibers $\mathcal{G}_x$ and $\mathcal{G}_{x'}$, $x \in \mathcal{R}$ of $r$ and $s$ are discrete countable sets and the
\(C^\ast\)-algebras $C^\ast(\mathcal{G})$ and $C^\ast_r(\mathcal{G})$ of this locally compact groupoid make good sense. They are the completions of the convolution algebra $C_r(\mathcal{G})$ of continuous functions with compact support on $\mathcal{G}$,

\[(f_1 \ast f_2)(\gamma) = \sum_{\gamma_1 \ast \gamma_2 = \gamma} f_1(\gamma_1) f_2(\gamma_2)\]

\[(f^\ast(\gamma) = \overline{f(\gamma^{-1})}\]

under the following respective norms:

\[\|f\|_{\text{max}} = \sup\{\|\pi(f)\| : \pi \in \text{Rep } \mathcal{G}\}\]

\[\|f\|_r = \sup_{x \in \mathcal{G}^{(0)}} \|\lambda_x(f)\|\]

where $\lambda_x$ is the left regular representation of $f$ in $\ell^2(\mathcal{G}^r)$ given by:

\[(\lambda_x(f)\xi)(\gamma) = \sum_{r(\gamma_1) = x} f(\gamma_1) \xi(\gamma_1^{-1}\gamma).\]

Now since the group $\mathbb{Q}_+^\ast$ is amenable it follows that the locally compact groupoid $\mathcal{G}$ is amenable in the sense of [Ren], so that the norms (17) and (18) coincide.

Next, given a representation $\pi$ of $\mathcal{H}$ we get a representation $\overline{\pi}$ of $C_c(\mathcal{G})$ as follows. We identify $\mathcal{H}$ with a subalgebra of $C_c(\mathcal{G})$ by checking that the following elements $\overline{\epsilon}(\gamma), \overline{\mu}_n$ of $C_c(\mathcal{G})$ satisfy the presentation of Proposition 18.

\[\overline{\epsilon}(\gamma)(b,a) = 0 \quad \text{unless} \quad a = 1.\]

\[\overline{\epsilon}(\gamma)(b,1) = \langle b, \gamma \rangle\]

(where $\langle b, \gamma \rangle$ is the pairing between $\mathcal{R}$ and $\mathbb{Q}/\mathbb{Z}$ given by Pontrjagin duality of abelian groups)

\[\overline{\mu}_n(b,a) = 0 \quad \text{unless} \quad a = n^{-1}; \quad \overline{\mu}_n(b,n^{-1}) = 1 \quad \forall b \in \mathcal{R}.\]
One checks directly the relations (a), . . . (f) of the presentation of \( \mathcal{H} \) using in particular the equality:

\[
(\gamma, b) = (\gamma, nb) \quad \forall \gamma \in \mathbb{Q}/\mathbb{Z} \quad \forall b \in \mathcal{R} \text{ such that } nb \in \mathcal{R}.
\]

Let then \( \pi \) be an involutive representation of \( \mathcal{H} \). We have shown above that the restriction of \( \pi \) to the group ring of \( \mathbb{Q}/\mathbb{Z} \) extends to a representation of \( C^*(\mathbb{Q}/\mathbb{Z}) = C^*_c(\mathbb{Q}/\mathbb{Z}) \). It thus follows using the \( \tilde{\mu}_n \), which together with \( C(\mathcal{R}) \) generate \( C_c(\mathcal{G}) \), that \( \pi \) extends to a representation \( \tilde{\pi} \) of the convolution algebra \( C_c(\mathcal{G}) \) of the locally compact groupoid \( \mathcal{G} \). The amenability of \( \mathcal{G} \) thus yields:

\[
\|\pi(f)\| \leq \sup_{x \in \mathcal{R}} \|\lambda_x(\tilde{f})\| \quad \forall f \in \mathcal{H}.
\]

The homomorphism \( h : \mathcal{G} \to \mathbb{Q}^+_\mathcal{R} \), \( h(b,a) = a \), yields for each \( x \in \mathcal{R} \) an injection of \( \mathcal{G}^x \) in \( \mathbb{Q}^+_\mathcal{R} \), which allows to consider the continuous field of Hilbert spaces \( \ell^2(\mathcal{G}^x) \), \( x \in \mathcal{R} \) as a subfield of the constant field with fiber \( \ell^2(\mathbb{Q}^+_\mathcal{R}) \). For any \( f \in \mathcal{H} \) the map \( x \to \lambda_x(\tilde{f}) \) is then strongly continuous with values in \( \mathcal{L}(\ell^2(\mathbb{Q}^+_\mathcal{R})) \). This can be checked directly for the \( \tilde{\mu}_n \) and the elements of \( C(\mathcal{R}) \).

It follows that for any \( f \in \mathcal{H} \) the function on \( \mathcal{R} \) given by \( x \to \|\lambda_x(\tilde{f})\| \), is lower semicontinuous in the sense that \( \{ x : \|\lambda_x(\tilde{f})\| > \alpha \} \) is an open set for any \( \alpha \). Thus:

\[
\sup_{x \in \mathcal{R}} \|\lambda_x(\tilde{f})\| = \text{Ess sup}_{x \in \mathcal{R}} \|\lambda_x(\tilde{f})\|
\]

and the right hand side is equal to \( \|f\|_{C^*_c(\Gamma; \Gamma_0)} \) so that the equality (9) follows.

5. Action of \( W \times \mathbb{R} \) on the \( C^* \)-algebra \( C_Q \)

Let \( (\sigma_t)_{t \in \mathbb{R}} \) be the action of \( \mathbb{R} \) on the \( C^* \)-algebra \( C_Q = C^*(P^+_Q, P^+_\mathbb{Z}) \) defined in Proposition 4. In terms of double classes \( X \) in \( \Gamma_0 \backslash \Gamma/\Gamma_0 \), \( \Gamma = P^+_Q \), \( \Gamma_0 = P^+_\mathbb{Z} \), one has

\[
\sigma_t(e_X) = k^{-it} e_X, \quad k = \frac{R(X)}{L(X)}
\]

and for the double class of \( g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \in P^+_Q \) one has

\[
k = a.
\]

In terms of the presentation (Proposition 18) of \( C_Q \) one has

\[
\sigma_t(\mu_n) = n^{it} \mu_n \quad \forall n \in \mathbb{N}^*, \quad \sigma_t(\epsilon(\gamma)) = \epsilon(\gamma)
\]
for any $\gamma \in \mathbb{Q}/\mathbb{Z}$ and any $t \in \mathbb{R}$.

Proposition 20. a) The $C^*$-subalgebra of $\mathcal{C}_\mathbb{Q}$ given by the fixed points of $\sigma$, $\mathcal{C}_\mathbb{Q}^\sigma = \{ x \in \mathcal{C}_\mathbb{Q} ; \sigma_t(x) = x \quad \forall t \in \mathbb{R} \}$ is the image of $C^*(\mathbb{Q}/\mathbb{Z})$ by the isomorphism associated to the homomorphism $\gamma \in \mathbb{Q}/\mathbb{Z} \rightarrow e(\gamma) \in \mathcal{C}_\mathbb{Q}$. 

b) The centralizer of the state $\varphi$ on $\mathcal{C}_\mathbb{Q}$ given by Lemma 16. is equal to $\mathcal{C}_\mathbb{Q}^\sigma = C^*(\mathbb{Q}/\mathbb{Z})$.

Proof. a) By construction $C^*(\mathbb{Q}/\mathbb{Z}) \subset \mathcal{C}_\mathbb{Q}^\sigma$. The action $\sigma$ of $\mathbb{R}$ on $\mathcal{C}_\mathbb{Q}$ is almost periodic and diagonal in the linear basis $(\varepsilon_X)$ of $\mathcal{H}$. The projection $P$ on the fixed points of $\sigma$, given by the almost periodic average of the $\sigma_t$, is the identity on the double classes $e_X^\gamma$, $\gamma \in \mathbb{Q}/\mathbb{Z}$, and vanishes, $P(\varepsilon_X) = 0$, on the double class of any $g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$, $a \neq 1$. The conclusion follows since $\mathcal{H}$ is dense on $\mathcal{C}_\mathbb{Q}$ and $P$ is norm continuous.

b) follows from a) and Lemma 16.

We shall now define a natural action by automorphisms of $\mathcal{C}_\mathbb{Q}$, of the idele class group $W = \mathcal{A}^*/\mathbb{Q}^*_+$. Recall first that we obtained $\mathcal{C}_\mathbb{Q}$ in Proposition 15 from the commutant of $P^+_\mathbb{Q}$ in $\ell^2(\Delta)$ where $\Delta = P_A/P_\mathbb{R} = P^+_\mathbb{Q}/P_\mathbb{Z}^+$. Let us show that $W = \mathcal{A}^*/\mathbb{Q}^*_+$ acts in a natural manner on the commutant of $P^+_\mathbb{Q}$ in $\ell^2(\Delta)$. The group $P_A$ acts on $\Delta$ and on $\ell^2(\Delta)$ and thus the commutant of $P^+_\mathbb{Q}$ is the same as the commutant of its closure $\overline{P^+_\mathbb{Q}}$ in $P_A$. One has (cf. Proposition 12), $\overline{P^+_\mathbb{Q}} = A \rtimes \mathbb{Q}^*_+$ which is a normal subgroup of $P_A$. Thus the quotient group:

$W = \mathcal{A}^*/\mathbb{Q}^*_+ = P_A/\overline{P^+_\mathbb{Q}}$

acts naturally on the commutant $P^+_\mathbb{Q}$ of $P^+_\mathbb{Q}$ in $\ell^2(\Delta)$, by

$\theta_u(x) = u x u^* \quad \forall x \in P^+_\mathbb{Q} \quad \forall u \in W$ .

(The choice of the representative $u \in P_A$ of the class of $u$ is irrelevant.)

This defines a strongly continuous action of the compact group $W$ on the von Neumann algebra $P^+_\mathbb{Q}$.

Proposition 21. a) The action $\theta$ of $W$ on $P^+_\mathbb{Q}$ leaves the dense $C^*$-subalgebra $\mathcal{C}_\mathbb{Q}$ globally invariant and is pointwise norm continuous on $\mathcal{C}_\mathbb{Q}$.

b) The fixed point subalgebra $\mathcal{C}_\mathbb{Q}^W$ in the $C^*$-algebra $C^*(\mathbb{N}^*)$ generated by the $\mu_n \in \mathcal{C}_\mathbb{Q}$.

c) The action of $W$ on $\mathcal{C}_\mathbb{Q}$ preserves the state $\varphi$ and commutes with the action $(\sigma_t)_{t \in \mathbb{R}}$.
Proof. Let us first show that $\mu_n$, i.e. the right convolution $r(f)$ in $\ell^2(\Gamma/\Gamma_0)$ by $f = n^{-1/2} \varepsilon_{X_n}$, belongs not only to the commutant of $P^+_Q$ (Proposition 15) but also to the commutant of $P_A$. By construction one has

$$(5) \quad \langle \mu_n^* \varepsilon_e, g \varepsilon_e \rangle = f(g) = n^{-1/2} \varepsilon_{X_n}(g) \quad \forall g \in \Gamma$$

and since the $g \varepsilon_e, g \in \Gamma/\Gamma_0$ form a basis of $\ell^2(\Delta)$ we get

$$(6) \quad \mu_n^* \varepsilon_e = n^{-1/2} g_n \varepsilon_e \quad , \quad g_n = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \in \Gamma .$$

Let us assume that $n = p$ is a prime number and show that $p^{1/2} \mu_p^*$ coincides with the operator in $\ell^2(\Delta) = \bigotimes_q (\ell^2(T_q), *)$ given by $\tilde{t}_p = 1 \otimes t_p \otimes 1 \otimes \ldots$, where $t_p$ is the hyperbolic translation of one unit of length towards the point at $\infty$ in the tree $T_p$ of Proposition 12.

Note that this hyperbolic translation is exactly $p$ to one so that $p^{-1/2} t_p$ is a coisometry on $\ell^2(T_p)$. Now both $\mu_p^*$ and $t_p$ commute with the action of $P^+_Q$ on $\ell^2(\Delta)$, and $\varepsilon_e$ is cyclic for $P^+_Q$. Thus the equality

$$p^{1/2} \mu_p^* \varepsilon_e = \tilde{t}_p \varepsilon_e$$

implies the equality of operators:

$$(7) \quad \mu_p^* = p^{-1/2} \tilde{t}_p .$$

Since $\tilde{t}_p$ belongs to the commutant of $P_A$ by construction we thus get that $\mu_p \in (P_A)'$ and

$$(8) \quad \theta_u(\mu_n) = \mu_n \quad \forall u \in W , \ n \in \mathbb{N^*} .$$

To prove 21 a) we just need to show that the action $\theta$ of $W$ leaves the $C^*$-subalgebra $C^*(\mathbb{Q}/\mathbb{Z}) = C(\mathcal{R})$ of $C_Q$ globally invariant and is pointwise norm continuous on $C^*(\mathbb{Q}/\mathbb{Z})$. This will follow from the following more precise statement.

Lemma 22. Let $f \in C(\mathcal{R}) = C^*(\mathbb{Q}/\mathbb{Z}) \subset C_Q$. Then

$$\theta_u(f)(b) = f(ub) \quad \forall b \in \mathcal{R} , \ u \in \mathcal{R}^* = W .$$

Proof. Let us consider (cf. [We2] p.257) the direct product decomposition:

$$\mathcal{A}^* = \mathbb{Q}^*_+ \times \left( \prod_p \mathcal{I}^*_p \right) = \mathbb{Q}^*_+ \times \mathcal{R}^*$$

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where the morphism \( r : A^* \to \mathbb{Q}^* \) is given by
\[
r(z) = \prod_p \left| z_p \right|_p^{-1} = \prod_p p^{\text{val}(z_p)}.\]

By construction, \( r \) is the identity on \( \mathbb{Q}^* \) and its kernel is \( \prod_p \mathbb{Z}^*_p = \mathcal{R}^* \), thus \( r \) is the projection on the first term of this decomposition as a product. We use the second projection to identify \( W \) with \( \mathcal{R}^* \).

The equality of abelian groups \( A/\mathcal{R} = \mathbb{Q}/\mathbb{Z} \) shows that the multiplication by an element \( u \in \mathcal{R}^* \) defines an automorphism \( m_u \) of \( \mathbb{Q}/\mathbb{Z} \) and to prove Lemma 22 one just has to show that for any \( \gamma \in \mathbb{Q}/\mathbb{Z} \) one has:

\[
(9) \quad \theta_u(\epsilon(\gamma)) = \epsilon(m_{u^{-1}} \gamma) \quad \forall u \in \mathcal{R}^*.
\]

Since \( \varepsilon_\epsilon \) is separating for \( P^+_\mathbb{Q} \), it is enough to check that \( \theta_u(\epsilon(\gamma)) \varepsilon_\epsilon = \epsilon(m_{u^{-1}} \gamma) \varepsilon_\epsilon \), i.e. that

\[
(10) \quad u \epsilon(\gamma) u^* \varepsilon_\epsilon = \epsilon(m_{u^{-1}} \gamma) \varepsilon_\epsilon.
\]

Since \( \varepsilon_\epsilon \) is fixed by \( P_\mathbb{R} \subset P_A \) it is fixed by the action of \( u^* \in \mathcal{R}^* \subset P_\mathbb{R} \). Moreover for any \( \delta \in \mathbb{Q}/\mathbb{Z} \) the action of \( \epsilon(\delta) \) on \( \varepsilon_\epsilon \) gives simply \( \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \varepsilon_\epsilon \). Thus (10) follows from the equality
\[
\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u^{-1} \end{bmatrix} = \begin{bmatrix} 1 & \delta u^{-1} \\ 0 & 1 \end{bmatrix}.
\]

This completes the proof of Lemma 22 and of Proposition 21 a). To prove 21 c) note that the action of \( \mathcal{R}^* \) in \( \ell^2(\Delta) \) fixes the vector \( \varepsilon_\epsilon \) which proves that the action of \( W \) on \( C^\mathbb{Q} \) preserves the state \( \varphi \). It obviously commutes with \( \sigma \) anyway. Let us prove 21 b). As \( W \) is a compact group we can consider the natural projection \( E \) of \( C^\mathbb{Q} \) on \( C^W \) given by:

\[
(11) \quad E(x) = \int_W \theta_u(x) du.
\]

By construction \( E \) is norm continuous and satisfies
\[
(12) \quad E(ab) = a E(x) b \quad \forall x \in C^\mathbb{Q}, \ a, b \in C^W.
\]

Since by (8) one has \( C^*(\mathbb{N}^*) \subset C^W \), it suffices using the linear basis \( \mu_n \epsilon(\gamma) \mu_m^* \) of \( \mathcal{H} \), to show that

\[
(13) \quad E(\epsilon(\gamma)) \in C^*(\mathbb{N}^*) \quad \forall \gamma \in \mathbb{Q}/\mathbb{Z}
\]

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to conclude that $C^*(\mathbb{N}^*) = C^*_{\mathbb{Q}}$.

Finally to prove (13) note that $E(e(\gamma)) \in C^*(\mathbb{Q}/\mathbb{Z}) = C(\mathcal{R})$ is given by a function $f \in C(\mathcal{R})$ such that:

\begin{equation}
 f(ub) = f(b) \quad \forall u \in \mathcal{R}^* \quad \forall b \in \mathcal{R}.
\end{equation}

This equality defines a $C^*$-subalgebra of $C(\mathcal{R}) = \bigotimes_{p \in \mathcal{P}} C(\mathcal{R}_p)$, which is identical with $C^*(\mathbb{N}^*) \cap C^*(\mathbb{Q}/\mathbb{Z})$. This can be seen locally by showing that

\begin{equation}
 f \in C(\mathcal{R}_p) \ , \ f(ub) = f(b) \quad \forall u \in \mathcal{R}_p^* \ , \ b \in \mathcal{R}_p
\end{equation}

implies that $f$ is a function of $| \cdot |_p$.

6. Classification of KMS$_\beta$ states for $\beta > 1$

We shall first construct involutive representations $\pi_\alpha$ of the Hecke algebra $\mathcal{H}$, labelled by the Galois group $G = \text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$ of the subfield $\mathbb{Q}^{\text{cycl}}$ of $\mathbb{C}$ generated by all roots of unity.

Let $\mathcal{H}$ be the Hilbert space $\ell^2(\mathbb{N}^*)$, with its canonical orthonormal basis $(\varepsilon_k)_{k \in \mathbb{N}^*}$.

**Proposition 23.** The following equalities define an involutive representation $\pi_1$ of the Hecke algebra $\mathcal{H} = \mathcal{H}(\Gamma, \Gamma_0)$ in $\ell^2(\mathbb{N}^*)$.

\begin{enumerate}
  \item $\pi_1(\mu_n) \varepsilon_k = \varepsilon_{nk} \quad \forall n, k \in \mathbb{N}^*$
  \item $\pi_1(e(\gamma)) \varepsilon_k = \exp(2\pi i k \gamma) \varepsilon_k \quad \forall k \in \mathbb{N}^* \ , \ \gamma \in \mathbb{Q}/\mathbb{Z}$.
\end{enumerate}

**Proof.** We just need to check that the relations a) . . . f) are fulfilled by the operators $\mu'_n = \pi_1(\mu_n)$ and $e'(\gamma) = \pi_1(e(\gamma))$ defined by a) . b). Since the map $k \to nk$ from $\mathbb{N}^*$ to $\mathbb{N}^*$ is injective the relation a) follows and moreover the adjoint $\mu'^*_n$ is given by:

\begin{equation}
 \mu'^*_n \varepsilon_k = \varepsilon_{k/n} \quad \text{if } n \mid k \quad \text{and 0 otherwise}.
\end{equation}

The relation b) is obvious. To check c) note that if $(n, m) = 1$ then $m \mid k \iff m \mid nk$ and when applied to $\varepsilon_k$ both $\mu'_n \mu'^*_m$ and $\mu'^*_n \mu'_m$ vanish or are equal to $\varepsilon_{nk/m}$. The relation d) is clear as well as e),

\begin{equation}
 e'(\gamma) \mu'_n \varepsilon_k = e'(\gamma) \varepsilon_{nk} = \exp 2\pi i (nk \gamma) \varepsilon_{nk} = \mu'_n e'(n \gamma) \varepsilon_k.
\end{equation}
Let us check f). First both sides applied to $\varepsilon_k$ vanish unless $n \mid k$. This is clear for the left side by (1), it is true for the right side because it is of the form:

$$e'(\delta_0) \frac{1}{n} \sum_{n\delta = 0} e'(\delta) , \quad n \delta_0 = \gamma$$

and $\sum_{n\delta = 0} e'(\delta) \varepsilon_k = 0$ unless $n \mid k$.

Next, when $n \mid k$ so that $k = qn$, one has:

$$\mu_n e'(\gamma) \mu_n^* \varepsilon_k = \exp(2\pi i q \gamma) \varepsilon_k$$

$$e'(\delta_0) \varepsilon_k = \exp(2\pi i k \delta_0) \varepsilon_k = \exp(2\pi i q \gamma) \varepsilon_k$$

for any $\delta_0 \in \mathbb{Q}/\mathbb{Z}$ such that $n \delta_0 = \gamma$.

**Proposition 24.** 1) For $\alpha \in \mathbb{G} = \text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$, the following equalities define an involutionary representation $\pi_\alpha$ of $\mathcal{H}$ in $\ell^2(\mathbb{N}^*)$:

- $\alpha) \pi_\alpha(\mu_n) \varepsilon_k = \varepsilon_{nk} \quad \forall n, k \in \mathbb{N}^*$
- $\beta) \pi_\alpha(e(\gamma)) \varepsilon_k = \alpha(\exp 2\pi i k \gamma) \varepsilon_k \quad \forall k \in \mathbb{N}^*, \gamma \in \mathbb{Q}/\mathbb{Z}.$

2) For any element $x \in \mathcal{H}(\mathbb{Q})$ of the $\mathbb{Q}$ algebra generated by the $\mu_n$ and the $e(\gamma)$, the matrix elements of $\pi_\alpha(x)$ satisfy:

$$\langle \pi_\alpha(x) \varepsilon_{k_1}, \varepsilon_{k_2} \rangle = \alpha(\pi_1(x) \varepsilon_{k_1}, \varepsilon_{k_2}) \quad \forall k_j.$$

**Proof.** 1) The above proof of Proposition 23 works with no change. The only important point is to check d) namely that $\pi_\alpha(e(\gamma))^* = \pi_\alpha(e(-\gamma))$. This follows because the complex conjugation $z \rightarrow \overline{z}$ commutes with any $\alpha \in \mathbb{G}$.

2) By construction the matrix elements of the $\pi_\alpha$ of the generators satisfy the required relation which is stable under the algebraic operations of matrices involving only finite sums of products.

Let then $H$ be the positive operator in $\ell^2(\mathbb{N}^*)$ corresponding to the time evolution described in section 2, namely

$$H \varepsilon_n = (\log n) \varepsilon_n \quad \forall n \in \mathbb{N}^*.$$

We already saw in section 2 that

$$\varepsilon^{it} \langle x, e^{-it} \rangle = \sigma_t(x) \quad \forall x \in C^*(\mathbb{N}^*), \forall t \in \mathbb{R}.$$
Since it is obvious that $H$ commutes with $\pi_\alpha(y)$ for any $y \in C^*(\mathbb{Q}/\mathbb{Z}) \subset C_\mathbb{Q}$, we thus obtain:

$$e^{i t H} \pi_\alpha(x) e^{-i t H} = \pi_\alpha(\sigma_t(x)) \quad \forall x \in C_\mathbb{Q}, \forall t \in \mathbb{R}.$$ 

We can now state:

**Theorem 25.** Let $\tilde{\pi}_\alpha$ be the canonical extension of the representation $\pi_\alpha$ of $\mathcal{H}$ to the $C^*$-algebra $C_\mathbb{Q}$, and let $\beta > 1$.

a) The following equality defines a KMS$_\beta$ state on $(C_\mathbb{Q}, \sigma_t)$:

$$\varphi_{\beta, \alpha}(x) = \zeta(\beta)^{-1} \text{Trace}(\tilde{\pi}_\alpha(x) e^{-\beta H}) \quad \forall x \in C_\mathbb{Q}.$$ 

b) The map $\alpha \rightarrow \varphi_{\beta, \alpha}$ is a homeomorphism of the Galois group $G$ of $\mathbb{Q}^{\text{cycl}}$ with the space of extreme points of the Choquet simplex of KMS$_\beta$ states on $(C_\mathbb{Q}, \sigma_t)$.

**Proof.** a) First by Proposition 19 we know that the representation $\pi_\alpha$ extends to a representation of $C_\mathbb{Q}$ and the equality (5) together with the finiteness of Trace $(e^{-\beta H}) = \zeta(\beta)$ gives a).

b) Let us fix $\beta > 1$ and show first that the map $\alpha \rightarrow \varphi_{\beta, \alpha}$ is injective. To show this note that each of the representations $\pi_\alpha$ of $C_\mathbb{Q}$ is irreducible and by construction each $\varphi_{\beta, \alpha}$ is a type $I_\infty$ factor state. Thus its GNS construction canonically determines the positive operator $H$, $0 \in \text{Sp} H$, as an unbounded operator affiliated to the weak closure of $C_\mathbb{Q}$. In particular $\varphi_{\beta, \alpha}$ determines canonically the 0-temperature state,

$$\varphi_{\infty, \alpha}(x) = \langle \pi_\alpha(x) \xi_1, \xi_1 \rangle.$$ 

When we restrict this state $\varphi_{\infty, \alpha}$ to the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ of $\mathbb{Q}/\mathbb{Z}$ with rational coefficients, we find the corresponding imbedding of the field $\mathbb{Q}^{\text{cycl}}$ of roots of unity in $\mathbb{C}$:

$$\varphi_{\infty, \alpha}(\Sigma \lambda_j \gamma_j) = \alpha(\Sigma \lambda_j \exp(2\pi i \gamma_j))$$

which of course determines $\alpha \in G$ uniquely.

Next each $\varphi_{\beta, \alpha}$ is a factor state and thus is an extreme point of the weakly compact convex set $K_\beta$ of KMS$_\beta$ states. Let $\mathcal{E}(K_\beta)$ be the space of extreme points of $K_\beta$. We have shown that the map $\alpha \rightarrow \varphi_{\beta, \alpha}$ is an injection of $G$ in $\mathcal{E}(K_\beta)$. This map is weakly continuous since, as $\beta > 1$, the series $\Sigma \alpha(\exp 2\pi i k \gamma) k^{-\beta}$ is uniformly convergent. It remains to show that this map is surjective. Note first that for any element $u$ of $W$ there exists a corresponding element $\alpha(u)$ of $G$ such that

$$\varphi_{\beta, 1} \circ \theta_u = \varphi_{\beta, \alpha(u)}$$

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and the map \( u \to \alpha(u) \) is an isomorphism of \( W \) with \( G \). Next there exists on \( C_\mathbb{Q} \) a unique KMS\(_{\beta} \) state which is \( W \)-invariant. This holds for any value \( \beta \in \mathbb{R} \) and follows from Proposition 21 b) and Proposition 8 a). Indeed, given such a state \( \varphi \) one has \( \varphi = \varphi \circ E \) where \( E \) is the projection \( E = \int_W \theta_u \, du \) of \( C_\mathbb{Q} \) on \( C_\mathbb{Q}^W \) and the restriction of \( \varphi \) to \( C_\mathbb{Q}^W = C^*(\mathbb{N}^*) \) is unique. Thus let \( \psi \) be any KMS\(_{\beta} \) state; one has

\[
\int_W \psi \circ \theta_u \, du = \int_W \varphi_{\beta, \alpha(u)} \, du
\]

since both sides are \( W \)-invariant KMS\(_{\beta} \) states. Now if \( \psi \in \mathcal{E}(K_\beta) \) is an extreme point this equality (9) gives two decompositions of the same state as a barycenter of measures over \( \mathcal{E}(K_\beta) \), which is a Choquet simplex (Proposition 2), so that

\[
\psi \circ \theta_u \in \{ \varphi_{\beta, \alpha(v)} : v \in W \} \quad \text{for almost all } u .
\]

Finally this implies that \( \psi = \varphi_{\beta, \alpha(u)} \circ \theta_u^{-1} \) for some \( u, v \in W \) and \( \psi \) is in the image of the map \( \alpha \to \varphi_{\beta, \alpha} \). Since the map \( \alpha \to \varphi_{\beta, \alpha} \) is continuous and bijective and \( G \) is compact, it is homeomorphism with its range \( \mathcal{E}(K_\beta) \) and this proves Theorem 25.

**Remarks 26.** 1) We shall give in the next section the general formula (for all values of \( \beta \)) for the \( W \)-invariant KMS\(_{\beta} \) state \( \varphi_\beta \) on \( C_\mathbb{Q} \), but we can already find the formula for \( \beta > 1 \) at this stage using the equality (9). First, using the linear basis \( (t_{n,m,\gamma}) \) of \( \mathcal{H} \subset C_\mathbb{Q} \) described in section 4 one has

\[
\varphi(t_{n,m,\gamma}) = 0 \quad \text{if} \quad n/m \neq 1 \quad \forall \varphi \in K_\beta
\]

thus it is enough to determine the restriction of \( \varphi_\beta \) to \( C_\mathbb{Q} \). For this one can for instance use the right hand side of (9) and the formula in Theorem 25 a) for \( \varphi_{\beta, \alpha} \). Let \( \gamma \in \mathbb{Q}/\mathbb{Z} \), \( n \in \mathbb{N}^* \), then one has

\[
\int_G \alpha(\exp(2\pi i n \gamma)) \, d\alpha = \frac{\mu(b/d)}{\varphi(b/d)}
\]

where \( \gamma = \frac{a}{b} \), \( (a, b) = 1 \) and \( (n, b) = d \) is the g.c.d of \( n \) and \( b \). Also \( \mu \) is the Moebius function and \( \varphi \) the Euler function. Let then \( \rho_\beta \) be the multiplicative function such that:

\[
\sum_{n \in \mathbb{N}^*, (n, b) = 1} n^{-\beta} = \rho_\beta(b) \zeta(\beta) .
\]

One has \( \rho_\beta(b) = \prod_{p \mid b, p \text{ prime}} (1 - p^{-\beta}) \).
The unique $W$-invariant KMS$_\beta$ state, given by (9) satisfies

$$
\varphi_\beta(\epsilon(\gamma)) = \sum_{d | b} \frac{\mu(b/d)}{\varphi(b/d)} \rho_\beta(b/d) d^{-\beta}
$$

where $b$ is the denominator of the irreducible fraction $\gamma = \frac{a}{b}$. The right hand side of (14) is a multiplicative function of $b$ and it is also given by:

$$
\varphi_\beta(\epsilon(\gamma)) = b^{-\beta} \prod_{p | b, p \text{ prime}} (1 - p^{\beta-1})(1 - p^{-1})^{-1}
$$

as can be seen by computing the right hand side of (14) when $b$ is a prime power.

We shall give another proof of (15) in the next section.

2) The statement of Theorem 25 b) also applies to the 0-temperature states $\varphi_{\infty, \alpha}$. For extreme such states the map $\varphi_{\infty, \alpha}$ restricted to $\mathbb{Q}(\mathbb{Q}/\mathbb{Z}) \subset C^*(\mathbb{Q}/\mathbb{Z})$ gives the associated embedding of the field $\mathbb{Q}^{\text{cycl}}$ of roots of unity in $\mathbb{C}$, as we have seen above.

7. **Uniqueness of KMS$_\beta$ states for $\beta \in [0,1]$**

In this section we shall show that for $\beta \in [0,1]$ there exists a unique KMS$_\beta$ state on the $C^*$-algebra $C_\mathbb{Q}$. We have seen in section 5, Proposition 21, that the $C^*$-subalgebra $C^*(\mathbb{N}^*) \subset C_\mathbb{Q}$ generated by the $\mu_n$ is the fixed point algebra of the action of $W$ on $C_\mathbb{Q}$:

$$
C^*(\mathbb{N}^*) = C^W_\mathbb{Q}.
$$

Since $W$ is a compact abelian group, its action on $C_\mathbb{Q}$ has discrete spectrum and we can consider for each character $\chi$ of $W$ the corresponding spectral subspace (cf. [Ped]),

$$
C_{\mathbb{Q}, \chi} = \{ x \in C_\mathbb{Q} : \theta_u(x) = \chi(u) x \quad \forall u \in W \}
$$

To prove the uniqueness of KMS$_\beta$ states on $C_\mathbb{Q}$ for $0 < \beta \leq 1$ we shall analyze the partial automorphisms of the type III$_1$ factor associated to $(C^*(\mathbb{N}^*), \varphi_\beta)$ and to a fixed non trivial character $\chi$ of $W$.

We shall show that these partial automorphisms are outer. Given an element $V$ of $C^*(\mathbb{Q}/\mathbb{Z}) = C(\mathcal{R})$ (resp. a character $\chi$ of $W$), we shall say that $V$ (resp. $\chi$) is localized in a subset $F$ of $\mathcal{P}$, the set of finite places of $\mathbb{Q}$, iff

$$
V \in ( \bigotimes_{p \in F} C(R_p) ) \otimes 1 \subset C(\mathcal{R})
$$
(resp. if $\chi$, seen as a character of $A^*$, factorizes through the projection $W \to \prod_{p \in F} Q^*_p$).

Let us now state the main lemma.

**Lemma 27.** Let $\beta \in [0, 1]$ and $\psi$ be a KMS$_\beta$ state on the $C^*$-dynamical system $(C_{Q}, \sigma_t)$. Then:

a) The restriction of $\psi$ to $C^*(N^*)$ is equal to $\varphi_\beta$.

b) Let $\chi$ be a non trivial character of $W$ and $V \in C^*(Q/\mathbb{Z})$ be a partial isometry, both localized in a finite set $F \subset P$, such that

$$\theta_g(V) = \chi(g) V \quad \forall g \in W.$$  

Then $\psi(Vx) = 0 \quad \forall x \in C^*(N^*)$.

c) The restriction of $\psi$ to the spectral subspaces $C_{Q_\chi}, \chi \neq 1$ is equal to 0.

**Proof.**

a) Since the restriction of $\sigma_t$ to $C^*(N^*)$ is the one parameter group of automorphisms of Proposition 8, the restriction of $\psi$ to $C^*(N^*)$ is a KMS$_\beta$ state and the conclusion follows from Proposition 8 a).

b) Let $E = V^*V$. As $C^*(Q/\mathbb{Z})$ is commutative one has $E = VV^*$. Also $E$ belongs to the algebra $C_{Q}^W = C^*(N^*)$. Let $\alpha$ be the automorphism of the reduced algebra $C^*(N^*)_E$ determined by the equality:

$$\alpha(x) = V x V^* \quad \forall x \in C^*(N^*)_E.$$  

Let $M$ be the factor (of type III$_1$) which is the weak closure of $C^*(N^*)$ in the G.N.S. representation of $\varphi_\beta$. Let us identify $C^*(N^*)$ with a weakly dense subalgebra of $M$ and extend the state $\varphi_\beta$ to a normal state $\tilde{\varphi}_\beta$ on $M$. As $V$ belongs to the fixed point algebra of $\sigma_t$, it belongs to the centralizer of $\psi$. It follows that the automorphism $\alpha$ of $C^*(N^*)_E$ preserves $\tilde{\varphi}_\beta$ and extends to an automorphism of $M_E$. Let us show that for $\beta \in [0, 1]$ this automorphism is outer. For any $q \in P \setminus F$ one has:

$$E \mu_q \in M_E \cdot \alpha(E \mu_q) = \chi(g_q) E \mu_q$$

where $g_q \in R^* = \Pi Z_p^*$ is given by its components

$$(g_q)_p = q \in \mathbb{Z} \cap Z_p^* \quad \text{if} \quad q \neq p, \quad (g_q)_q = 1.$$  

To prove (5) note that for any $f \in C(R) = C^*(Q/\mathbb{Z})$ and $n \in N^*$ one has (cf. Proposition 18 e))

$$f \mu_n = \mu_n f_n \quad \text{where} \quad f_n(b) = f(nb) \quad \forall b \in R.$$  

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Thus if \( f \) is localized in \( F \subset \mathcal{P} \) and \( q \not\in F \) one gets \( \theta_{g_q}(f) = f_q \).

(8) \[
    f_{\mu_q} = \mu_q \theta_{g_q}(f).
\]

Applying this to \( f = V \) one gets \( V_{\mu_q} = \chi(g_q) \mu_q V \), i.e. one gets (5). Let us view \( \chi \) as a character of \((\mathbb{Z}/m\mathbb{Z})^*\) where the prime factors of \( m \) all belong to \( F \). Then we can simply write \( \chi(q) \) instead of \( \chi(g_q) \), for \( q \not\in F \). It follows from (5) that modulo inner automorphisms the automorphism \( \alpha \) is the infinite tensor product

(9) \[
    \alpha = \bigotimes_{q \in F, \chi(q)} \rho_{q, \chi(q)} \quad \text{in} \quad M_{F^*} = \bigotimes_{q \in F, \chi(q)} (M_q, \varphi_{\beta, q})
\]

where for any complex number \( \lambda, |\lambda| = 1 \), we let \( \rho_{p, \lambda} \) be the automorphism of \( \tau_p \) such that

(10) \[
    \rho_{p, \lambda}(\mu_p) = \lambda \mu_p.
\]

This automorphism is a special case of \( \sigma_{\tau, p} \) and preserves the state \( \varphi_{\beta, p} \) by construction.

For any \( \chi \in \hat{W} \) localized on \( F \) we let \( \hat{\theta}_\chi \) be the element of \( \text{Out} M = \text{Aut}(M)/\text{Int}(M) \) determined by the class of

(11) \[
    \left( \bigotimes_{q \in F} \text{id} \right) \otimes \bigotimes_{q \not\in F, \chi(q)} \rho_{q, \chi(q)}.
\]

**Lemma 28.** \( \hat{\theta}_\chi \) is inner relative to \( \varphi_{\beta} \) iff the following infinite product converges absolutely in absolute value:

\[
    \prod_{p \in \mathcal{P}} (1 - p^{-\beta})(1 - \chi(p)p^{-\beta})^{-1}.
\]

**Proof.** One has in the type \( I_\infty \) factor \( M_p \) associated to \( (\tau_p, \varphi_{\beta, p}) \) a unitary implementing the automorphism \( \rho_{p, \chi(p)} \), it is given by the diagonal operator with eigenvalues the \( \chi(p)^3 \).

Evaluating the state \( \varphi_{\beta, p} \) on this unitary gives:

\[
    (1 - p^{-\beta}) \sum_{n = 0}^{\infty} \chi(p)^n p^{-n^3} = (1 - p^{-\beta})(1 - \chi(p)p^{-\beta})^{-1}
\]

and the result follows from general criterions (cf. [Co]).

This shows using Dirichlet's theorem [Ser1] that \( \hat{\theta}_\chi \) is outer (\( \chi \) non trivial) when \( \beta \leq 1 \) and is inner (since \( \varphi_{\beta} \) is a type \( I_\infty \) factor state) for \( \beta > 1 \).
This lemma shows that, for $\beta \in [0, 1]$, the automorphism $\alpha$ of $M_E$ given by (4) is outer:

$$\{ y \in M_E ; \quad y \alpha(x) = x y \quad \forall x \in M_E \} = \{ 0 \} .$$

Now let $L$ be the linear form on $M_E$ given by

$$L(x) = \psi(V x) \quad \forall x \in C^*(\mathbb{N}^\ast)_E .$$

The Schwartz inequality $|L(x)|^2 \leq \psi(E) \psi(x^* x)$ shows that it is a normal linear functional on $M_E$. Let $u$ be the partial isometry $u \in M_E$ of its polar decomposition: $L = u |L|.$

The KMS$_\beta$ condition for $\psi$ applied to the pair $V x, y; x \in C^*(\mathbb{N}^\ast)$, $y \in C^*(\mathbb{N}^\ast)$, shows that $L$ satisfies the $\alpha$-twisted KMS$_\beta$ condition, where $L(\sigma_t(y) x)$ is replaced by $L(\sigma_t(y) \alpha(x))$.

Now as both $V$ and $\psi$ are $\sigma_t$ invariant so is $L$ and hence so are $u$ and $|L|$. It follows that the Radon-Nikodym derivative $(D|L| : D \bar{\varphi}_\beta)_t$ belongs to the centralizer of $\bar{\varphi}_\beta$ and is of the form $h^u$ with

$$|L| (x) = \bar{\varphi}_\beta(h x) \quad \forall x \in M_E .$$

From the twisted KMS$_\beta$ condition we then get:

$$z \overline{\alpha(z)} \quad \forall z \in M_E$$

which implies by (12) that $h u = 0$ and that $L = 0$.

This proves Lemma 27 b). Let us prove 27 c). It is enough for that purpose, given a character $\chi \in \hat{W}$ localized in $F \subset \mathcal{P}$, to find a sequence $V_n$ of partial isometries $V_n \in C^*(\mathbb{Q}/\mathbb{Z}) = C(\mathcal{R})$, localized in $F$ and such that:

$$\theta\beta(V_n) = \chi(g) V_n \quad \forall g \in W : \bar{\varphi}_\beta(V_n V_n^*) \overset{n \to \infty}{\to} 1 .$$

Then, given $a \in C_{\mathbb{Q}, \chi}$, one has:

$$\psi(a) = \lim_{n \to \infty} \psi(V_n V_n^* a) = 0$$

because $V_n^* a \in C_{\mathbb{Q}, 1} = C^*(\mathbb{N}^\ast)$ and Lemma 27 b) applies. Finally the construction of the $V_n$ is reduced by Lemma 22 to the construction of continuous functions $V_n \in C(\prod_{p \in F} \mathbb{Z}_p)$ such that:

$$V_n(g b) = \chi(g) V_n(b) \quad \forall b \in \bigcap_{p \in F} \mathbb{Z}_p \quad g \in \bigcap_{p \in F} \mathbb{Z}_p^* .$$

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and such that the $|V_n|$ are uniformly bounded and converge pointwise to 1; this is immediate and the existence of the partial isometries $V_n$ follows.

**Corollary 29.** For any $\beta \in [0, 1]$ there exists at most one KMS$_\beta$ state on $C\mathcal{Q}$.

**Proof.** The group $W$ is a compact group so that the direct sum of the spectral subspaces $C\mathcal{Q}_\lambda$ is dense in $C\mathcal{Q}$ and this determines $\psi$ uniquely by Lemma 27.

We shall now construct this unique KMS$_\beta$ state $\psi_\beta$ on $C\mathcal{Q}$ in a geometric manner using the action on the product of trees of section 3. The construction will follow from the following general lemma applied to the $C^*$-module $\mathcal{E} = C^*(G)e$ over $C^*(\mathbb{N}^*)$ and the time evolution $\sigma_t$ of $C^*(\mathbb{N}^*)$.

**Lemma 30.** Let $C$ be a unital $C^*$-algebra, $\mathcal{E}$ a $C^*$-module over $C$, $(\sigma_t)_{t \in \mathbb{R}}$ a one-parameter group of automorphisms of $C$, $\beta \in ]0, \infty[$, $\varphi_\beta$ a KMS$_\beta$ state on $C$, and $\mathcal{H}_{\varphi_\beta}$ the Hilbert space of the GNS construction for $\varphi_\beta$.

a) Let $\mathcal{H}_\beta$ be the completion of $\mathcal{E}$ for the inner product given by:

$$\langle \xi, \eta \rangle_\beta = \varphi_\beta(\langle \xi, \eta \rangle) \quad \forall \xi, \eta \in \mathcal{E}.$$  

Then the action of the endomorphisms $\text{End}_C(\mathcal{E})$, on $\mathcal{E}$ extends by continuity to $\mathcal{H}_\beta$.

b) There exists a unique representation $\rho$ of $C^0$ (the opposite algebra of $C$) in $\mathcal{H}_\beta$ such that for any $\xi \in \mathcal{H}_\beta$ and $a \in C$ in the domain of $\sigma_{1/2}$ one has:

$$\rho(a)\xi = \xi \sigma_{1/2}(a).$$

This representation commutes with the left action of $\text{End}_C(\mathcal{E})$.

**Proof.** The Hilbert space $\mathcal{H}_\beta$ is the tensor product of $C^*$ modules

$$\mathcal{H}_\beta = \mathcal{E} \otimes_C \mathcal{H}_{\varphi_\beta}$$

so that the first assertion follows. The second assertion also follows, using $\mathcal{H}_{\varphi_\beta}$ as a left Hilbert algebra and the stabilization theorem of Kasparov [Ka].

We apply this lemma with $C = C^*(\mathbb{N}^*)$, $\mathcal{E} = C^*(G)e$, and $\sigma_t \in \text{Aut} C$ given by the time evolution (Proposition 7 3)) of $C^*(\mathbb{N}^*)$. As $\mathcal{E}$ is a space of functions on $\Delta$, so is each of the $\mathcal{H}_\beta$, and for each $\alpha \in \Delta$ we let $\varepsilon_\alpha$ be the characteristic function of $\{\alpha\} \subset \Delta$. The vectors $\varepsilon_\alpha$, $\alpha \in \Delta$, are of unit length in each $\mathcal{H}_\beta$ and always span a dense subspace of $\mathcal{H}_\beta$. For $\beta = 1$ they give an orthonormal basis, so that $\mathcal{H}_1 = \ell^2(\Delta)$. In order to compute
the inner product $\langle \varepsilon_\alpha, \varepsilon_{\alpha'} \rangle_\beta$ in $\mathcal{H}_\beta$ we shall first work locally, i.e. we fix the local field $K = \mathbb{Q}_p$ and apply Lemma 30 to $C = C^*(P_K)_\tau$, the reduced $C^*$-algebra of $P_K$ relative to the projection $e = 1_{P_R}$, while the $C^*$ module is $\mathcal{E} = C^*(P_K)_\tau$. We use on $C$ the state $\varphi_{\beta,p}$. Then Lemma 30 yields an inner product on the space of functions with finite support on the tree $T_p = P_K/P_R$.

Let us now compute explicitly this inner product on the tree $T$ associated to any value $\beta$. Thus $K$ is a local field, $K = \mathbb{Q}_p$, and we first note that transporting $\varphi_{\beta,p}$ by the canonical isomorphism of $\tau_p$ with the reduced $C^*$-algebra $C^*(P_K)_\tau$, its value on a $P_R$ bi-invariant function $f(s)$, $s \in P_K$ is given by (cf. formula (9) of section 3),

$$
\varphi_{\beta,p}(f) = \left( \sum_{k > 0} p^{k(1 - \beta)} f \left( \begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix} \right) \right) (1 - p^{\beta - 1}) + f \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right).
$$

Let $g \in P_K$, $g = \begin{bmatrix} 1 & n_0 \\ 0 & h_0 \end{bmatrix}$. The inner product $\langle g \varepsilon_0, \varepsilon_0 \rangle_\beta$, with $\varepsilon_0$ corresponding to the base point, is equal to $\varphi_{\beta,p}(f)$, where the function $f$ is associated by formula (13) of section 3 to the right cosets $P_R$ and $gP_R$ in $P_K/P_R$:

$$
f(s) = m(g P_R \cap P_R s^{-1}) \quad \forall s \in P_K
$$

where $m$ is the left Haar measure on $P_K$. We just need to evaluate $f(s)$ for $s = \begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix}$.

One has

$$
P_R s^{-1} = \left\{ \begin{bmatrix} 1 & n \\ 0 & h \end{bmatrix} : \begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix} : \text{val}(n) \geq 0 , \text{val}(h) = 0 \right\} =
$$

$$
\left\{ \begin{bmatrix} 1 & p^{-k} + n \\ h \end{bmatrix} : \text{val}(n) \geq 0 , \text{val}(h) = 0 \right\}.
$$

All its elements $\begin{bmatrix} 1 & m \\ 0 & a \end{bmatrix}$ satisfy $\text{val}(a) = 0$. This implies that $P_R s^{-1} \cap g P_R \neq \emptyset$ only if $\text{val}(h_0) = 0$. Thus:

$$
\langle g \varepsilon_0, \varepsilon_0 \rangle = 0 \quad \text{if} \quad \text{val}(h_0) \neq 0 \quad \text{(for} \quad g = \begin{bmatrix} 1 & n_0 \\ 0 & h_0 \end{bmatrix} \in P_K) \).
$$

Assume now that $\text{val}(h_0) = 0$; replacing $g$ by $g \begin{bmatrix} 1 & 0 \\ 0 & h_0^{-1} \end{bmatrix}$ we can assume that $h_0 = 1$ without changing $g \varepsilon_0$. One has

$$
g P_R = \left\{ \begin{bmatrix} 1 & n_0 \\ 0 & 1 \end{bmatrix} : \begin{bmatrix} 1 & n_1 \\ 0 & h_1 \end{bmatrix} : \text{val}(n_1) \geq 0 , \text{val}(h_1) = 0 \right\} =
$$

$$
\left\{ \begin{bmatrix} 1 & n_1 + h_1 n_0 \\ h_1 \end{bmatrix} : \text{val}(n_1) \geq 0 , \text{val}(h_1) = 0 \right\}.
$$
One has \( gP_R \cap P_R s^{-1} \neq \emptyset \) only if \( \text{val}(n_0) = -k \). Let us assume that \( \text{val}(n_0) = -k \). Then we need to compute the multiplicative Haar measure of the set of \( h_1 \in \mathbb{R}^* \) such that \( h_1n_0 = p^{-k}n_0^{-1} + n_0^{-1}R = p^{-k}n_0^{-1} + p^kR \). The additive and multiplicative Haar measures coincide on \( \mathbb{R}^* \) up to an overall coefficient \( d^* h = \left( 1 - \frac{1}{p} \right)^{-1} dh \). Thus we get the equality, with \( g = \begin{bmatrix} 1 & n_0 \\ 0 & 1 \end{bmatrix} \)

\[
(24) \quad f \left( \begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix} \right) = (1 - p^{-1})^{-1} p^{-k} \quad \text{if} \quad k = -\text{val}(n_0) \\
\]

\[
f \left( \begin{bmatrix} 1 & p^{-k} \\ 0 & 1 \end{bmatrix} \right) = 0 \quad \text{if} \quad k \neq -\text{val}(n_0) \, .
\]

This together with formula (19) gives the equality

\[
(25) \quad \varphi_{\beta,p}(f) = p^{-k\beta} (1 - p^{\beta-1})(1 - p^{-1})^{-1} \quad k = -\text{val}(n_0)
\]

i.e.

\[
(26) \quad \langle g \varepsilon_0, \varepsilon_0 \rangle_{\beta} = p^{-k\beta} (1 - p^{\beta-1})(1 - p^{-1})^{-1} \, ,
\]

where \( g = \begin{bmatrix} 1 & n_0 \\ 0 & 1 \end{bmatrix} \), \( k = -\text{val}(n_0) \).

The next step is to understand the geometric meaning of the orbit of \( \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} L_0 \right\} \) in the tree \( T_p \) and of the value \( k = -\text{val}(n) \). Since the action of \( P_K \) on the tree \( T_p \) fixes a point at \( \infty \) it preserves the horocycles corresponding to this point. These horocycles are the equivalence classes of the relation \( R_\infty : L \sim L' \) iff \( \exists q \quad t^q L = t^q L' \) where \( t \) is the hyperbolic translation of one unit towards the point at \( \infty \).

We first check that two lattices \( L, L' \) are \( R_\infty \) equivalent iff they are on the same orbit of the subgroup \( \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \), \( n \in K \) of \( P_K \). This subgroup is a normal subgroup and hence defines an equivalence relation which is stable under the left action of \( P_K \). The map \( t \) is given by:

\[
(27) \quad t(g \, L_0) = g \, g_p \, L_0 \quad \forall \, g \in P_K \quad \text{with} \quad g_p = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \, .
\]

More generally \( t^q(g \, L_0) = g \, g_p^q \, L_0 \quad \forall \, g \in P_K \). Thus \( g_1 \, L_0 \sim g_2 \, L_0 \) (\( R_q \)) iff \( g_1 \, g_p^q \, L_0 = g_2 \, g_p^q \, L_0 \), i.e.

\[
(28) \quad g_2^{-1} \, g_1 \in g_p^q \, P_R \, g_p^{-q} \, .
\]
This holds iff \( g_2^{-1}g_1 \) is of the form \[
\begin{bmatrix}
1 & m \\
0 & h
\end{bmatrix},
\text{val}(h) = 0, \text{val}(m) \geq -q.
\]
Thus \( g_1 L_0 \sim g_2 L_0 \ (R_\infty) \) iff \( g_2^{-1}g_1 \in K \gg R^* \). Two lattices \( g_1 L_0 \) and \[
\begin{bmatrix}
1 & n \\
0 & 1
\end{bmatrix} g \ L_0
\]
obviously satisfy this relation and conversely, if \( g_1 L_0 \sim g_2 L_0(R_\infty) \) we may write \( g_2 \) as \[
\begin{bmatrix}
1 & n \\
0 & 1
\end{bmatrix}
g_1 \) without affecting \( g_j L_0 \).

Let us then understand the value of \(-\text{val}(m)\) as a function of the distance between \( L_0 \) and \( L = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} L_0 \). Let \( k = -\text{val}(m) \). We know that \( t^k(L_0) = t^k(L) \) and that this fails for \( k-1 \). Hence \( d(L_0, L) = 2k \). We thus get

**Lemma 31.** Let \( L, L' \in T \) be two lattices.

a) If they belong to different horocycle equivalence classes, they are orthogonal for \( (\cdot)_\beta \).

b) If they belong to the same horocycle class with distance \( d(L, L') = 2k > 0 \) one has:

\[
(\varepsilon_L, \varepsilon_{L'})_\beta = p^{-k\beta}(1 - p^\beta - 1)(1 - p^{-1})^{-1}.
\]

c) If \( L = L' \), then \( (\varepsilon_L, \varepsilon_{L'})_\beta = 1 \).

**Remarks.**
a) For \( \beta \to +\infty \) the above inner product converges to a non zero value only if \( L = L' \) or if \( L \neq L' \) but \( L \sim L'(R_1) \) in which case it goes to \(-p^{-1}(1 - p^{-1})^{-1}\).

b) For \( \beta \to 0 \) the inner product converges to 1 on each horocycle equivalence class, which are then reduced to a single point in \( \mathcal{H}_\beta, \beta = 0 \).

Let us now compute the corresponding inner product on \( \Delta = \Pi(T_p, L_0) = P_Q^+/P_{T}^+ \). We again call \( L_0 \) the base point.

Given \[
\begin{bmatrix}
1 & n \\
0 & h
\end{bmatrix} = g, \ n \in Q, \ h \in Q^*_+, \] to get a non zero inner product, \( (g L_0, L_0) \), we need that for each place \( p \), \( g_p L_0 \sim L_0(R_\infty) \) and hence that \( \text{val}(g_p) = 0 \). Thus \( h = 1 \). Let then \( N = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in Q \right\} \) and let us understand the inner product on the orbit \( N L_0 \). We have a basis \( \varepsilon_x \) parametrized by \( x \in Q/\mathbb{Z} \),

\[
\varepsilon_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} L_0.
\]

The inner product \( (\varepsilon_x, \varepsilon_0)_\beta \) is then given, using Lemma 31, by

\[
(\varepsilon_x, \varepsilon_0)_\beta = \prod_{\begin{subarray}{c} p \in P \\ k_p \neq 0 \end{subarray}} p^{-k_p \beta}(1 - p^\beta - 1)(1 - p^{-1})^{-1}
\]
where \(x = a/b\), \((a, b) = 1\) and \(b = \Pi_p p^k\) is the decomposition of \(b\) as a product of prime powers.

More generally, it is invariant by translations, i.e. \(\langle \varepsilon_x, \varepsilon_y \rangle_\beta = \langle \varepsilon_{x-y}, \varepsilon_0 \rangle_\beta\), thus the positivity involved is the fact that the function given by (29) is of positive type on the group of roots of unity: \(\mathbb{Q}/\mathbb{Z}\). This function is the function \(\psi_\beta\) of Theorem 5.

One has \(\langle g_1 L_0, g_2 L_0 \rangle_\beta = 0\) if \(g_2^{-1} g_1 \notin \mathbb{N}\).

Let us then take an arbitrary orbit \(N g L_0, g \in \text{P}_\mathbb{Q}^+\). We need to compute \(\langle (1, x) g L_0, (1, y) g L_0 \rangle_\beta\) where \(x, y \in \mathbb{Q}\) and \((1, x), (1, y)\) are the corresponding elements of \(\mathbb{N}\). One has \(\langle (1, x) g L_0, (1, y) g L_0 \rangle = \langle g \varepsilon_x, \varepsilon_y \rangle = \langle \varepsilon_{x'y}, \varepsilon_0 \rangle\) where \((1, x') = g^{-1}(1, x)g\) and \((1, y') = g^{-1}(1, y)g\). Thus we see that the orbits \(\mathbb{N} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \varepsilon_0, k \in \mathbb{Q}^*_+,\) are pairwise orthogonal for \(\langle \cdot , \cdot \rangle_\beta\) and the inner product is, up to a relabelling, given by (29) on each of them.

We are now ready to describe the Hilbert spaces \(\mathcal{H}_\beta\) associated by Lemma 30 to the \(\mathbb{C}^*\) module \(\mathcal{E} = B\varepsilon\) over \(\mathbb{C}^*(\mathbb{N}^*)\) and the KMS state \(\varphi_\beta\), and then get the commutant of \(\text{P}_\mathbb{Q}^+\) in \(\mathcal{H}_\beta\) as a unitary representation of the \(\mathbb{C}^*\)-algebra \(\mathcal{C}_\mathbb{Q}\).

**Proposition 32.** a) Let \(\mathcal{H}_\beta\) be the Hilbert space completion of the \(\mathbb{C}^*\) module \(\mathcal{E} = B\varepsilon\) over \(e\mathbb{C}^*(\text{P}_\mathbb{A})\varepsilon = \mathbb{C}^*(\mathbb{N}^*)\), with state \(\varphi_\beta\). Then \(\mathcal{H}_\beta\) has a natural basis indexed by \(\text{P}_\mathbb{Q}^+ / \text{P}_\mathbb{A}^+\), and its inner product is invariant under left translations by \(\text{P}_\mathbb{Q}^+\) and given by

\[
\langle \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \varepsilon_e, \varepsilon_e \rangle = 0 \quad \text{unless} \quad a = 1
\]

\[
\langle \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \varepsilon_e, \varepsilon_e \rangle = \psi_\beta(b)
\]

where \(\psi_\beta\) is the function of positive type defined in Theorem 5.

b) The \(\mathbb{C}^*\)-algebra \(\mathcal{C}_\mathbb{Q}^0\) admits a representation in \(\mathcal{H}_\beta\) given by the right convolution with \(e^{\beta/2} f\) for any \(\text{P}_\mathbb{A}^+\) bi-invariant function \(f\) on \(\text{P}_\mathbb{A}^+\).

c) The vector \(\varepsilon_0 = \text{class of } \text{P}_\mathbb{A}^+\) is cyclic for \(\text{P}_\mathbb{A}^+\), separating for \(\mathcal{C}_\mathbb{Q}\), \(\overline{\mathcal{C}_\mathbb{Q} \varepsilon_0}\) is the fixed points of \(\text{P}_\mathbb{A}^+ \subset \text{P}_\mathbb{Q}^+\) and \((\mathcal{C}_\mathbb{Q})''\) is the commutant of \(\text{P}_\mathbb{Q}^+\) in \(\mathcal{H}_\beta\).

d) The vector \(\varepsilon_0\) defines a KMS\(\beta\) state on \(\mathcal{C}_\mathbb{Q}\).

**Proof.** The proof of a) follows from (29). We let \(\mathcal{H}_{\beta,1}\) be the subspace of \(\mathcal{H}_\beta\) generated by the orbit \(N \varepsilon_0\) of \(\varepsilon_0\) under the normal subgroup \(N\) of \(\text{P}_\mathbb{Q}^+\). More generally for \(k \in \mathbb{Q}^*_+\), we let

\[
\mathcal{H}_{\beta,k} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \mathcal{H}_{\beta,1}.
\]

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The subspaces \( \mathcal{H}_{\beta,k} \) are pairwise orthogonal and \( \mathcal{H}_\beta \) is their direct sum.

Let us prove b). We know that the action by right convolution of the Hecke algebra \( \mathcal{H} \) of \( P_\mathbb{Q}^+ \)-biinvariant function on \( P_\mathbb{Q}^+ \) yields a representation of \( \mathcal{H}_0 \) on the linear span of the natural basis \( \varepsilon_x, x \in P_\mathbb{Q}^+ / P_\mathbb{Q}^+ \) of \( \mathcal{H}_\beta \). This is still true if we twist this action by the (non involutive) automorphism of \( \mathcal{H} \) given by multiplication by \( \delta^{\beta/2} \).

Thus we just need to show that the new representation of \( \mathcal{H}_0 \) in \( \mathcal{H}_\beta \) is involutive.

When we restrict this representation to the group ring of \( \mathbb{Q} / \mathbb{Z} \) we obtain (as \( \delta = 1 \) on double classes in \( \mathbb{Q} / \mathbb{Z} \)) that the corresponding representation of \( \mathbb{Q} / \mathbb{Z} \) is given by

\[
\rho(\gamma) \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \varepsilon_0 = \begin{bmatrix} 1 & b + \gamma \\ 0 & a \end{bmatrix} \varepsilon_0.
\]

Thus \( \rho(\gamma) \) is diagonal in the decomposition \( \mathcal{H}_\beta = \mathcal{H}_0 \oplus \mathcal{H}_{\beta,k} \) and its restriction to \( \mathcal{H}_{\beta,k} \) is unitary since the left action of \( N \) on \( \mathcal{H}_{\beta,k} \) is unitary.

When we restrict the representation \( \rho \) of \( \mathcal{H}_0 \) to the involutive subalgebra generated by the \( \mu_n \), its unitarity follows from Lemma 30 b). The explicit computation of the isometry \( U_p = \rho(\mu_p^*) \) in \( \mathcal{H}_\beta \) associated to \( \mu_p^* \), \( p \) a prime, is the following. We let as above \( t_p \) be the hyperbolic translation of one unit of length towards the point at \( \infty \) in the tree \( T_p \). We let it act trivially on the other trees. One then obtains

\[
U_p \varepsilon_\alpha = \mu^{\beta/2-1} \sum_{t_p(\alpha')=\alpha} \varepsilon_{\alpha'}.
\]

One can check directly using Lemma 31 that \( U_p \) is indeed an isometry.

We have thus shown that \( \rho \) is an involutive representation of \( \mathcal{H}_0 \) in \( \mathcal{H}_\beta \) and by Proposition 19 it extends to a representation of \( C_\mathbb{Q}^0 \) in \( \mathcal{H}_\beta \). This proves b). By construction \( \varepsilon_0 \) is cyclic for \( P_\mathbb{Q}^+ \). Since the above action of \( \mathcal{H}_0 \) (and \( C_\mathbb{Q}^0 \)) commutes with \( P_\mathbb{Q}^+ \) the vector \( \varepsilon_0 \) is separating for \( C_\mathbb{Q}^0 \).

The proof of the last statement of c) is the same as in the case \( \beta = 1 \) (cf. Lemma 17).

The proof of d) is the same as that of Lemma 16.

Combining Corollary 29 with Proposition 32 d) we obtain that for \( \beta \in [0,1] \) the state on \( C_\mathbb{Q} \) given by

\[
\varphi(x) = (\rho(x), \varepsilon_0, \varepsilon_0) \quad \forall x \in C_\mathbb{Q}
\]

is the unique KMS\(_{\beta} \) state. This together with 32 a) completes the proof of Theorem 5.
Remarks 33. a) Let on $\mathcal{H}_\beta$, $\Delta$ be the operator of multiplication by $k^\beta$ on $\mathcal{H}_{\beta,k}$. There exists a unique weight $\psi_\beta$ (up to a multiplicative constant) on $(P^+_Q)^\prime\prime$ with modular automorphism group given by

$$\sigma^{\psi_\beta}_t(\cdot) = \Delta^{it} \cdot \Delta^{-it}. \quad (34)$$

For each $m \in \mathbb{N}^*$, the vector $\varepsilon_m = \begin{bmatrix} 1 & 0 \\ 0 & m^{-1} \end{bmatrix} \varepsilon_e$ is separating for $C''_Q$ but not cyclic. Its cyclic span $C'_Q \varepsilon_n$ defines a projection $E_m \in (P_Q)^{\prime\prime}$ which belongs to the centralizer of $\psi_\beta$ and on which $\psi_\beta$ is finite. On the subspace $E_m$ the modular operator of the pair $(P''_Q/E_m, C''_Q$ and vector $\varepsilon_m)$ is the restriction of $\Delta$.

The subspace $E_m$ is the space of fixed points of the subgroup $\begin{bmatrix} 1 & mI \\ 0 & 1 \end{bmatrix}$ of $P^+_Q$. These subspaces form a nested family (i.e. $E_m \subset E_{m'}$ if $m$ divides $m'$) which is total in $\mathcal{H}_\beta$.

b) We used throughout this paper the pairs of groups $P^+_Q$, $P^+_Z$ instead of the pair $P_Q$, $P_Z \subset P_Q$. The relation between the corresponding $C^*$-dynamical systems $C^*(P^+_Q, P^+_Z)$, $\sigma_t$ and $C^*(P_Q, P_Z)$, $\sigma_t$ is quite simple. Indeed the latter is just the fixed point $C^*$-algebra of the involution $\alpha$ of the former given by the complex conjugation $\varepsilon \to \overline{\varepsilon}$ viewed as an element of $W = \text{Gal}(Q^{\text{cycl}})$. This is easy to check because the double class $X$ modulo $P_Z$ of an element $g \in P_Q$, $g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$ is the same as the double class modulo $P_Z$ of $g \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$, $\varepsilon = \text{Sign}(a)$ which allows to assume $a > 0$. This shows that $P_Z$ biinvariant functions on $P_Q$ yield all the $P^+_Z$ invariant functions on $P^+_Q$ which are invariant under the involution:

$$g \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} g \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad g \in P^+_Q. \quad (35)$$

One thus obtains the $\sigma_t$ equivariant equality

$$C^*(P_Q, P_Z) = C^*(P^+_Q, P^+_Z)^{\alpha} \quad (36)$$

and one can rewrite the main theorem of this paper in terms of $C^*(P_Q, P_Z)$.

c) In this paper we have ignored the place at infinity in our treatment of KMS$_\beta$ states or weights and in the construction of $C^*_Q$ from the action on the product of trees (section 2). We obtained, for the finite places, the action of $P_K$ on the tree of $SL(2, K)$ as well as the relevant inner product on functions on the tree (section 5) from the understanding of KMS$_\beta$ weights on $C^*(P_K)$ and of the reduction by the projection $e \in C^*(P_K)$, $e = 1_{P_K}$. 

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At the infinite place the $C^*$-dynamical system at hand is $C^*(P_\mathbb{R})$, $\sigma$, where $P_\mathbb{R}$ is the group of matrices

$$(37) \quad \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} ; \quad b \in \mathbb{R}, \quad a \in \mathbb{R}^*$$

and where $\sigma$ is given by the module $\delta$ of $P_\mathbb{R}$,

$$(38) \quad \sigma_t(f)(g) = \delta(g)^{-it} f(g) \quad \forall f \in L^1(P_\mathbb{R}), \quad t \in \mathbb{R}.$$ 

The $C^*$-algebra of $P_\mathbb{R}$ is, using the identification of $\mathbb{R}$ with its Pontrjagin dual group, given by

$$(39) \quad C^*(P_\mathbb{R}) = C_0(\mathbb{R}) \rtimes \mathbb{R}^*$$

where the action of $\mathbb{R}^*$ is by homotheties. This action has two orbits, $\mathbb{R}/\{0\}$ and $\{0\}$ and to the equivariant exact sequence of $C^*$-algebras:

$$(40) \quad 0 \to C_0(\mathbb{R}/\{0\}) \to C_0(\mathbb{R}) \to \mathbb{C} \to 0$$

corresponds the exact sequence of crossed products:

$$(41) \quad 0 \to K \to C^*(P_\mathbb{R}) \to C^*(\mathbb{R}^*) \to 0$$

similar to the exact sequence of the Toeplitz $C^*$-algebra. Here the two sided ideal $K$ is the elementary $C^*$-algebra of compact operators. The representation theory of $C^*(P_\mathbb{R})$ immediately follows from (7) and besides the characters of $C^*(\mathbb{R}^*)$ which yield one dimensional representations of $C^*(P_\mathbb{R})$ one has a unique infinite dimensional irreducible representation $\pi$. This representation can be described as follows. One lets $\mathcal{H} = L^2(\mathbb{R})$ with

$$(42) \quad (\pi(g)\xi)(t) = |a|^{1/2} \xi(at-b) \quad \forall t \in \mathbb{R}, \quad \xi \in L^2(\mathbb{R})$$

where $g = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$ as above, belongs to $P_\mathbb{R}$.

For each $\beta \in ]0, \infty[$ there exists up to normalization a unique KMS$_\beta$ weight on $C^*(P_\mathbb{R})$, it is given by

$$(43) \quad \varphi_\beta(f) = \text{Trace}(\pi(f) \Delta^{-\beta/2}) \quad \forall f \in C^*(P_\mathbb{R})^+$$

where $\Delta = -\frac{d^2}{d\tau^2}$ is the Laplacian, an unbounded selfadjoint operator in $L^2(\mathbb{R})$.

By construction $\varphi_\beta$ is a type $I_\infty$ factor weight which is a dominant weight $([C], [C-T])$ on the corresponding type $I_\infty$ factor. It follows that for any factor $M$ with weight $\psi$ the
centralizer of $\psi \otimes \varphi_\beta$ is the associated semifinite von Neumann algebra of the continuous decomposition of $M$, i.e. the crossed product by the modular automorphism group $\sigma_\psi$.

\[(M \otimes I_\infty)_{\psi \otimes \varphi_\beta} = M \rtimes_\sigma \mathbb{R} .\]

In our case, with $\beta \in ]0,1[$, it is natural to take for $M$ the commutant of $P^+_\mathbb{Q}$ acting in the Hilbert space $\mathcal{H}_\beta$ (cf. Proposition 32). To obtain the crossed product (44) it is then natural to use at the infinity place the Hilbert space $\mathcal{H}_\beta^\infty$ of the GNS construction of the weight $\varphi_\beta$ on $C^*(P_\mathbb{R})$. In the tensor product $\mathcal{H}_\beta \otimes \mathcal{H}_\beta^\infty$ one has a natural product action of the group $P_A$ over the adeles $A = A \times \mathbb{R}$. The crossed product (44) is then contained in the commutant of $P^+_\mathbb{Q}$ which is a discrete subgroup of $P_A$.

References


