Universal Charge Distribution
and Nonlocal Quantum Electrodynamics

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ABSTRACT

It is assumed that electric charge of point-like objects carries non-locality defined by fundamental length and has distribution over space. From physical requirements a unique form of charge distribution is found that in turn gives rise to a change in the Coulomb law at short distances and leads to modification of the photon propagator. A nonlocal gauge transformation connected with extended charge is presented, which allows us to construct gauge-invariant nonlocal quantum electrodynamics free from ultraviolet divergences.

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1. Introduction

Physical and space-time (or geometric) understandings of the origin of electric charge is an unsolved problem of modern physics. There are two hypothetical assumptions about very nature of electrical charge. First, from the physical point of view, the need in extended, fundamental objects, as opposed to point-like constituents, for explanation of nonlocality^[1] in quantum mechanics and for construction of unified field theory^[2] of all interactions including gravitation, allows us to connect electric charge (distribution of which) with fundamental length^[3] (a size of extended objects) and to understand it as topological defects (modes) on the string world-sheet (See^[4], for discussion). Second, in a deeper level, where the quantum fluctuations in the geometry of space are so great at small distances that even the topology fluctuates, makes "wormholes" and traps lines of force, as it has been supposed by A.Wheeler (See^[5]), the electrical and nuclear charges give evidence for the presence of a submicroscale structure of space-time resembling a foam-like structure which is on the whole homogeneous. Thus, it seems, the fluctuations in the geometry make the topological defects (or a multiple connected topology) which provide a natural description for the electric charge as electric lines of force trapped in the topology of a multiple connected space. Our approach belongs to^{the} first direction and is modest, where we will attempt to find^a unique form of the electric charge distribution associated to the universal fundamental length and to construct nonlocal gauge invariant quantum electrodynamics. In the given scheme, the Coulomb law is changed at

short distances and the photon propagator turns out to be modified and theory becomes nonlocal and finite in both classical and quantum levels. The constructive subject of our scheme is still in a somewhat primitive state by comparison with sophistication of our understanding of conventional "point"field theory. Results are arrived by a variety of techniques which we shall survey in this paper. Our exposition is following. In Section 2 we will start with Poisson's equation for a point-like charge as a basis of the construction of the local quantum field theory in which ultraviolet divergences are presented. In Section 3 we will introduce the fundamental length into physics via charge distribution over space and the infinitely sharp delta-functions, involved in the definition of the point-like charge distribution, are smeared out over the extension of fundamental objects. Poisson's equation for extended charge is obtained and a unique form of the charge density is found here.

The Coulomb law and the photon propagator are modified in consistent with the local theory of point-like elementary constituents. Further, in order to construct nonlocal quantum electrodynamics (QED) we shall introduce in Section 4 a nonlocal gauge transformation induced by extended electric charge distribution. The Efimov^[6] nonlocal S-matrix for QED is obtained in Section 5. Sections 6 and 7 deals with the regularization procedure and the gauge invariance for the S-matrix. Finally, in Sections 8 and 9, we will study simplest primitive Feynman diagrams and obtain restriction on the value of the fundamental length in the nonlocal QED. In Appendix A we will expound some mathematical computations.

2. A Point-like Charge and The Coulomb Law

Let us consider the point-like charge e with the distribution $\rho(\vec{r}) = \delta(\vec{r})$ in space. Here $\delta(\vec{r})$ is the Dirac δ -function. Then the Poisson equation for its potential is

$$\Delta\varphi_c = -e\rho(\vec{r}) \quad (2.1)$$

and the solution of which is the Coulomb Law

$$\varphi(\vec{r}) = \frac{e}{4\pi} \int \frac{d\vec{r}' \rho(\vec{r} - \vec{r}')}{|\vec{r}'|} = \frac{e}{4\pi} \int \frac{d\vec{r}' \delta(\vec{r} - \vec{r}')}{|\vec{r}'|} = \frac{e}{4\pi} \frac{1}{r} \quad (2.2)$$

It is well known that Eq. (2.1) and potential (2.2) are a priori of classical and quantum theories of the electromagnetic "point" interaction. In both them there exist divergences, for example, a self-energy of the point-like classical charge

$$\begin{aligned} w &= \frac{e}{2} \int d^3r \rho(\vec{r}) \varphi_c(\vec{r}) = \frac{1}{2} \int d^3r (\text{grad } \varphi_c(\vec{r}))^2 = \\ &= \frac{1}{2} \int d^3r E^2 = \frac{e^2}{4\pi} \int_0^\infty \frac{dr}{r^2} \end{aligned} \quad (2.3)$$

goes to infinity and in QED the local Green function of the photon

$$\Delta_{\mu\nu}(x) = -g_{\mu\nu} \Delta(x) \quad (2.4)$$

where

$$\Delta(x) = \frac{1}{(2\pi)^4 i} \int d^4 p e^{i p x} \tilde{\Delta}(p^2) \quad (2. 5)$$

$\tilde{\Delta}(p^2) = (-p^2 - i \epsilon)^{-1}$, $p^2 = p_0^2 - \vec{p}^2$, has singularity at the point $x=0$. In the

Euclidean metric the expression (2. 4) takes the form

$$\Delta_E(x) = \frac{1}{4\pi^2} \frac{1}{x_E^2}; \quad x_E^2 = x_4^2 + \vec{x}^2 = -x^2, \quad x^2 = x_0^2 - \vec{x}^2 \quad (2. 6)$$

It should be noted that in the static limit, Fourier transform of (2. 5) is related to the Coulomb potential (2. 2) by

$$\varphi_c = \frac{e}{4\pi} \frac{1}{r} = \frac{e}{(2\pi)^3} \int d^3 p e^{i \vec{p} \cdot \vec{r}} \frac{1}{\frac{\vec{p}^2}{p}} \quad (2. 7)$$

$$\text{or} \quad \frac{1}{\frac{\vec{p}^2}{p}} = \frac{1}{e} \int d^3 r e^{-i \vec{p} \cdot \vec{r}} \varphi_c(\vec{r}) \quad (2. 8)$$

The latter should be understood as an improper integral (See Appendix A). As seen above, interrelationships (2. 6) and (2. 7) mean that the concept of the point-like charge [its singular potential (2.2)] gives rise to appearance of singularities in the local quantum field theory and vice versa.

3. A Charge Distribution and The Fundamental Length

Recently, the majority of physicists believe that in nature there exists some new fundamental constant of the dimension of length, together with such constants as the velocity of light c and the Planck constant $\hbar$. One assumes that this new universal constant leads to a principle change in our concepts of physical world, and in particular, the concept of space-time and locality (causality). Introducing such a constant into physics is needed to the understanding of the nonlocal nature of quantum physics^[3] and to description of extended, fundamental objects such as strings, superstrings^[2] as a basis of the unified theories of all interactions. Here we attempt to define this fundamental length by using physical characteristics of the electric charge. It may be that the very existence of a fundamental length is caused by the notion of the electric charge distribution over space. To realize this idea, we should smear out the infinitely sharp delta-function, involved in the definition of idealized concept of the point-like charge, by the following change

$$e \delta(\vec{r}) \Rightarrow e \rho_1(\vec{r}) \quad (3. 1)$$

where first consistent scheme is

$$\lim_{l \rightarrow 0} \rho_1(r) = \delta(\vec{r}) \quad (3. 2)$$

Here the distribution $\rho_1(\vec{r})$ describes the extended electric charge due to the existence of the fundamental length. We assume that the

charge distribution $\rho_1(\vec{r})$ has an universal characteristic and is independent on concrete properties of elementary constituents (say, electron, quarks and etc.) which carry the electrical charge.

The change (3.1) leads to the "nonlocal" Poisson equation

$$\Delta \varphi_1(\vec{r}) = -e \rho_1(\vec{r}) \quad (3. 3)$$

and its solution is

$$\varphi_1(\vec{r}) = \frac{e}{4\pi} \int \frac{d\vec{r}' \rho_1(\vec{r} - \vec{r}')}{|\vec{r}'|} \quad (3. 4)$$

This is a modified form of the Coulomb law at short distances.

It is obviously that in accordance with the correspondance principle, self-energy (2. 3) and nonlocal photon propagator (2. 4) are finite for $\rho_1(\vec{r})$. At this ,the nonlocal Coulomb potential is related to a nonlocal photon propagator by

$$\varphi_1(\vec{r}) = \frac{e}{(2\pi)^3} \int d^3p e^{i\vec{p} \cdot \vec{r}} D(\vec{p}^2) \quad (3. 5)$$

and
$$D_1(\vec{p}^2) = \frac{1}{e} \int d^3r e^{-i\vec{p} \cdot \vec{r}} \varphi_1(\vec{r}) \quad (3. 6)$$

in the static limit.

Now we attempt to find a unique form for the distribution $\rho_1(\vec{r})$. For this purpose, we consider the Green function of the photon field:

$$D_{\mu\nu}(x) = \frac{i}{(2\pi)^4} g_{\mu\nu} \int d^4p e^{ipx} D_1(p^2) \quad (3. 7)$$

where its Fourier transform $D_1(p^2)$ in the static limit is given by formula (3. 6):

$$D_1(\vec{p}^2) = \frac{1}{e} \int d^3r e^{-i\vec{p}\vec{r}} \frac{e}{4\pi} \int d\vec{r}' \frac{\rho_1(\vec{r} - \vec{r}')}{|\vec{r}'|} \quad (3. 8)$$

in accordance with (3. 4). Here

$$\rho_1(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d^3q e^{-i\vec{q}(\vec{r} - \vec{r}')} \tilde{\rho}_1(\vec{q}) \quad (3. 9)$$

$$\text{and} \quad \frac{1}{|\vec{r}'|} = \frac{1}{2\pi^2} \int d^3p e^{i\vec{p}\vec{r}'} \frac{1}{p^2} \quad (3. 10)$$

where $\tilde{\rho}_1(\vec{q})$ is the Fourier transform of the charge density $\rho_1(\vec{r})$. It is easy to calculate that

$$D_1(\vec{p}^2) = \frac{1}{e} \frac{1}{\vec{p}^2} \tilde{\rho}_1(\vec{p}) \quad (3. 11)$$

On the other hand, propagator (3. 7) is defined by using some nonlocal photon field $A_\mu^1(x)$:

$$D_{\mu\nu}(x - y) = \langle 0 | T \left\{ A_\mu^1(x) A_\nu^1(y) \right\} | 0 \rangle \quad (3. 12)$$

where generalized field $A_\mu^1(x)$ should be determined from the interaction vertex:

$$\mathcal{L}_{in}(x) = e A_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x) \Rightarrow e \rho(x) A'_\mu(x) \bar{\psi}(x) \gamma^\mu \psi(x) \quad (3. 13)$$

A modified form of the interaction vertex between electromagnetic and charged fields arises due to the charge distribution $\rho_1(\vec{x})$ and also (at the same time) a modification of the Coulomb law, which gives rise to a change of electromagnetic field

$$A_\mu(x) \Rightarrow A'_\mu(x) \Rightarrow \rho_1(x) A_\mu(x)$$

The factorization form (3. 11) allows us to introduce smeared out electromagnetic field $A_\mu^1(x)$:

$$A_\mu^1(x) = \rho_1^2(x) \otimes A_\mu(x) = \int d^4 y \rho_1^2(x - y) A_\mu(y)$$

and the generalized form of the interaction vertex

$$\mathcal{L}_{in}(x) = A_\mu^1(x) j_\mu(x) \quad (3. 14)$$

(their strict deduction will be given below), where

$$A_\mu^1(x) = \int d^4 y \rho_1^2(x - y) A_\mu(y) \quad (3. 15)$$

and $j_\mu(x) = e \bar{\psi}(x) \gamma^\mu \psi(x)$ is the local current of the Fermion field

$\psi(x)$. Quantity $\rho(x)$ in (3. 12) and (3. 14) is the generalized form of the charge density in the four-dimensional case. Generally speaking, $\rho_1(x)$ is a generalized function in the Minkowski space. In the Euclidean metrix, $\rho(x_E)$ possesses the probability measure satisfying condition

$$\int d^4x_E \rho_1^2(x_E) = 1 \quad (3. 16)$$

With the choice (3. 15), the photon propagator (3. 12) turns out to be

$$D_{\mu\nu}(x-y) = \frac{i}{(2\pi)^4} g_{\mu\nu} \int d^4p e^{ip(x-y)} \frac{[\tilde{\rho}_1^2(p)]^2}{-p^2 - i\epsilon} \quad (3. 17)$$

where $p^2 = p_0^2 - \vec{p}^2$, $\tilde{\rho}_1^2(p)$ is the Fourier transform of the generalized charge density $\rho_1^2(x)$.

Thus, the expected charge distribution $\rho_1(\vec{r})$ should obey conditions (3. 2)- (3. 6) and the equality

$$\tilde{\rho}_1(\vec{p}) = \left[\tilde{\rho}_1^2(\vec{p}, p_0) \right]^2 \Big|_{p_0 \rightarrow 0} \quad (3. 18)$$

i. e. in the static limit.

Theorem 1. The nonlocal charge distribution of the Gaussian form

$$\rho_1(\vec{r}) = \frac{1}{\pi^{3/2} l^3} \exp \left\{ -\frac{\vec{r}^2}{l^2} \right\} \quad (3. 19)$$

and its Euclidean extension

$$\rho_1^2(x_E) = \frac{4}{\pi^2 l^4} \exp \left\{ -\frac{2}{l^2} (\vec{x}^2 + x_4^2) \right\} \quad (3. 20)$$

satisfy all above conditions.

Proof is verified by direct calculations.

1. First of all, expressions (3. 19) and (3. 20) satisfy the normalization conditions

$$\int d^3r \rho_1(\vec{r}) = \frac{4\pi}{\pi^{3/2}l^3} \int_0^\infty dr r^2 e^{-r^2/l^2} = \frac{4}{\sqrt{\pi}l^3} \frac{1}{2(2\frac{1}{l^2})} \sqrt{\pi l^2} = 1$$

and

$$\int d^4x_E \rho_1^2(x_E) = \frac{4}{\pi^2 l^4} \pi^2 \int_0^\infty du u e^{-\left(\frac{2u}{l^2}\right)} = \frac{4}{l^4} \frac{1}{\left(\frac{2}{l^2}\right)^2} \Gamma(2) = 1$$

2. Their Fourier transforms are

$$\begin{aligned} \tilde{\rho}_1(\vec{p}) &= \int d^3r e^{-i\vec{p}\cdot\vec{r}} \rho_1(\vec{r}) = \frac{1}{\pi^{3/2}l^3} \frac{4\pi}{p} \int_0^\infty dr r \sin(pr) e^{-r^2/l^2} = \\ &= \frac{4}{\sqrt{\pi}pl^3} \frac{p\sqrt{\pi}}{4\left(\frac{1}{l^3}\right)} \exp\left[-\frac{\vec{p}^2 l^2}{4}\right] = \exp\left[-\frac{\vec{p}^2 l^2}{4}\right] \end{aligned} \quad (3. 21)$$

$$\begin{aligned} \tilde{\rho}_1^2(p_E) &= \int d^4x_E e^{-ip_E x_E} \rho_1^2(x_E) = \frac{4}{\pi^2 l^4} \frac{4\pi^2}{p} \int_0^\infty dx x^2 J_1(px) e^{-\frac{2x^2}{l^2}} = \\ &= \frac{16}{pl^4} \frac{p}{\left(\frac{4}{l^2}\right)^2} \exp\left[-\frac{p^2}{4\left(\frac{2}{l^2}\right)}\right] = \exp\left[-\frac{p_E^2 l^2}{8}\right] \end{aligned} \quad (3. 22)$$

or in the pseudo-Euclidean metric $p_E^2 = -p_{pE}^2$

$$\tilde{\rho}_1^2(p) = \exp\left[\frac{p^2 l^2}{8}\right], \quad p^2 = p_0^2 - \vec{p}^2 \quad (3. 23)$$

From this, it is easily seen that

$$\rho_1^2(x) = \frac{1}{(2\pi)^4} \int d^4p e^{-ipx} \tilde{\rho}_1^2(p) = \exp \left\{ + \frac{a l^2}{8} \right\} \delta^{(4)}(x) \quad (3. 24)$$

where

$$a = \frac{\partial^2}{\partial \vec{x}^2} - \frac{\partial^2}{\partial x_0^2}$$

Thus, the extended form of the charge distribution (3. 24) in the Minkowski space is just the generalized distribution investigated in [6]. From the explicit formulas (3. 21) and (3. 23) one can see that the equality (3. 18) holds automatically.

3. The point-like charge and its local theory are obtained as a consequence of (3. 2) with distribution (3. 19), where

$$\lim_{l \rightarrow 0} \frac{1}{\pi^{3/2} l^3} \exp \left(- \frac{\vec{r}^2}{l^2} \right) = \delta(\vec{r})$$

4. The modified Coulomb law (3. 4) with (3. 19) is

$$\varphi_l(r) = \frac{e}{4\pi r} \phi(r/e) \quad (3. 25)$$

where $\phi(x)$ is the probability integral

$$\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}$$

5. By direct calculations (see Appendix A) it is easy to show that the Poisson equation (3. 3) with (3. 19) and (3. 25) is valid identically.

6. Direct and inverse Fourier transforms (3. 5) and (3. 6) with the charge density (3. 19) give interrelationship between the propagator of the photon (3. 11) in the static limit and the changed Coulomb law (3. 25).

7. In our scheme, the self-energy of the extended charge is finite

$$w_1 = \frac{e}{2} \int d^3r \rho_1(\vec{r}) \varphi_1(\vec{r}) = \frac{\alpha}{\sqrt{2\pi}} , \quad \alpha = e^2 / 4\pi$$

and the photon propagator $D_{\mu\nu}(x) = -g_{\mu\nu} D(x)$,

$$D(x) = \frac{1}{(2\pi)^4 i} \int d^4p e^{ipx} \tilde{D}(P^2)$$

has no singularities at the point $x=0$

$$D(0) = \frac{1}{(2\pi)^4} 2\pi^2 \int_0^\infty du \frac{u}{2} \frac{e^{-u^2/4}}{u} = \frac{1}{16\pi^2} \int_0^\infty du e^{-u^2/4} = \frac{1}{4\pi^2} \frac{1}{1^2}$$

Thus, we have obtained all necessary formulas which are sufficient to construct nonlocal gauge invariant quantum electrodynamics.

4. A Nonlocal Gauge Transformation and

The Nonlocal Electromagnetic Interaction

It is well known that interaction of charged fields $\varphi_j(x)$ with electromagnetic field $A_\mu(x)$ is defined by the requirement of the gauge invariance. This means that physical content of the description

of the electromagnetic field by using potentials $A_\mu(x)$ does not change under the gauge transformation:

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu f(x) \quad (4. 1)$$

since the electromagnetic tensor of the field

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x)$$

is invariant under the gauge transformation (4. 1). It is usually assumed that the interaction of charged fields $\phi_j(x)$ with the electromagnetic field $A_\mu(x)$ is invariant with respect to the group of the gauge transformations:

$$\begin{aligned} A_\mu(x) &\longrightarrow A_\mu(x) + \partial_\mu f(x) \\ \phi_j(x) &\longrightarrow \phi_j(x) \exp\{i q_j f(x)\} \\ \phi_j^*(x) &\longrightarrow \phi_j^*(x) \exp\{-i q_j f(x)\} \end{aligned} \quad (4. 2)$$

with an arbitrary function $f(x)$. Here q_j means a charge of the fields $\phi_j(x)$. Invariance of the total Lagrangian $L(\phi_j, \phi_j^*, A_\mu)$ with respect to the gauge group (4. 2) leads to the charge conservation:

$$\partial_\mu j_\mu(x) = 0 \quad (4. 3)$$

where

$$j_\mu(x) = i \sum_j q_j \left\{ \frac{\delta L}{\delta(\partial_\mu \phi_j^*(x))} \phi_j^*(x) - \frac{\delta L}{\delta(\partial_\mu \phi_j(x))} \phi_j(x) \right\} \quad (4. 4)$$

It should be noted that gauge transformation (4. 2) means locality of the interaction of the electromagnetic field with charged fields. A unique electromagnetic characteristic of the field $\phi_j(x)$ is its

charge q_j which enters into the transformation (4. 2). The explicit form of the interaction Lagrangian of the electromagnetic field with charged fields is defined by using the principle of minimality which asserts that one gets the change

$$\begin{aligned}\partial_\mu \phi_j(x) &\Rightarrow \left\{ \partial_\mu - i q_j A_\mu(x) \right\} \phi_j(x) \\ \partial_\mu \phi_j^*(x) &\Rightarrow \left\{ \partial_\mu + i q_j A_\mu(x) \right\} \phi_j^*(x)\end{aligned}\quad (4. 5)$$

under the action of the operator ∂_μ on the fields $\phi_j(x)$ and $\phi_j^*(x)$.

This is generally accepted procedure which leads to all difficulties encountered in the local quantum field theory.

In order to generalize the theory in the case of the nonlocal electromagnetic interaction due to extended electrical charge distribution we consider, first, the stationary picture when fields $\phi_j(x)$ do not depend on the time variable. At this, instead of the gauge group transformation (4. 2) the following nonlocal transformation are assumed:

$$\begin{aligned}A_0(\vec{x}) &\Rightarrow A_0(\vec{x}) \\ \vec{A}(\vec{x}) &\Rightarrow \vec{A}(\vec{x}) + \vec{\partial} f(\vec{x}) \\ \phi_j(\vec{x}) &\Rightarrow \phi_j(\vec{x}) \exp \left\{ i q_j \int d\vec{y} \rho_l^2(\vec{x} - \vec{y}) f(\vec{y}) \right\} \\ \phi_j^*(\vec{x}) &\Rightarrow \phi_j^*(\vec{x}) \exp \left\{ -i q_j \int d\vec{y} \rho_l^2(\vec{x} - \vec{y}) f(\vec{y}) \right\}\end{aligned}\quad (4. 6)$$

where $\rho_l(\vec{x})$ is the charge distribution defined in (3. 19). In this case, the current vector (4. 4) takes the form

$$\vec{J}_l(\vec{x}) = \sum_j \int d\vec{y} \rho_l^2(\vec{x} - \vec{y}) \vec{j}_j(\vec{y}) \quad (4. 7)$$

Here

$$\vec{j}_j(\vec{x}) = iq_j \left\{ \frac{\delta L}{\delta(\vec{\partial} \vec{\phi}_j(\vec{x}))} \vec{\phi}_j(\vec{x}) - \frac{\delta L}{\delta(\vec{\partial} \vec{\phi}_j(\vec{x}))} \vec{\phi}_j(\vec{x}) \right\}$$

is the local current, and therefore the electromagnetic interaction of the type of $\vec{A}(\vec{x}) \vec{J}_l(\vec{x})$ means that the vector potential $\vec{A}(\vec{x})$ is associated with the local current $\vec{j}_j(\vec{x})$ of the j-th charged particle through some spatial formfactor $\rho_l^2(\vec{x} - \vec{y})$; that is just the charge distribution with the fundamental length l.

For the general case, when fields $\phi_j(x)$ and $A_\mu(x)$ depend on the space-time variables $x^\mu = x^0, \vec{x}$, it is natural to use the following nonlocal gauge transformations

$$A_\mu(x) \Rightarrow A_\mu(x) + \partial_\mu f(x)$$

$$\phi_j(x) \Rightarrow \phi_j(x) \exp \left\{ i q_j \int d^4 y \rho_l^2(x - y) f(y) \right\}$$

$$\varphi_j^*(x) \Rightarrow \varphi_j^*(x) \exp \left\{ -i q_j \int d^4 y \rho_l^2(x-y) f(y) \right\} \quad (4. 8)$$

where $\rho_l^2(x)$ is the generalized charge distribution (3. 24). These transformations grant the conservation of an extended electric current:

$$J_\mu^l(x) = i \sum_j q_j \int dy \rho_l^2(x-y) \left\{ \frac{\delta L}{\delta(\partial_\mu \varphi_j^*(y))} \varphi_j^*(y) - \frac{\delta L}{\delta(\partial_\mu \varphi_j(y))} \varphi_j(y) \right\}$$

The local variant is obtained in the limit $l \Rightarrow 0$.

or $\rho_l^2(x-y) \mid_{l \Rightarrow 0} = \delta(x-y)$

In the nonlocal case, the usual procedure of the change (4. 5) takes the form

$$\begin{aligned} \partial_\mu \varphi_j(x) &\Rightarrow \left\{ \partial_\mu - i q_j \int dy \rho_l^2(x-y) A_\mu(y) \right\} \varphi_j(x) \\ \partial_\mu \varphi_j^*(x) &\Rightarrow \left\{ \partial_\mu + i q_j \int dy \rho_l^2(x-y) A_\mu(y) \right\} \varphi_j^*(x) \end{aligned} \quad (4. 9)$$

We now turn to the construction of the nonlocal QED with the fundamental universal charge density (3. 19) and the smeared out photon field (3. 15).

5. The Efimov Nonlocal S- Matrix for the Nonlocal QED

As seen above, in the case of the interaction between electromagnetic and the Dirac electron-positron fields, the total Lagrangian of classic fields has the form

$$L(x) = L_A^0(x) + L_e^0(x) + L_{in}(x) \quad (5. 1)$$

where

$$L_A^0(x) = - \frac{1}{2} \partial_\nu A_\mu(x) \partial_\nu A_\mu(x) \quad (5. 2)$$

$$L_e^0(x) = \bar{\psi}(x) (i \hat{\partial} - m) \psi(x) \quad \hat{\partial} = \gamma^\mu \partial_\mu \quad (5. 3)$$

$$L_{in}(x) = e \bar{\psi}(x) \hat{A}_\mu^I(x) \psi(x) \quad (5. 4)$$

Here

$$A_\mu^I(x) = \int dy \rho_I^2(x - y) A_\mu(y) \quad (5. 5)$$

e is the electron charge and $\rho_I(x)$ is its distribution.

The Lagrangian of the free electromagnetic field $L_A^0(x)$ is written in the form in accordance with the Lorentz condition $\partial_\mu A_\mu(x) = 0$.

For QED the gauge group (4. 8) acquires the form

$$A_\mu(x) \Rightarrow A_\mu(x) + \partial_\mu f(x)$$

$$\psi(x) \Rightarrow \psi(x) \exp \left[ie \int dy \rho_I^2(x - y) f(y) \right] \quad (5. 6)$$

$$\bar{\psi}(x) \Rightarrow \bar{\psi}(x) \exp \left\{ -ie \int dy \rho_1^2(x-y) f(y) \right\}$$

Formally, the S-matrix can be written in the form of the T-products. ^[6]:

$$S = T_{\Lambda}^{\delta} \exp \left\{ ie \int dx \bar{\psi}(x) \hat{A}^1(x) \psi(x) \right\} \quad (5.7)$$

where the symbol T_{Λ}^{δ} means the so-called Wick T-product or T*-operation (see ^[7], for example) and the upper and lower cases δ, Λ correspond to some intermediate regularization procedures (defined below) which make finite all matrix elements of the perturbation theory, and δ, Λ are parameters of the regularization. The limits $\Lambda \Rightarrow \infty$ and $\delta \Rightarrow 0$ mean a removal of the regularizations.

In order to construct the perturbation series for the S-matrix (5.7) by prescription of the usual local theory, it is necessary to change (in the Feynman diagrams):

$$\begin{aligned} \Delta_{\mu\nu}(x-y) &\Rightarrow D_{\mu\nu}(x-y) = g_{\mu\nu} D(x-y) = \langle 0 | T \left[A_{\mu}^1(x) A_{\nu}^1(y) \right] | 0 \rangle = \\ &= \int dy_1 \int dy_2 \rho_1^2(x-y_1) \rho_1^2(y-y_2) \langle 0 | T \left[A_{\mu}(y_1) A_{\nu}(y_2) \right] | 0 \rangle = \\ &= -g_{\mu\nu} \frac{1}{(2\pi)^4 i} \int dp \frac{\left[\tilde{\rho}_1^2(p^2) \right]^2}{-p^2 - i\epsilon} e^{-ip(x-y)} \end{aligned} \quad (5.8)$$

and to keep the usual local Fermion propagator

$$S(x - y) = \langle 0 | T \{ \psi(x) \bar{\psi}(y) \} | 0 \rangle = \frac{1}{(2\pi)^4 i} \int dp \frac{1}{\hat{p} - m - i\epsilon} e^{-ip(x - y)} \quad (5.9)$$

The calculation of the matrix elements for the charged lepton loops will be undertaken using the following regularization procedure.

6. An Intermediate Regularization

Procedure

The construction of the perturbation series for the S-matrix is possible only within the framework of a regularization procedure. In the nonlocal quantum electrodynamics it is sufficient to regularize the nonlocal photon propagator and closed fermion loops. Thus, for the regularized photon propagator in the momentum space one gets [see ^[6] for detail]

$$\tilde{D}^{\delta}(k^2) = \frac{l^2}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \frac{v(\xi)}{\sin \pi \xi} e^{\delta \xi^2} [l^2(-k^2 - i\epsilon)]^{\xi-1} \quad (6.1)$$

where we have used the Mellin representation

$$V(-p^2 l^2) = \exp \left[\frac{p^2 l^2}{4} \right] = \frac{1}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} d\xi \frac{v(\xi)}{\sin \pi \xi} l^{2\xi} [-p^2]^{\xi} \quad (6.2)$$

$$v(x) = \frac{1}{\Gamma(1+\xi)} 2^{-2\xi}, \quad 0 < \alpha < 1$$

for the Fourier transform of the charge density $[\rho_1^2(x)]^2$.

For regularization of fermion propagators we will use the so-called Pauli-Villars gauge-invariant procedure. It means that causal

fermion propagators are regularized not separately but in closed spinor loops:

$$\sum_j c_j \text{Sp} \left[\gamma S_{M_j}(x_1 - x_2) \gamma S_{M_j}(x_2 - x_3) \dots \right] \quad (6. 3)$$

where the coefficients c_j satisfy following conditions ^[6]

$$\begin{aligned} c_1 + c_2 + c_3 &= -1 \\ c_1 \Lambda_1 + c_2 \Lambda_2 + c_3 \Lambda_3 &= -1 \\ c_1 \ln \Lambda_1 + c_2 \ln \Lambda_2 + c_3 \ln \Lambda_3 &= d \end{aligned} \quad (6. 4)$$

$M_j^2 = m^2 \Lambda_j$, Λ_j ($j = 1, 2, 3$) are large dimensionless parameters

($\Lambda_j = \Lambda + \epsilon_j$, $\Lambda \gg 1$; $0 < \epsilon_j < 1$), and d is some finite number which must be chosen from the normalization condition of the physical charge of the electron.

Thus, the regularization introduced here makes it possible to pass to the Euclidean metric in any diagrams of the perturbation theory.

We recall that the unique formfactor (6. 2) decreases only in the Euclidean direction, i. e., when $p^2 \Rightarrow -\infty$. Therefore we shall investigate the Feynman diagrams in the Euclidean momentum space. At the end of all calculations it is necessary to remove this intermediate regularization, i. e., to pass to the limit $\delta \Rightarrow 0$.

Moreover, spinor loops are finite in the limit $\Lambda \Rightarrow \infty$ in accordance with the conditions (6. 4).

Finally, it should be noted that the limit

$$S = \lim_{\delta \Rightarrow 0} \lim_{\{\Lambda_j \Rightarrow \infty\}} S_{\Lambda}^{\delta} \quad (6. 5)$$

exists and obtained in such a way, that the S-matrix is unitary and satisfies a macrocausality condition investigated by Efimov ^[6].

In the nonlocal quantum electrodynamics the interaction Lagrangian has formally the same form as in the local theory:

$$\begin{aligned} L_{in}(x) = & e : \bar{\psi}(x) \hat{A}^1(x) \psi(x) : + e (Z_1 - 1) : \bar{\psi}(x) \hat{A}^1(x) \psi(x) : - \delta m : \bar{\psi}(x) \psi(x) : \\ & + (Z_2 - 1) : \bar{\psi}(x) (i \hat{\partial} - m) \psi(x) : - (Z_3 - 1) \frac{1}{4} : F_{\mu\nu}(x) F_{\mu\nu}(x) : \end{aligned} \quad (6. 6)$$

where renormalized constants z_1, z_2, z_3 and δm are finite and $z_1 = z_2$ in accordance with the Ward identity.

7. Gauge Invariance of the Nonlocal S-matrix

A requirement of gauge invariance for the nonlocal S-matrix, i. e., invariance with respect to the transformation

$$A_\mu(x) \Rightarrow A_\mu(x) + \partial_\mu f(x) \quad (7. 1)$$

with an arbitrary function $f(x)$, can be written in the form

$$\partial / \partial x_{\mu_1} \dots \partial / \partial x_{\mu_n} \left(\delta^n S / \delta A_{\mu_1}(x_1) \dots \delta A_{\mu_n}(x_n) \right) = 0 \quad (7. 2)$$

where fermion operators of the electron field satisfy the free motion equation. For the proof (7. 2) it is sufficient to consider the case $n=1$, i. e.,

$$\partial_\mu \frac{\delta S}{\delta A_\mu(x)} = 0 \quad (7. 3)$$

Let us carry out a formal proof in terms of the representation

$$S = T \exp \left\{ i \int dx L_{in}(x) \right\} \quad (7. 4)$$

Suppose that representation (7. 4) ensures the construction of the perturbation series with the causal functions (5. 8) and the S-matrix is decomposed in series on normal products of field operators satisfying free motion equations. Thus, making use of (7. 4) one gets

$$\begin{aligned} \frac{\delta S}{\delta A_{\mu}(x)} &= i T \left\{ \left(\frac{\delta}{\delta A_{\mu}(x)} \int dy L_{in}(y) \right) S \right\} = i \int dx' \rho^2(x - x') T \left\{ \left(\frac{\delta}{\delta A_{\mu}(x')} \int dy L_{in}(y) \right) S \right\} \\ &= i \int dx' \rho^2(x - x') T \left\{ e \bar{\psi}(x') \gamma_{\mu} \psi(x') S \right\} \quad (7. 5) \end{aligned}$$

Further, take into account the following equalities

$$\begin{aligned} T \{ \psi(x) S \} &= \psi(x) S + \int dy S(x - y) T \{ i e \hat{A}^1(y) \psi(y) S \}, \\ i \hat{\partial} \{ \psi(x) S \} &= T \left\{ \left[m \psi(x) - i e \hat{A}^1(x) \psi(x) \right] S \right\} \\ i \partial_{\mu} \left\{ \bar{\psi}(x) \gamma_{\mu} S \right\} &= T \left\{ \left[- m \bar{\psi}(x) + i e \bar{\psi}(x) \hat{A}^1(x) \right] S \right\} \quad (7. 6) \end{aligned}$$

These relations are valid if the perturbation theory is constructed in accordance with the Wick theorem with chronological pairing of the fermion operators (5. 9), and the S-matrix depends on field operators satisfying free motion equations.

In terms of relations (7. 6) one gets

$$\partial_\mu \frac{\delta S}{\delta A_\mu(x)} = \int dy \rho^2(x-y) \frac{\partial}{\partial y_\mu} \frac{\delta S}{\delta A_\mu^1(y)} = \int dy \rho^2(x-y) *$$

$$T\left\{\left[m(\bar{\psi}(y)\psi(y)) - ie\bar{\psi}(y)\hat{A}^1(x)\psi(y) - m(\bar{\psi}(y)\psi(y)) + ie\bar{\psi}(y)\hat{A}^1(x)\psi(y)\right]S\right\} = 0 \quad (7.7)$$

So, the S-matrix is gauge invariant within the given formal consideration.

8. The Calculation of the Primitive Feynman Diagrams

Let us calculate the matrix elements for the S-matrix corresponding to the primitive diagrams (Fig. 1) which are divergent in the usual local quantum electrodynamics.

Fig. 1. The primitive Feynman diagrams
in Non. local QED

8. 1. The Diagram of Self-Energy (Figure 1a)

The corresponding term in the S-matrix can be written in the form

$$- i: \bar{\psi}(x) \Sigma_1(x-y) \psi(y) \quad (8.1)$$

where

$$\Sigma_1(x-y) = -ie^2 \gamma_\mu S(x-y) \gamma_\mu D(x-y)$$

Passing to momentum representation and making use of the our regularization procedure δ which allows us to go to the Euclidean metric by the turning $k_0 \Rightarrow \exp\left(i \frac{\pi}{2}\right) k_4$, one gets in the limit $\delta \Rightarrow 0$:

$$\begin{aligned} \widetilde{\sum}_1(p) &= \lim_{\delta \rightarrow 0} (-ie^2) \int dx e^{ipx} \gamma_\mu S(x) \gamma_\mu D^\delta(x) = \\ &= \frac{e^2}{(2\pi)^4} \int dk_E \frac{V(k_E^2 |^2)}{k_E^2} \gamma_\mu^{(E)} \frac{m - \hat{p}_E + \hat{k}_E}{m^2 + (p_E - k_E)^2} \gamma_\mu^{(E)} \end{aligned} \quad (8.2)$$

Here $p_E = (-i p_0, \vec{p})$, $\gamma^{(E)} = (-i \gamma_0, \vec{\gamma})$, $k_E = (k_4, \vec{k})$

so that $p_E k_E = p_4 k_4 + \vec{p} \vec{k} = -i p_0 k_4 + \vec{p} \vec{k}$

$$\begin{aligned} \hat{p}_E &= (p_E \gamma^{(E)}) = p_4 \gamma_4 + \vec{p} \vec{\gamma} = -p_0 \gamma_0 + \vec{p} \vec{\gamma} = -(p\gamma) = -\hat{p} \\ \hat{k}_E &= k_4 \gamma_4 + \vec{k} \vec{\gamma} = -i \gamma_0 k_4 + \vec{k} \vec{\gamma} \end{aligned} \quad (8.3)$$

$$\gamma_\mu^{(E)} \gamma_\nu^{(E)} + \gamma_\nu^{(E)} \gamma_\mu^{(E)} = -2 \delta_{\mu\nu}, \quad (\delta_{11} = \delta_{22} = \delta_{33} = \delta_{44} = 1)$$

and

$$\gamma_\mu^{(E)} \hat{p}_E \gamma_\mu^{(E)} = -2 \hat{p}_E, \quad \hat{p}_E^2 = -p_E^2 = p^2$$

Taking into account the Mellin representation (6.2) for the form-factor $V(k_E^2 |^2)$ and after some calculations we have

$$\widetilde{\sum}_1(p) = \frac{e^2}{8\pi} \frac{1}{2i} \int_{-\alpha-i\infty}^{-\alpha+i\infty} \frac{d\xi v(\xi) (m^2 |^2)^\xi}{(\sin \pi \xi)^2 \Gamma(1+\xi)} F(\xi, p) \quad (8.4)$$

where

$$F(\xi, p) = \frac{1}{\Gamma(1 - \xi)} \int_0^1 du \left(\frac{1 - u}{u} \right)^\xi \left(1 - \frac{p^2}{m^2} u \right)^\xi (2m - \hat{p}u) \quad (8.5)$$

is regular function in the half-plane $\text{Re}\xi > -1$. Assuming the value $m^2|l^2$ to be small, one can obtain the following expression for the self-energy

$$\begin{aligned} \widetilde{\Sigma}_l(p) &= \frac{e^2}{8\pi^2} \int_0^1 du (2m - u\hat{p}) \ln \left(1 - \frac{p^2}{m^2} u \right) - \frac{e^2}{16\pi^2} \cdot \\ &\left[\left(3 \ln \frac{1}{m^2|l^2} + 3v'(0) + 3\psi(1) + 1 \right) + 4 m^2|l^2 v(1) \left(\ln \frac{1}{m^2|l^2} - \frac{v'(1)}{v(1)} - \frac{5}{12} \frac{p^2}{m^2} \right) \right] - \\ &\frac{e^2}{16\pi^2} (m - \hat{p}) \left[\left(\ln \frac{1}{m^2|l^2} - v'(0) + 1 \right) - m^2|l^2 v(1) \frac{p^2}{3m^2} \right] + O((m^2|l^2)^2) \quad (8.6) \end{aligned}$$

Let us calculate the correction to the electron mass that is

$$\delta m = m_0 - m = - \widetilde{\Sigma}(m) = \frac{3}{4\pi} \alpha \left\{ \chi + O(1) \right\} m \quad (8.7)$$

where $\chi = \ln \frac{1}{m^2|l^2}$

As seen above, obtained expression for $\widetilde{\Sigma}(p)$ is consistent with the usual result in the local quantum electrodynamics.

8. 2. The Vacuum Polarization Diagram

The term of the scattering matrix corresponding to this diagram (Fig. 1b) has the form

$$- i: A_{\mu}(x) \Pi_{\mu\nu}(x - y) A_{\nu}(y): \quad (8. 8)$$

where

$$\Pi_{\mu\nu}(x - y) = -ie^2 \text{Sp} \left\{ \gamma_{\mu} S(x - y) \gamma_{\nu} S(y - x) \right\} \quad (8. 9)$$

Let us use the proposed method of the regularization (6. 3) and obtain in the momentum space

$$\text{reg } \Pi_{\mu\nu}(x - y) = \frac{1}{(2\pi)^4} \int d^4p e^{-ip(x - y)} \text{reg } \widetilde{\Pi}_{\mu\nu}(p) \quad (8. 10)$$

Here

$$\begin{aligned} \text{reg } \Pi_{\mu\nu}(p) &= \frac{e^2}{(2\pi)^4 i} \int dk \sum_{j=0}^3 c_j \text{Sp} \left\{ \gamma_{\mu} \frac{1}{M_j - \hat{k} - i\varepsilon} \gamma_{\nu} \frac{1}{M_j - (\hat{k} - \hat{p}) - i\varepsilon} \right\} = \\ &= (p_{\mu} p_{\nu} - g_{\mu\nu} p^2) \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \left\{ \ln \left(1 - x(1-x) \frac{p^2}{m^2} \right) + \sum_{j=1}^3 c_j \ln \left(\Lambda_j - x(1-x) \frac{p^2}{m^2} \right) \right\} \end{aligned} \quad (8. 11)$$

In virtue of the condition (6, 4) one obtains in the limit $\Lambda \Rightarrow \infty$

$$\widetilde{\Pi}_{\mu\nu}(p) = \lim_{\Lambda \rightarrow \infty} \lim_{\varepsilon_j \rightarrow 0} \text{reg } \widetilde{\Pi}_{\mu\nu}(p) = (g_{\mu\nu} p^2 - p_{\mu} p_{\nu}) \widetilde{\Pi}(p^2) \quad (8. 12)$$

where

$$\tilde{\Pi}(p^2) = \tilde{\Pi}_r(p^2) + \frac{e^2}{12\pi^2} d$$

and

$$\tilde{\Pi}_r(p^2) = \frac{e^2}{12\pi^2} p^2 \int_{4m^2}^{\infty} \frac{dp^2}{p^2(p^2 - p^2 - i\epsilon)} \sqrt{1 - \frac{4m^2}{p^2}} \left(1 + \frac{2m^2}{p^2}\right) \quad (8.13)$$

Thus, within the framework of our regularization procedure the polarization operator $\tilde{\Pi}(p^2)$ is finite at the removal of the regularization and coincides with the renormalized expression in the usual local electrodynamics, if we choose $d=0$ for an arbitrary constant of the regularization. In this case $\tilde{\Pi}(p^2)$ is normalized by the condition

$$\Pi(0) = \tilde{\Pi}_r(0) = 0$$

It means that the constant d should define renormalized electron charge and the choice $d=0$ corresponds to the fact that at least in the second order of the perturbation theory the charge renormalization does not take place, i. e., physical charge of the electron e coincides with the bare one e_0 .

8. 3. The Vertex Operator

Let us consider the diagram shown in Fig. 1c. Its matrix element is

$$ie: \bar{\psi}(x) \Gamma_{\mu}(x, z; y) \psi(z) A_{\mu}(y): \quad (8. 14)$$

where we have introduced a vertex function of the third order

$$\Gamma_{\mu}(x, z; y) = ie^2 \gamma_{\nu} S(x - y) \gamma_{\mu} S(y - z) \gamma_{\nu} D(x - z) \quad (8. 15)$$

Taking into account momentum variables as shown in Fig. 1c and passing to the momentum representation one can obtain in the Euclidean metric

$$\begin{aligned} \tilde{\Gamma}_{\mu}(p_1, p) &= \lim_{\delta \Rightarrow 0} ie^2 \int dy \int dz e^{ipz + iqy} \gamma_{\nu} S(y) \gamma_{\mu} S(z - y) \gamma_{\nu} D^{\delta}(z) = \\ &= - \frac{e^2}{(2\pi)^4} \int \frac{dk_E V((p_E - k_E)^2 |^2) \gamma_{\nu} (m - \hat{k}_E - \hat{q}_E) \gamma_{\mu} (m - \hat{k}_E) \gamma_{\nu}}{(p_E - k_E)^2 [m^2 + (k_E + q_E)^2] (m^2 + k_E^2)} \quad (8. 16) \end{aligned}$$

Let us carry out integration over virtual momentum k_E in terms of the generalized Feynman parameterization:

$$\begin{aligned} b_1^{n_1} b_2^{n_2} \dots b_j^{n_j} &= \frac{\Gamma(n_1 + \dots + n_j)}{\Gamma(n_1) \dots \Gamma(n_j)} \int_0^1 dx_1 \dots \int_0^1 dx_j \delta\left(1 - \sum_{i=1}^j x_i\right) * \\ * \quad &x_1^{n_1-1} \dots x_j^{n_j-1} \left[\sum_{i=1}^j x_i b_i \right]^{-n_1 - \dots - n_j} \quad (8. 17) \end{aligned}$$

Again passing to the Minkowski metric in according to the conditions $(p_{iE} p_{jE}) = - (p_i p_j)$, one gets

$$\tilde{\Gamma}_\mu(p_1, p) = -\frac{e^2}{8\pi} \frac{1}{2i} \int_{-\alpha+i\infty}^{-\alpha-i\infty} \frac{v(\xi) (m^2)^{\xi}}{(\sin \pi \xi)^2 \Gamma(1+\xi)} F_\mu(\xi; p_1, p) \quad (8.18)$$

where

$$F_\mu(\xi; p_1, p) = \gamma_\mu F_1(\xi; p_1, p) + F_{2\mu}(\xi; p_1, p)$$

Here

$$F_1(\xi; p_1, p) = \frac{1}{\Gamma(1-\xi)} \iiint_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma) \alpha^{-\xi} Q^\xi$$

$$F_2(\xi; p_1, p) = \frac{1}{\Gamma(-\xi)} \iiint_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma) \alpha^{-\xi} Q^{\xi-1} * \quad (8.19)$$

$$\frac{1}{m^2} \left[m^2 \gamma_\mu - 2mq_\mu + 4m(\beta q_\mu - \alpha p_\mu) + (\alpha \hat{p} - \beta \hat{q}) \gamma_\mu \hat{q} + (\alpha \hat{p} - \beta \hat{q}) \gamma_\mu (\alpha \hat{p} - \beta \hat{q}) \right]$$

$$Q = \beta + \gamma - \alpha \gamma \frac{p^2}{m^2} - \beta \gamma \frac{q^2}{m^2} - \alpha \beta \frac{(p+q)^2}{m^2}$$

We now study the vertex function (8.18) for two cases. First, when $q=0$ and p has an arbitrary value. Second, q is an arbitrary quantity and p, p_1 are situated on the mass shell. In the first case, assuming $q=0$ in the formula (8.19) and after some calculations we have

$$F_\mu(\xi; p, p) = \frac{1}{\Gamma(1-\xi)} \int_0^1 du \left(\frac{1-u}{u} \right)^\xi \left(1-u \frac{p^2}{m^2} \right)^\xi \left[u \gamma_\mu + \frac{2\xi u p_\mu (2m - u \hat{p})}{m^2 - u p^2} \right] \quad (8.20)$$

Comparing the obtained formula with the expression (8. 5) for the self-energy of the electron, it is easy to note that

$$F_{\mu}(\xi; p, p) = - \frac{\partial}{\partial p_{\mu}} F(\xi, p) \quad (8. 21)$$

From this we can obtain a very important conclusion. In the nonlocal quantum electrodynamics constructed by using the concept of the extended charge density the Ward-Takahashi identity is valid

$$\tilde{\Gamma}_{\mu}(p, p) = - \frac{\partial}{\partial p_{\mu}} \widetilde{\Sigma}(p) \quad (8. 22)$$

In the second case, one can put

$$\bar{u}(\vec{p}_1) \tilde{\Gamma}_{\mu}(p_1, p) u(\vec{p}) = \bar{u}(p_1) \Lambda_{\mu}(q) u(\vec{p}) \quad (8. 23)$$

where $\bar{u}(\vec{p}_1)$ and $u(\vec{p})$ are solutions of the Dirac equations:

$$(\hat{p} - m) u(\vec{p}) = 0, \quad \bar{u}(\vec{p}_1) (\hat{p}_1 - m) = 0.$$

Substituting the vertex function (8. 18) into (8. 23) and after some transformations, we have

$$\bar{u}(\vec{p}_1) F_{\mu}(\xi; p_1, p) u(\vec{p}) = \bar{u}(p_1) \Lambda_{\mu}(\xi, q) u(\vec{p}) \quad (8. 24)$$

Here

$$\Lambda_{\mu}(\xi, q) = \gamma_{\mu} f_1(\xi, q^2) + \frac{i}{2m} \sigma_{\mu\nu} q_{\nu} f_2(\xi, q^2),$$

$$\sigma_{\mu\nu} = \frac{1}{2i} (\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu})$$

$$f_j(\xi, q^2) = \frac{1}{\Gamma(1-\xi)} \int_0^1 \int_0^1 \int_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma) \alpha^{-\xi} L^{\xi-1} g_j(\alpha, \beta, \gamma, q^2)$$

$$L = \lambda\alpha + (1-\alpha)^2 - \beta\gamma \frac{q^2}{m^2} \quad (8. 25)$$

$$g_1(\alpha, \beta, \gamma, q^2) = \left[(1-\alpha)^2(1-\xi) + 2\alpha\xi \right] - \left[\beta\gamma + \xi(\alpha+\beta)(\alpha+\gamma) \right] \frac{q^2}{m^2}$$

$$g_2(\alpha, \beta, \gamma, q^2) = 2\alpha\xi (1-\alpha)$$

In order to avoid infrared divergences in the vertex function we have here introduced the parameter $\lambda = \mu_{ph}^2 / m^2$ taking into account the "mass" of the photon.

Finally, we obtain

$$\Lambda_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i}{2m} \sigma_{\mu\nu} q_\nu F_2(q^2) \quad (8. 26)$$

where

$$F_j(q^2) = - \frac{e^2}{8\pi} \frac{1}{2i} \int_{-\alpha-i\infty}^{-\alpha+i\infty} \frac{d\xi v(\xi) (m^2 |^2)^\xi}{(\sin \pi\xi)^2 \Gamma(1+\xi)} f_j(\xi, q^2)$$

It is easy to verify that the vertex function $\Lambda_\mu(q)$ satisfies the gauge invariant condition:

$$q_\mu \bar{u}(\vec{p}_1) \Lambda_\mu(q) u(\vec{p}) = 0 \quad (8. 27)$$

Let us write first terms of the decomposition for functions

$F_1(q^2)$ and $F_2(q^2)$ over two small parameters $m^2 l^2$ and q^2/m^2 :

$$F_1(q^2) = \frac{\alpha}{4\pi} \left[\chi - 2\sigma - v'(0) + \frac{9}{2} (-6c - 3m^2 l^2 v(1)) \right] + \frac{\alpha}{2\pi} \frac{q^2}{m^2} \cdot \left[\frac{2}{3} \left(\frac{1}{2} \sigma - \frac{3}{8} \right) + \frac{m^2 l^2}{3} \left[v(1) \left(-\chi + 2c - \frac{13}{6} \right) + v'(1) \right] \right] \quad (8. 28)$$

where $\sigma = \ln(m^2/\mu_{ph}^2)$, $c=0,577215...$ is the Euler constant, $\alpha = e^2/4\pi$ and χ is defined above in (8. 7)

$$F_2(q^2) = - \frac{\alpha}{2\pi} \left[1 - \frac{2}{3} v(1) m^2 l^2 \right] \quad (8. 29)$$

The term in (8. 28) independent on q^2 defines the renormalized constant Z_1 and is subject to renormalization. Other terms may be defined from experimental data which will be discussed below.

Now, following Efimov ^[6] we consider the role of renormalized constants in the nonlocal QED, which we have introduced in the Lagrangian (6. 6). The self-energy operator with the renormalized constants δm and Z_2 is written in the form

$$\tilde{\Sigma}_r(p) = \left\{ A(p^2)m + B(p^2)\hat{p} \right\} + \delta m - (Z_2 - 1)(\hat{p} - m) \quad (8. 30)$$

Here structure functions of the mass operator (8. 4) and (8. 5) are denoted through $A(p^2)$ and $B(p^2)$. Representation (8. 30) is valid in any order of the perturbation theory. Constants δm and Z_2 are chosen by the condition

$$\lim_{q \Rightarrow 0} \bar{u}(\vec{p}) \frac{\tilde{\Sigma}_r(p+q)}{m - (\hat{p} + \hat{q})} u(\vec{p}) = 0 \quad (8.31)$$

where q is some four-vector such that $(pq) \neq 0$. The vector p lies on the mass shell, i. e., $p^2 = m^2$ and $\hat{p} u(\vec{p}) = m u(\vec{p})$. Substituting (8.30) into (8.31) and using the properties of the solutions of the Dirac equation

$$\bar{u}(\vec{p}) \gamma_\mu u(\vec{p}) = \frac{p_\mu}{m} \bar{u}(\vec{p}) u(\vec{p})$$

we obtain

$$\begin{aligned} \delta m &= -m(A(m^2) + B(m^2)) \\ Z_2^{-1} &= B(m^2) + 2m^2 (A'(m^2) + B'(m^2)), \end{aligned} \quad (8.32)$$

$$A'(m^2) = \left. \frac{dA(p^2)}{dp^2} \right|_{p^2 = m^2}$$

Substituting defined values of the renormalized constants into (8.30) one gets the following expression for the operator of mass

$$\begin{aligned} \tilde{\Sigma}_r(p) &= m(A(p^2) - A(m^2)) + (B(p^2) - B(m^2)) \hat{p} - \\ &- 2m^2 (A'(m^2) + B'(m^2)) (\hat{p} - m) \end{aligned} \quad (8.33)$$

The vertex function with renormalized constant has the form

$$\tilde{\Gamma}_{r\mu}(p_1, p) = \tilde{\Gamma}_\mu(p_1, p) + (z_1 - 1) \gamma_\mu \quad (8.34)$$

The Ward-Takahashi identity (8.22) should be satisfied for the renormalized quantities $\tilde{\Sigma}_r(p)$ and $\tilde{\Gamma}_{r\mu}(p_1, p)$, and therefore

$$Z_2 = Z_1 \quad (8. 35)$$

It should be noted again that all renormalized constants $\delta m, Z_1$, and Z_2 are finite and functions on the elementary length l in the nonlocal theory. At this, δm and Z_2 are chosen from normalization condition of the vertex operator (8. 31) and the quantity Z_1 is defined from the Ward-Takahashi identity.

The renormalized operator of the vacuum polarization is written in the form

$$\tilde{\Pi}_{\mu\nu}^r(p) = (g_{\mu\nu} p^2 - p_\nu p_\mu) \left[\tilde{\Pi}_F^r(p^2) + \frac{\alpha}{3\pi} d + Z_3 - 1 \right] \quad (8. 36)$$

Choosing $Z_3 = 1 - \frac{\alpha}{3\pi} d$ one obtains normalization $\Pi_F(0) = 0$ for $\tilde{\Pi}_{\mu\nu}^r$.

This condition grants that charge e in the interaction Lagrangian is the physical observable charge and $\alpha = e^2 / 4\pi = 1/137$.

9. An Experimental Restriction on the Value of the Fundamental Length.

Recently, no effect has been found experimentally which could not be confined within the local quantum electrodynamics. Tests of locality are usually performed by using very high precision experiments in atomic physics and in high energy lepton-lepton scattering processes. In this section we calculate nonlocal corrections to the anomalous magnetic moments of leptons, the Lamb shift, cross sections of the electromagnetic processes

$e^-e^- \rightarrow e^-e^-$, $e^+e^- \rightarrow e^+e^-$, $e^+e^- \rightarrow \mu^+\mu^-$, and obtain restriction on the value of the fundamental length.

9. 1. The Corrections to the Anomalous

Magnetic Moment (AMM) of the Leptons

Nonlocal contribution to the AMM is defined from the vertex function $\Lambda_\mu(q)$ in (8. 26) containing the term with $\sigma_{\mu\nu}q_\nu$, namely the formula (8. 29) gives

$$\Delta\mu = \frac{\alpha}{2\pi} \left[1 - \frac{2}{3} v(1) m^2 l^2 \right] \quad (9. 1)$$

and its first term corresponds to the Schwinger correction obtained

in the local QED, where $v(1) = \frac{1}{4}$ [see formula (6. 2)]. At present,

experimental values ^[8] of the AMM of the electron and muon are

$$\Delta\mu_{\text{exp}}^{(e)} = 1.001\,159\,652\,193 \pm 0.000\,000\,000\,010$$

$$\Delta\mu_{\text{exp}}^{(\mu)} = 1.001\,165\,923 \pm 0.000\,000\,008 \quad (9. 2)$$

and are fully described by the local QED. Comparing the correction (9. 1) with the experimental errors in (9. 2) one can obtain

$$\begin{aligned} l &\leq 8.76 \times 10^{-15} \text{ cm} & \text{for} & \Delta\mu_{\text{exp}}^{(e)} \\ l &\leq 1.2 \times 10^{-15} \text{ cm} & \text{for} & \Delta\mu_{\text{exp}}^{(\mu)} \end{aligned} \quad (9. 3)$$

9. 2. The Lamb Shift

According to the standard calculation ^[9] the difference between energy levels $2S_{1/2}$ and $2P_{1/2}$ for the hydrogen atom, due to the change of functions F_1 and F_2 in (8. 26) is given by

$$\Delta E_l(2S_{1/2} - 2P_{1/2}) = \alpha^2 Ry \left| m^2 F_1'(0) - \frac{1}{2} F_2(0) \right| \quad (9. 4)$$

where $Ry = m\alpha^2/2$ is the Rydberg constant.

Making use of the formulas (8. 28) and (8. 29) one obtains the following expression for the correction due to nonlocality of the electron charge

$$\Delta E_l(2S_{1/2} - 2P_{1/2}) = \frac{\alpha^3}{6\pi} Ry m^2 l^2 \left[v(1) \left(\ln m^2 l^2 + 2c - \frac{3}{2} \right) + v'(1) \right] \quad (9. 5)$$

or with the function (6. 2)

$$\Delta E_l(2S_{1/2} - 2P_{1/2}) = -\frac{\alpha^3}{6\pi} Ry \frac{m^2 l^2}{4} \left[\ln \frac{1}{m^2 l^2} + \frac{5}{2} + 2\ln 2 - 3c \right] \quad (9. 6)$$

$c=0,577216$.

The experimental value of the Lamb shift

$$(\Delta E)_{\text{exp}} = (1057.912 \pm 0, 011) \text{ MHz/sec} \quad (9. 7)$$

and is well explained by the local QED ^[9].

Therefore

$$|\Delta E_l(2S_{1/2} - 2P_{1/2})| \leq 0. 011 \text{ MHz/sec}$$

and substituting formula (9. 6) into this equality we get

$$l \leq 3 \times 10^{-13} \text{ cm} \quad (9. 8)$$

9. 3. Electron Scatterings at High-Energies

More strict restrictions on the value of the fundamental length may be obtained from the experiments on the electron scattering at high-energies. Since, electromagnetic processes of the type of $e^-e^- \rightarrow e^-e^-$, $e^+e^- \rightarrow e^+e^-$, and $e^+e^- \Rightarrow \mu^+\mu^-$ are described by lower orders of the perturbation theory (Fig. 2) even at high-energies attainable up to now.

$$e^-e^- \rightarrow e^-e^-, e^+e^- \rightarrow e^+e^-, \text{ and } e^+e^- \Rightarrow \mu^+\mu^-$$

Processes in the low order of the perturbation theory

A ratio of cross sections calculated by local and nonlocal theories is

$$\frac{\sigma_{\text{nonlocal}}}{\sigma_{\text{local}}} = \left[V(-se^2) \right]^2 \sim 1 + 2v(1) sl^2 \quad (9. 9)$$

where $s = (p_1 + p_2)^2 = (2E)^2 = W^2$, $w=2E$ is the total energy in the system of centre mass. Estimation based on the formula (9. 9) and experimental data ^[10] is very simple and gives the restriction of the order:

$$l \leq 10^{-16} \text{ cm} \quad (9. 10)$$

Finally, it should be noted that generally speaking restrictions (9. 3), (9. 8) and (9. 10) imply that the leptons as elementary constituents carrying extended electric charge are point-like particles with radii smaller than 10^{-16} cm .

Appendix A

1. The Mellin representation (6. 2) for an entire function is useful to calculate improper integrals. For example,

$$I_1 = \int_0^{\infty} dx \frac{\sin ax}{x} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2i} \int_{-\alpha + i\infty}^{-\alpha - i\infty} d\xi \frac{1}{\sin \pi \xi} \frac{a^{1+2\xi}}{\Gamma(2+2\xi)} \int_{\varepsilon}^{\infty} dx x^{2\xi} =$$

$$= - \lim_{\varepsilon \rightarrow 0} \frac{1}{2i} \int_{-\alpha + i\infty}^{-\alpha - i\infty} d\xi \frac{1}{\sin \pi \xi} \frac{a^{1+2\xi}}{\Gamma(2+2\xi)} \frac{\varepsilon^{1+2\xi}}{1+2\xi}, \quad \begin{matrix} a > 0 \\ (\alpha > 0) \end{matrix} \quad (A.1)$$

Displacing the contour integration to the right and calculating the residues one can easily see that the residue at the point $\xi = -1/2$ gives

$$I_1 = \frac{\pi}{2} \quad (A.2)$$

in the limit $\varepsilon \rightarrow 0$.

Similarly, we have

$$I_2 = \int_0^{\infty} dx \sin(ax^2) = - \lim_{\varepsilon \rightarrow 0} \frac{1}{2i} \int_{-\alpha + i\infty}^{-\alpha - i\infty} d\xi \frac{1}{\sin \pi \xi} \frac{1}{\Gamma(2+2\xi)} \frac{a^{1+2\xi} \varepsilon^{4\xi+3}}{3+4\xi} = \frac{\sqrt{\pi}}{2\sqrt{2} a}$$

and

(A. 3)

$$I_3 = \int_0^\infty dx \sin ax = - \lim_{\epsilon \rightarrow 0} \frac{1}{2i} \int_{-\alpha - i\infty}^{-\alpha + i\infty} d\xi \frac{1}{\sin \pi \xi} \frac{1}{\Gamma(2+2\xi)} \frac{a^{1+2\xi} \epsilon^{2\xi+2}}{2+2\xi} = \frac{1}{a}$$

(A. 4)

2. The modified Coulomb potential

$$\phi_1(r) = \frac{e}{4\pi} \frac{1}{r} \phi(r/l) \quad (\text{A. 5})$$

satisfies the Poisson equation

$$\Delta \phi_1(r) = \frac{-1}{\pi^{3/2} l^3} \exp(-r^2/l^2) \quad (\text{A. 6})$$

identically. Here we have used an another representation for the probability integral

$$\phi(r/l) = 1 - \frac{2}{\sqrt{\pi}} \exp[-r^2/l^2] I_0 \quad (\text{A. 7})$$

$$I_0 = \int_0^\infty dt (t^2 + r^2)^{-1/2} (t/l) \exp[-t^2/l^2] \quad (\text{A. 8})$$

and the following integral forms

$$I_1 = \int_0^\infty dt (t/l)(r^2 + l^2)^{-3/2} \exp(-t^2/l^2) = \frac{1}{l} \frac{1}{r} - \frac{2}{l^2} I_0 \quad (\text{A. 9})$$

$$I_2 = \int_0^\infty dt (t/l)(r^2 + l^2)^{-5/2} \exp(-t^2/l^2) = \frac{1}{3l} \frac{1}{r^3} - \frac{2}{3l^3} \frac{1}{r} + \frac{4}{3l^4} I_0 \quad (\text{A.10})$$

which are arisen from the Laplacian $\Delta \phi_1(r)$.

3. The equality

$$W = \frac{1}{2} \int d^3r \rho_i(\vec{r}) \phi_i(\vec{r}) = \frac{1}{2} \int d^3r \vec{E}_i^2(\vec{r}) \quad (\text{A. 11})$$

$$\vec{E} = -\vec{\nabla} \left(\frac{e}{4\pi} \frac{1}{r} \phi(r/l) \right)$$

for the self-energy of the extended electrical charge gives following useful formulas

$$\int_0^\infty dy \frac{\phi^2(y)}{y^2} = \frac{\sqrt{2}}{\sqrt{\pi}} \left[\sqrt{2} \ln \frac{1+\sqrt{2}}{\sqrt{2}-1} - 1 \right] \quad (\text{A. 12})$$

or

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{[2(n_1+n_2)-1]!!}{(2n_1+1)!! (2n_2+1)!!} 2^{-(n_1+n_2)} = \sqrt{2} \ln \frac{1+\sqrt{2}}{\sqrt{2}-1} - 1 \quad (\text{A. 13})$$

The latter means that an explicit sum of the symmetrical double series (A. 13) exists.

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