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**THE FEYNMAN PROBLEM AND
THE INVERSE PROBLEM FOR POISSON DYNAMICS**

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ABSTRACT

We review the Feynman proof of the Lorentz force equations, as well as its generalization to the dynamics of particles with internal degrees of freedom. In addition, we discuss the inverse problem for Poisson dynamics and the inverse problem of the calculus of variations. It is proved that the only classical dynamics compatible with localizability and the existence of second order differential equations on tangent bundles over arbitrary configuration spaces, is necessarily of the Lagrangian type. Furthermore, if the dynamics is independent of the velocity of test particles, it must correspond to that of a particle coupled to an electromagnetic field and/or a gravitational field. The same ideas are carried out for particles with internal degrees of freedom. In this case, if we insist on a weak localizability condition and the existence of a second order Hamiltonian differential equation, then the dynamics results from a singular Lagrangian. (Here we assume in addition that the dynamics satisfies a regularity condition.) These results extend those of Feynman and provide the conditions which guarantee the existence of a Lagrangian description. They are applied to systematically discuss Feynman's problem for systems possessing Lie groups as configuration spaces, with internal variables modeled on Lie algebras of groups. Finally, we illustrate what happens when the condition of localizability is dropped. In this regard, we obtain alternative Hamiltonian descriptions of standard dynamical systems. These non-standard solutions are discussed within the framework of Lie-Poisson structures.

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1 Introduction

Recently Dyson [Dy90] reported on an unpublished result of Feynman showing that the only interaction for a quantum particle compatible with localizability and the canonical commutation relation $[q^i, p_j] = i\hbar\delta_j^i$, is the Lorentz force law of electrodynamics. After the communication of this result, modifications of Feynman's procedure have been used to "prove" several other interactions. They include the descriptions of relativistic and non relativistic particles, spinning particles in an electromagnetic field [Le90], particles with isospin moving in a Yang-Mills field, and particles in gravity. [Ta92] (see also [St93], [La93], [Ch94].)

According to Dyson, Feynman's real interest was not simply recovering the standard minimal coupling to electrodynamics, but rather in finding new kinds of particle dynamics, possibly those which are not expressible in terms of a Lagrangian. In this sense, Feynman's result is a no-go theorem, showing that unless some well founded physical conditions are removed, there are no interactions apart from the one associated with electrodynamics.

The majority of modifications to Feynman's procedure involve introducing additional dynamical degrees of freedom (associated with spin or isospin) which we refer to as *internal* degrees of freedom. Internal degrees of freedom are traditionally associated with Lie groups, but this need not be the case as has been illustrated in Refs. [So92] and [St93]. Unlike spatial coordinates (or *external* degrees of freedom), internal coordinates are generally associated with first order equations of motion. The standard dynamics for the internal degrees can be recovered from the generalization of Feynman's procedure. On the other hand, the procedure also allows for some nonstandard solutions.

It has been noted recently that the same conditions used in Feynman's approach imply that the Helmholtz conditions for the inverse problem of the calculus of variations are satisfied. This shows the existence of a Lagrangian description for such systems from a different perspective. [Ho90], [Hu92] In fact, it was already observed in [No82] that localizability plus regularity implies the existence of a (local) Lagrangian description for the dynamics.

In this paper we will reexamine Feynman's approach to particles dynamics. *Here in order to avoid unnecessary complication due to operator ordering we shall discuss only the classical analogue of the Feynman procedure.* The inclusion of internal degrees of freedom will also be considered. For systems with and without internal variables, we shall find necessary and sufficient conditions that guarantee the existence of a Lagrangian description for the system. The form of the Lagrangian function will be given when these conditions are met. The solution involves a discussion of the inverse problem of Poisson dynamics and the inverse problem of the calculus of variations. Thus, we geometrize Feynman's problem and set it in the context of the geometry of Poisson manifolds, Poisson dynamics and Lagrangian dynamics. We will be able to extract the geometrical meaning of the different hypotheses used in the various applications of Feynman's procedure. We also discuss the geometrical meaning of localizability, the Newtonian character of the Poisson brackets, the independence of the velocity, as well as the different role played by each of them. This discussion will lead us to several geometrical versions of the inverse problem of the calculus of variations, both for regular and singular Lagrangians. In this sense, three different facets of the dynamics of physical systems are tied together: Feynman's problem, the inverse problem for Poisson dynamics and the inverse problem of the calculus of variations.

Our article is divided into three parts. In Part I we review the generalized Feynman

procedure for particles with and without internal degrees of freedom in a general context. Some properties of the dynamics of internal variables will be discussed in §3, and continued in §10. In §3 we shall also examine descriptions of internal variables which are relevant for quantum groups [Ta89], [Ho91]. Internal variables are combined with external ones in §4 and §8 where standard Lagrangian realizations are shown.

In Part II we systematically explore the previous formulation of Feynman's problem in arbitrary configuration spaces. We will show that dynamical systems which are described by second order differential equations (SODE's), and which have localizable Poisson brackets on velocity phase spaces, must, at least locally, have a Lagrangian description. Hence, again locally, we are back to the solution found by Feynman. This no-go theorem will be proved using geometrical methods and for that, some technology on Poisson manifolds and tangent bundles will be needed. (This is reviewed in §5 and §6.) Similar results have been obtained by [Ho90], [No82] and [Bl73]. However, as is shown in the examples of Part I, we can extend them to systems possessing internal degrees of freedom. Such systems will be defined on extended spaces (typically, vector or principal bundles over TQ). In this way, the no-go theorem of Feynman is circumvented and non-trivial physical Poisson brackets can be found. In general, the relevant Poisson brackets cannot be derived from a Lagrangian function. In fact, the bundle where they are defined may not be a tangent bundle. An extension of Feynman's procedure could then be formulated as follows: Find a Lagrangian realization for the Poisson brackets defining a particle with internal degrees of freedom on an extension of the velocity phase space. In general, this problem has not been solved. We will show that if the Poisson brackets among the internal variables satisfy an additional regularity property, then there is a symplectic realization of the system. Furthermore, there exists a Lagrangian realization for it on an extended space, the Lagrangian however being singular. To show this result, we use an extension of the inverse problem of the calculus of variations for coupled systems of SODE's and first order differential equations [Ib91]. It extends previous formulations in geometrical terms (see for instance [Mo90] and references therein).

In Part III we will systematize some of the examples discussed in Part I, showing that they are representative of entire families of systems whose construction relies on the geometry of Lie groups, principle fibre bundles and the inverse problem of Poisson dynamics. Part III will also be devoted to the discussion of some examples and applications of the geometrical treatment of Feynman's problem beyond those examples found in Part I. For instance, in §9 we adapt Feynman's procedure to particle dynamics on Lie groups, in particular the group $SU(2)$. The corresponding free system leads to the standard Hamiltonian formulation of a rigid rotator. After this we will abandon Feynman's procedure and show that there exists an alternative Hamiltonian description of the rigid rotator [Ma93], [St94]. For this alternative description, localizability is no longer required and it is therefore canonically inequivalent to the description given in earlier. Furthermore, rotations are canonical symmetries in the standard description, while they are Poisson-Lie group symmetries in the alternative description.

Part I

Poisson dynamics and Feynman's Procedure

2 Feynman's problem and the inverse problem of Poisson dynamics

Poisson dynamics

Taking the point of view that classical mechanics should be considered as a limit of quantum mechanics, the appropriate setting for the description of classical systems would be in terms of Poisson brackets and a Hamiltonian function. The former is regarded as the classical limit of the commutator of quantum observables, and the latter is the classical limit of the corresponding quantum evolution operator. Upon taking the classical limit of the quantum system, the carrier space could inherit a nontrivial topology. (For example, for the case of a spinning particle belonging to an irreducible representation of $SU(2)$ the classical phase space is a 2 dimensional sphere.) The classical limit of a quantum system is then said to be a Hamiltonian system defined on a Poisson manifold, i.e., a manifold M equipped with a Poisson bracket.

More precisely, let $\mathcal{F} = \mathcal{F}(M)$ be the algebra of classical observables on the manifold M . $\mathcal{F}(M)$ may be obtained as a certain commutative limit ($\hbar \rightarrow 0$) of the non commutative algebra of quantum observables $\mathcal{O}_\hbar$ of a quantum system. The Poisson bracket $\{.,.\}$ on M is defined by

$$\{F, G\} = \lim_{\hbar \rightarrow 0} \frac{1}{i\hbar} [F_\hbar, G_\hbar],$$

where $F = \lim_{\hbar \rightarrow 0} F_\hbar$ for every $F_\hbar \in \mathcal{O}_\hbar$. $\{.,.\}$ is a skewsymmetric bilinear map on $\mathcal{F}$ such that it satisfies Jacobi's identity and Leibnitz's rule:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0, \quad (1)$$

$$\{F, GH\} = G\{F, H\} + \{F, G\}H. \quad (2)$$

This constitutes the formal definition of a Poisson structure on the classical state space M .

Definition 1 A Poisson structure on a manifold M is a skew-symmetric $\mathbf{R}$ -bilinear map, denoted $\{.,.\} : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow \mathcal{F}(M)$, such that:

- a) $(\mathcal{F}(M), \{.,.\})$ becomes a real Lie algebra, i.e., it satisfies the Jacobi identity eq. (1)
- b) The map $X_F = \{., F\}$ is a derivation of the associative algebra $\mathcal{F}(M)$ of functions on M , i.e., it satisfies Leibnitz rule eq. (2).

The pair $(M, \{.,.\})$ is called a Poisson manifold. When the only locally defined functions F such that $\{F, G\} = 0, \forall G \in \mathcal{F}(M)$ are the constant functions, the Poisson manifold is said to be non-degenerate or regular. If there exist globally defined nonconstant functions C such that $\{C, G\} = 0, \forall G \in \mathcal{F}(M)$, they will be called Casimir functions.

The best known example of a Poisson manifold is provided by $\mathbf{R}^{2n}$ with the ordinary Poisson bracket:

$$\{F, G\} = \frac{\partial F}{\partial x^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x^i},$$

where $x^i, p_i, i = 1, \dots, n$, are canonical position and momentum variables used to parametrize $\mathbf{R}^{2n}$. Notice that in this case there are no nontrivial Casimirs.

It could happen that the algebra of classical observables on M constructed as classical limits of quantum observables will define a Lie subalgebra $\mathcal{F}$ of $\mathcal{F}(M)$. (In fact, this is what happens for $\mathbf{R}^{2n}$.) The implication of this is that the classical space we are dealing with it is not exactly M but a quotient of it. The subalgebra $\mathcal{F}$ defines a quotient space of M in the following way: two points m, m' on M are equivalent if $f(m) = f(m')$ for all $f \in \mathcal{F}$. This quotient space can be pathological from the differentiable point of view. In any case, we can declare that the algebra of functions on the quotient space of M with respect to this equivalence relation is precisely $\mathcal{F}$. For the case of $\mathbf{R}^n$, the subalgebra $\mathcal{F}$ is dense in $\mathcal{F}(\mathbf{R}^{2n})$ with respect to some appropriate topology. Then it will not define a subspace of $\mathbf{R}^{2n}$.

The derivation property is very important and there are many examples of Lie algebra structures in $\mathcal{F}(M)$ for which this property does not hold. For instance, if X is an arbitrary vector field on M , then $\{F, G\}$ defined by $\{F, G\} = F\mathcal{L}_X G - G\mathcal{L}_X F$ does not satisfy the Leibnitz rule. ($\mathcal{L}_X$ denotes the Lie derivative with respect to X .) The Leibnitz rule means that for every function F , there exists a vector field, to be denoted X_F , such that $X_F G = \{G, F\}$. In a set of local coordinates for P, ξ^a , the vector field X_F is written

$$X_F = X_F(\xi^a) \frac{\partial}{\partial \xi^a} = \{\xi^a, F\} \frac{\partial}{\partial \xi^a},$$

where summation on repeated indices is understood. The system of differential equations determining its integral curves looks like Hamilton equations:

$$\frac{d\xi^a}{dt} = \{\xi^a, F\}, \quad (3)$$

where the function F plays the role of Hamiltonian and t parametrizes the curves. Therefore the vector fields X_F are called *Hamiltonian vector fields* and F is said to be the *Hamiltonian* of X_F . The coordinate expression for the Poisson bracket $\{F, G\}$ is

$$\{F, G\} = X_G F = \{\xi^a, G\} \frac{\partial F}{\partial \xi^a}.$$

Since $X_{\xi^a} = \{\xi^b, \xi^a\} \partial / \partial \xi^b$, we see that

$$\{\xi^a, G\} = -\{G, \xi^a\} = -\{\xi^b, \xi^a\} \frac{\partial G}{\partial \xi^b} = \{\xi^a, \xi^b\} \frac{\partial G}{\partial \xi^b},$$

from which the expression for $\{F, G\}$ becomes

$$\{F, G\} = \{\xi^a, \xi^b\} \frac{\partial F}{\partial \xi^a} \frac{\partial G}{\partial \xi^b} = \Lambda^{ab}(\xi) \frac{\partial F}{\partial \xi^a} \frac{\partial G}{\partial \xi^b}.$$

This shows that in order to compute the Poisson bracket of any pair of functions in a particular set of local coordinates, it suffices to know the fundamental Poisson brackets

$$\Lambda^{ab} = \{\xi^a, \xi^b\}, \quad (4)$$

corresponding to the coordinate functions themselves.

The preceding expression for $\{F, G\}$ shows that its value at a point $m \in M$ does not depend on G but on dG , and therefore the Poisson structure furnishes a twice contravariant skew-symmetric tensor Λ by

$$\Lambda(dF, dG) = \{F, G\}.$$

More specifically, if $\bar{\xi} = \phi(\xi)$ are some new coordinates in M , then

$$\bar{\Lambda}^{ab} = \{\bar{\xi}^a, \bar{\xi}^b\} = \{\phi^a, \phi^b\} = \frac{\partial \phi^a}{\partial \xi^c} \{\xi^c, \xi^d\} \frac{\partial \phi^b}{\partial \xi^d} = \Lambda^{cd} \frac{\partial \phi^a}{\partial \xi^c} \frac{\partial \phi^b}{\partial \xi^d},$$

which shows that the components of Λ change like those of a twice-contravariant skew-symmetric tensor with coordinate expression

$$\Lambda = \Lambda^{ab} \frac{\partial}{\partial \xi^a} \wedge \frac{\partial}{\partial \xi^b}. \quad (5)$$

The tensor Λ , called in what follows a Poisson tensor, will play a central role in the description of the classical system. As will be reviewed in §5, Jacobi's identity for the Poisson bracket defined by Λ is equivalent to the vanishing of a 3-tensor. The latter is the Schouten bracket of Λ with itself, denoted by $[\Lambda, \Lambda] = 0$.

Any function $H \in \mathcal{F}$ will define a dynamical system on M by the formula

$$\frac{dF}{dt} = \{F, H\}, \quad (6)$$

or in local coordinates

$$\frac{d\xi^a}{dt} = \Lambda^{ab} \frac{\partial H}{\partial \xi^b}. \quad (7)$$

The Hamiltonian vector field X_H defined by H is written in terms of the Poisson tensor as:

$$X_H = -\Lambda(dH). \quad (8)$$

In some cases the Poisson bracket can be associated with a symplectic structure, §5 (iff Λ is nondegenerate), or a Lagrangian structure, §6 (for classical phase spaces M of the form TQ , i.e., tangent bundles of classical configuration spaces). Then we recover the more familiar symplectic and Lagrangian formalisms.

Inverse problem for Poisson dynamics

From the above point of view, it is natural to ask when a classical dynamical system Γ can be given a description in terms of some Poisson bracket and some Hamiltonian function. Here we do not make *a priori* assumptions on the Poisson tensor Λ , and hence on the form of the canonical relations eq. (4), except those coming from the physics of the quantum systems under consideration. It is important to realize that the Hamiltonian system, if it exists, will depend on the Poisson tensor Λ because of eq. (8).

The above problem is called the inverse problem of Poisson dynamics, but stated in such general terms it is hopeless, difficult, or useless because of the lack of uniqueness of solutions and the lack of understanding of their physical meaning. One way to improve the problem is to make some further assumptions on the Poisson manifold M . For instance,

we can assume that $M = \mathbb{R}^{2n}$ and that the classical system is described by a second order differential equation (SODE). In this way we obtain new problems that could be called restricted inverse problems for Poisson dynamics. Obviously attached to these problems is the question of uniqueness of the Poisson description for the given classical system. This was already raised by Wigner in 1950 [Wi50].

If the inverse problem of Poisson dynamics is solved for a given dynamical system Γ on M and a Poisson tensor Λ is found, we can ask if there exists a symplectic realization for it, that is, a symplectic manifold (P, Ω) and a Poisson map $\Phi: P \rightarrow M$. Such a symplectic realization exists locally, but in general there are obstructions to obtaining globally defined ones [We83], [We87].

Together with the problem of finding symplectic realizations for Poisson manifolds, arises the problem of finding Lagrangian realizations. That is, does there exist a Lagrangian for a given dynamical system? Once a symplectic realization has been found, and if the symplectic manifold is a tangent bundle, we will apply the inverse problem of the calculus of variations to look for a Lagrangian describing the system. The conditions assuring the existence of such Lagrangians, both local and global [Mo90], [Ib90], are well known and can be systematically applied to the problems at hand, Thms. 1, 6.

Feynman's Problem

An alternative to the inverse problem for Poisson dynamics as discussed above, consists in asking as Feynman did, which dynamical laws are compatible with certain basic physical assumptions. One of Feynman's aims was to obtain dynamical laws beyond those in the Lagrangian formalism. Starting with just a few postulates for the quantum mechanics of a single particle, Feynman was able to recover the standard interaction with an electromagnetic field. The postulates which we will here review include assuming the existence of position and velocity operators with commutation relations consistent with particle localizability. (In order to avoid unnecessary complication due to operator ordering we shall be concerned only with the classical analogue of Feynman's procedure.) This program failed, as described by Dyson [Dy90], because the conditions imposed on the Poisson brackets were restrictive enough to show that only electromagnetic forces were compatible with them.

Feynman's problem deals with a full family of dynamical systems, those being SODE's on a fixed velocity phase space TQ . The classical Feynman problem can then be stated in a more general setting as the problem of finding all Poisson tensors on TQ such that they have Hamiltonian vector fields which correspond to SODE's and such that the systems are localizable.

Let us now recall Feynman's derivation of the Lorentz force for a point particle. For the choice of phase space variables we take $(\xi^a) = (x^i, v^i)$ for $i = 1, 2, 3$, x^i denoting position and v^i velocity. The fundamental brackets will be given by eq. (4), and the equations of motion associated with a Hamilton function H are given by eq. (7). The analogue of Feynman's postulates gives some information about Λ^{ab} and the equations of motion. The assumptions on the Poisson brackets are that

$$\{x^i, x^j\} = 0 \quad (9)$$

and

$$m\{x^i, v^j\} = \delta_{ij}. \quad (10)$$

In terms of Poisson tensors Λ , we are postulating that Λ has the following form on $\mathbb{R}^6$:

$$\Lambda = \frac{1}{m} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial v^i} + f^{ij}(\vec{x}, \vec{v}) \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial v^j}, \quad (11)$$

where the form of f^{ij} is to be determined. Eq. (9) is the classical analogue of the condition of localizability, while m in (10) refers to the particle's mass. The equations of motion were postulated to have the form

$$\frac{d}{dt} x^i = \{x^i, H\} = v^i \quad (12)$$

$$\frac{d}{dt} v^i = \{v^i, H\} = \frac{1}{m} F^i(\vec{x}, \vec{v}). \quad (13)$$

Notice that eq. (12) implies that the Hamiltonian dynamical system is a SODE. For convenience we will at first assume that F^i has no explicit time dependence.

The problem which was then posed was to find dynamics compatible with assumptions (9), (10), (12) and (13). Alternatively this means that one should determine the Hamilton function H and the remaining Poisson brackets defining Λ_{ab} , which here are just $\{v^i, v^j\}$. A unique solution to this problem gives the coupling to electromagnetic fields, as we now review.

One first defines the magnetic field B^i according to $f^{ij} = \frac{1}{m^2} \epsilon^{ijk} B^k$, or

$$m^2 \{v^i, v^j\} = \epsilon^{ijk} B^k. \quad (14)$$

It is then easy to show that B^i is a function of only the spatial coordinates x^i . For this, one writes down the Jacobi identity

$$\{x^i, \{v^j, v^k\}\} + \{v^j, \{v^k, x^i\}\} + \{v^k, \{x^i, v^j\}\} = 0, \quad (15)$$

and applies (10) and (14) to find that $\{x^i, B^j\} = 0$. Hence B^i must be independent of the particle velocity v^i . It also has zero divergence, which follows from the Jacobi identity involving v^i, v^j and v^k , and thus we recover one of the Maxwell equations.

Next one takes the time derivatives of Poisson brackets (10) and (14). The former gives

$$\{v^i, v^j\} + \{x^i, v^j\} = 0,$$

or

$$\{x^i, F^j\} = -\frac{1}{m} \epsilon^{ijk} B^k(\vec{x}), \quad (16)$$

from which it follows that F^i is at most linear in the velocities

$$F^i = E^i(\vec{x}) + \epsilon^{ijk} v^j B^k(\vec{x}). \quad (17)$$

Eq. (17) is the Lorentz force law (where the electric charge is set equal to one) and it here defines the electric field $E^i(\vec{x})$.

From the time derivative of (14) one gets

$$m \epsilon^{ijk} \{F^i, v^j\} = \dot{B}^k. \quad (18)$$

Next apply (17) and the divergenceless condition on B^i to simplify (18) to

$$m \epsilon^{ijk} \{E^i, v^j\} = \dot{B}^k - m \{B^k, v^i\} v^i. \quad (19)$$

But the right hand side of (19) vanishes because we have assumed B^i to have no explicit time dependence, and so we are just left with

$$\epsilon^{ijk} \{E^i, v^j\} = 0, \quad (20)$$

implying that E^i has zero curl, as is the case for time independent magnetic fields.

Upon repeating this calculation for time dependent fields, one instead recovers the Faraday law and thus both of the homogeneous Maxwell equations. In this case both the velocity v^i and the force F^i are allowed to have explicit dependence on t , so that eq. (13) is generalized to

$$\frac{d}{dt} v^i = \{v^i, H\} + \frac{\partial v^i}{\partial t} = \frac{1}{m} F^i(\vec{x}, \vec{v}, t). \quad (21)$$

The analysis then proceeds as in the above with the static fields $E^i(\vec{x})$ and $B^i(\vec{x})$ replaced by $E^i(\vec{x}, t)$ and $B^i(\vec{x}, t)$. Now the right hand side of (19) is $\partial B^i / \partial t$ and hence the Faraday law is recovered.

The Hamilton function for the above system is

$$H = \frac{m}{2} \vec{v}^2 + \phi(\vec{x}, t),$$

where $\phi(\vec{x}, t)$ is the standard scalar potential. From it we construct the electric field according to,

$$\vec{E} = -\vec{\nabla} \phi + m \frac{\partial \vec{v}}{\partial t}.$$

3 Internal dynamics

Here we shall be interested in the inverse problem for Poisson dynamics, i.e.: What are the possible Poisson structures and Hamiltonian functions associated with a given dynamical system? For this we shall concentrate on one particular system – that of a spinning particle interacting with a magnetic field. In addition to giving the standard Hamiltonian formulation of this system, we shall give examples of nonstandard formulations.

In this section we shall ignore the spatial degrees of freedom of the particle and rather only study the spin variables. The spatial coordinates will be included in the following section.

Let us denote the spin variables by S_i , $i = 1, 2, 3$. They span some 3-manifold M . (We initially impose no constraints on S_i .) Since S_i correspond to *internal* degrees of freedom (as opposed to the spatial coordinates discussed in the previous section) we should not make the same assumptions as Feynman used for getting dynamics on Minkowski space. That is, we should not in general assume localizability (i.e., that the Poisson brackets of S_i with themselves are zero), nor should we assume that the equations of motion are second order in time derivatives. So let us instead start with the most general Poisson tensor

$$\Lambda = \epsilon_{ijk} F^k(\vec{S}) \frac{\partial}{\partial S_i} \wedge \frac{\partial}{\partial S_j}. \quad (22)$$

Then, the Jacobi identity requires that $\vec{F} \cdot (\vec{\nabla} \times \vec{F}) = 0$, where the gradient $\vec{\nabla}$ is with respect to $\vec{S}$. If we introduce the 1-form $\alpha = F^i dS_i$, the condition reads: $\alpha \wedge d\alpha = 0$. Locally, α can then be written $\alpha = f_0 df_1$, f_0 and f_1 being functions of S_i and we have

$$\{S_i, S_j\} = f_0(\vec{S}) \epsilon_{ijk} \frac{\partial f_1}{\partial S_k}.$$

As a result f_1 has zero Poisson bracket with all S_i and it therefore defines a classical Casimir function. If we now require that the dynamics be such that there is a Hamilton function $H = H(S_i)$, then f_1 is also a constant of the motion and therefore

$$\dot{S}_i \frac{\partial f_1}{\partial S_i} = 0. \quad (23)$$

For dynamics which is non trivial (ie., $\dot{S}_i \neq 0$), eq. (23) implies a condition on f_1 . To see what this is, let us now specialize to the dynamical system of a spin interacting with a magnetic field $\vec{B}$. The standard equation of motion for such a system gives the precession of the spin S_i ,

$$\dot{S}_i = \mu \epsilon_{ijk} S_j B^k, \quad (24)$$

where μ denotes the magnetic moment. From (23) and (24) one finds that

$$\frac{\partial f_1}{\partial S_k} = \rho_1 S_k + \sigma_1 B^k, \quad (25)$$

and so f_1 must be of the form

$$f_1 = f_1(\vec{S}^2, \vec{S} \cdot \vec{B}). \quad (26)$$

The Hamilton function $H(S_i)$ must have an analogous form. To see this, substitute (24) into $\dot{S}_i = \{S_i, H\}$ to get:

$$f_0(\vec{S}) \epsilon_{ijk} \frac{\partial H}{\partial S_j} \frac{\partial f_1}{\partial S_k} = \mu \epsilon_{ijk} S_j B^k. \quad (27)$$

It follows that

$$\frac{\partial H}{\partial S_k} = \rho_H S_k + \sigma_H B^k,$$

and thus that the most general solution for H is of the form

$$H = H(\vec{S}^2, \vec{S} \cdot \vec{B}). \quad (28)$$

From (27) we get the following restriction on the derivatives of f_1 and H :

$$f_0(\rho_H \sigma_1 - \rho_1 \sigma_H) = \mu. \quad (29)$$

The functions f_1 and H given by (26) and (28) subject to the constraint (29) define a family of Poisson brackets and Hamiltonian functions all of which lead to the same equations of motion (24) for the spin. They correspond in general to canonically inequivalent descriptions for the dynamics. We first review the standard Hamiltonian description.

Standard Formalism

The standard canonical formalism for a spinning particle is recovered when we write

$$f_0 = \frac{1}{2}, \quad f_1 = \vec{S}^2, \quad H = -\mu \vec{S} \cdot \vec{B}.$$

For this choice the Poisson bracket algebra (22) corresponds to the $\mathfrak{su}(2)$ Lie algebra. Reducible representations of $\mathfrak{su}(2)$ then arise upon performing a canonical quantization of this system. Alternatively, to obtain irreducible representations for $\mathfrak{su}(2)$ one must impose a constraint on the variables S_i . Namely, one must require that the classical Casimir function f_1 takes on certain constant values (and as a result S_i will then span S^2). [Ba82]

Alternative Formalisms

There exist alternative Hamiltonian formulations associated which yield the classical equation of motion (24), which are canonically inequivalent to the standard one. One such formulation results from the choice [So92], [St93]

$$f_0 = \frac{1}{2}, \quad f_1 = S_1^2 + S_2^2 + \frac{1}{2\lambda} \left[\frac{\cosh 2\lambda S_3}{\sinh \lambda} - \frac{1}{\lambda} \right] \quad (30)$$

and

$$H = -\mu \lambda S_3, \quad (31)$$

where with no loss of generality we have taken the magnetic field to be in the third direction. Here λ is a 'deformation' parameter as the standard formalism is recovered when $\lambda \rightarrow 0$. For non zero λ , (31) leads to

$$\{S_2, S_3\} = S_1, \quad \{S_3, S_1\} = S_2, \quad (32)$$

$$\{S_1, S_2\} = \frac{1}{2} \frac{\sinh 2\lambda S_3}{\sinh \lambda}. \quad (33)$$

Unlike in the standard formulation, the Poisson bracket algebra of observables S_i does not correspond to $\mathfrak{su}(2)$ or any Lie algebra.

These brackets are a classical realization of the quantum commutation relations for generators of the $U_q(\mathfrak{sl}(2))$ Hopf algebra [Ta89]. Let us denote the latter generators by I_+ , I_- and I_0 . Their commutation relations are

$$[I_0, I_{\pm}] = \pm I_{\pm}, \quad [I_+, I_-] = \frac{q^{2I_0} - q^{-2I_0}}{q - q^{-1}}. \quad (34)$$

These commutation relations reduce to the $\mathfrak{su}(2)$ Lie algebra relations in the limit $q \rightarrow 1$. To obtain the above classical system, one can replace the quantum operators I_+ , I_- and I_0 by $S_1 + iS_2$, $S_1 - iS_2$ and S_3 , respectively, q by $\exp \lambda$, and the commutation relations (34) by i times Poisson brackets.

There is another choice for f_0 and f_1 which is known to correspond to the classical limit of the $U_q(\mathfrak{sl}(2))$ Hopf algebra. It is:

$$f_0 = \frac{\lambda}{4} S_3, \quad f_1 = (S_1)^2 + (S_2)^2 + (S_3)^2 + (S_3)^{-2}. \quad (35)$$

It leads to the following brackets for S_i :

$$\begin{aligned} \{S_2, S_3\} &= \frac{\lambda}{2} S_1 S_3, \quad \{S_3, S_1\} = \frac{\lambda}{2} S_2 S_3, \\ \{S_1, S_2\} &= \frac{\lambda}{2} \left((S_3)^2 - (S_3)^{-2} \right), \end{aligned} \quad (36)$$

or if we define $S_{\pm} = S_1 \pm iS_2$, these relations can be expressed by

$$\{S_{\pm}, S_3\} = \pm \frac{i\lambda}{2} S_{\pm} S_3, \quad \{S_+, S_-\} = -i\lambda \left((S_3)^2 - (S_3)^{-2} \right). \quad (37)$$

Now in order to obtain the equations of motion for the spin (24) one can choose the Hamiltonian

$$H = -\frac{2\mu B}{\lambda} \ln S_3,$$

where again we have chosen the magnetic field to be along the third direction.

To show that the Poisson structure defined in (37) has something to do with the classical limit of the $U_q(\mathfrak{sl}(2))$ Hopf algebra generated by I_+ , I_- and I_0 , let us define new operators S_+ , S_- and S_0 according to

$$S_{\pm} = -\sqrt{q}(q - q^{-1})I_{\pm}, \quad S_0 = q^{I_0}.$$

Then the commutation relations (34) can be expressed according to

$$S_3 S_{\pm} = q^{\pm 1} S_{\pm} S_3, \quad [S_+, S_-] = q(q - q^{-1}) \left((S_3)^2 - (S_3)^{-2} \right). \quad (38)$$

Next we introduce Planck's constant $\hbar$ by setting $q = \exp \frac{\lambda \hbar}{2}$ and expand (38) in powers of $\hbar$. The result is:

$$[S_3, S_{\pm}] = \pm \frac{\lambda \hbar}{2} S_{\pm} S_3 + O(\hbar^2), \quad [S_+, S_-] = \lambda \hbar \left((S_3)^2 - (S_3)^{-2} \right) + O(\hbar^2). \quad (39)$$

The lowest order terms in the commutation relations (39) for S_+ , S_- and S_0 correspond with $i\hbar$ times the Poisson brackets (37) of S_+ , S_- and S_0 and it is in this regard that (37) is the classical limit of $U_q(\mathfrak{sl}(2))$.

4 Combining Internal and External Structures

Here we adapt Feynman's procedure to examine the dynamics of particles which trace out trajectories in space-time, as well as in some internal space M . Physically, coordinates of the latter space can be associated with an extrinsic spin of the particle (as discussed in the preceding section) or with some internal degrees of freedom like isospin or color. In the following general treatment we let the manifold M be of arbitrary dimension. Further, the algebra of observables spanning M need not be associated with any Lie algebra (as was the case in the alternative formulations of spin given in the previous section). The treatment given here thus allows for the possibility of finding new kinds of particle dynamics. We recover some familiar systems upon specializing to the case where the Poisson bracket algebra for the internal variables corresponds to a Lie algebra; for instance, the Wong equations [Wo70]

describing the coupling of particles to a Yang-Mills field. We can also recover the magnetic dipole coupling of a spinning particle with a magnetic field when we take the Poisson bracket algebra on M to be $\mathfrak{su}(2)$. We will postpone the discussion of the general setting for these questions as well as the general results to §8. Of course the general treatment will include particles on curved space-time [Ta92]. Again in §11 the discussion will again be specialized to the case of M being a Lie group and a cotangent bundle of a group.

Free System

We first consider free motion on $\mathbb{R}^3 \times M$. The manifold M now denotes a d -dimensional internal space which we parametrize by I_a , $a = 1, \dots, d$. Free motion on $\mathbb{R}^3$ corresponds to

$$\ddot{x}^i = 0, \quad i = 1, 2, 3, \quad (40)$$

while on M one standardly writes

$$\dot{I}_a = 0. \quad (41)$$

We parametrize the corresponding phase space of the system by x^i , v^i and I_a . In writing down a Poisson description on $T\mathbb{R}^3 \times M$ we let

$$m\{x^i, v^j\} = \delta_{ij}; \quad \{I_a, I_b\} = C_{ab} \quad (42)$$

define the non zero Poisson brackets for the system, where C_{ab} may be any general function on $\mathbb{R}^3 \times M$. Notice that the zero brackets $\{x^i, x^j\} = \{x^i, I_a\} = 0$ reflect the localizability of the system and the simultaneous measurability of inner variables with external coordinates (see §8 for more details). From the Jacobi identity involving I_a , I_b and x^i , we get that $\{x^i, C_{ab}\} = 0$ and consequently C_{ab} can have no dependence on v^i .

The Poisson tensor defined by (42) reads as:

$$\Lambda = \frac{1}{m} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial v^i} + \frac{1}{2} C_{ab} \frac{\partial}{\partial I_a} \wedge \frac{\partial}{\partial I_b}. \quad (43)$$

The Jacobi identity $[\Lambda, \Lambda] = 0$, implies that $[\Lambda_I, \Lambda_I] = 0$, Λ_I being the second term in (43). As a result C_{ab} satisfies the following condition:

$$0 = C_{af} \frac{\partial C_{bc}}{\partial I_f} + C_{bf} \frac{\partial C_{ca}}{\partial I_f} + C_{cf} \frac{\partial C_{ab}}{\partial I_f}, \quad (44)$$

and Λ_I defines a Poisson tensor itself on $\mathbb{R}^3 \times M$. For the case where the Poisson brackets of I_a define a Lie algebra, we have $C_{ab} = c_{ab}{}^f I_f$, $c_{ab}{}^f$ being the corresponding structure constants. Then the Jacobi identity is satisfied due to

$$0 = c_{af}{}^d c_{bc}{}^f + c_{bf}{}^d c_{ca}{}^f + c_{cf}{}^d c_{ab}{}^f.$$

Finally, the free motion (40) and (41) results from (42) along with the Hamilton function

$$H = \frac{m}{2} v^2.$$

Interactions

Now we ask: What kind of interaction can be introduced into the system by generalizing the Poisson brackets and Hamilton function? [St93] In order to better define this question we generalize the assumptions (9-13) of Feynman. We assume equations of motion of the form

$$\begin{aligned}\dot{x}^i &= v^i, \\ \dot{v}^i &= \frac{1}{m} F^i(\vec{x}, \vec{v}, I), \\ \dot{I}_a &= f_a(\vec{x}, \vec{v}, I),\end{aligned}\quad (45)$$

while for the Poisson brackets we take

$$\{x^i, x^j\} = 0, \quad \{x^i, I_a\} = 0, \quad m\{x^i, v^j\} = \delta^{ij}, \quad (46)$$

and

$$\{I_a, I_b\} = C_{ab}, \quad m^2\{v^i, v^j\} = \mathcal{F}^{ij}, \quad m\{v^i, I_a\} = \mathcal{A}_a^i, \quad (47)$$

where the functions F^i , f_a , $\mathcal{F}^{ij}$, $\mathcal{A}_a^i$ and C_{ab} are initially undetermined. This corresponds to defining a Poisson tensor which is not a direct sum, since it will contain cross terms. After the brackets have been determined, we once again demand that the equations of motion are Hamiltonian. Free dynamics is recovered if all functions except for C_{ab} vanish.

It is clear that we have two problems here. The first one is to find conditions on the functions C_{ab} , $\mathcal{F}^{ij}$ and $\mathcal{A}_a^i$ such that the Poisson brackets satisfy the Jacobi identity. The second one is to ensure that the functions are compatible with the requirement that the dynamics defined by (45) be Hamiltonian.

We begin by taking the time derivative of (46). This can be done under the assumption that the evolution equation is Hamiltonian with respect to the Poisson tensor defined by the Poisson brackets (46) and (47). Upon using (45) we get

$$\begin{aligned}\{x^i, v^j\} + \{v^i, x^j\} &= 0, \\ \{v^i, I_a\} + \{x^i, f_a\} &= 0, \\ m\{v^i, v^j\} + \{x^i, F^j\} &= 0.\end{aligned}\quad (48)$$

Of course the first equation is identically satisfied. From the second equation we notice that if f_a depends non trivially on velocity we need $\mathcal{A}_a^i \neq 0$. Similarly, from the third equation, if F^i depends non trivially on velocity we need $\mathcal{F}^{ij} \neq 0$. Now using the Jacobi identity one finds that the coordinates x^i have zero brackets with the functions C_{ab} , $\mathcal{A}_a^i$ and $\mathcal{F}^{ij}$, and consequently none of these functions can have a dependence on velocity. The Poisson tensor will then be said to be velocity independent. (This will be discussed further in §6.)

For $\mathcal{F}^{ij} \neq 0$ or $\mathcal{A}_a^i \neq 0$, we must modify the free Poisson tensor Λ (43) to

$$\Lambda = \frac{1}{m} \mathcal{D}_i \wedge \frac{\partial}{\partial v^i} + \frac{1}{2} C_{ab} \frac{\partial}{\partial I_a} \wedge \frac{\partial}{\partial I_b} + \frac{1}{2m^2} \mathcal{F}^{ij} \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial v^j}, \quad (49)$$

where

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} - \mathcal{A}_a^i(\vec{x}, I) \frac{\partial}{\partial I_a}.$$

The Hamiltonian vector field associated with dx^i is $\Lambda(dx^i) = \partial/\partial v^i$. Being Hamiltonian then implies that $\mathcal{L}_{\partial/\partial v^i} \Lambda = 0$, and we once again have the result that C_{ab} , $\mathcal{A}_a^i$ and $\mathcal{F}^{ij}$ can have no v^i dependence.

This result further implies that functions F^i and f_a appearing in the equations of motion (45) are at most first order in the velocity. From (48)

$$F^i = \mathcal{F}^{ij}(\vec{x}, I) v^j + \mathcal{F}^{i0}(\vec{x}, I), \quad f_a = -\mathcal{A}_a^i(\vec{x}, I) v^i - \mathcal{A}_a^0(\vec{x}, I). \quad (50)$$

Here we have introduced new functions $\mathcal{F}^{i0}$ and $\mathcal{A}_a^0$ which are by definition independent of velocity.

By expanding the Jacobi identity

$$[\Lambda, \Lambda] = 0, \quad (51)$$

we obtain additional conditions on C_{ab} , $\mathcal{A}_a^i$ and $\mathcal{F}^{ij}$ and consequently additional conditions on the functions F^i and f_a . From (51) we obtain

$$\frac{1}{m^2} [\mathcal{D}_i \wedge \frac{\partial}{\partial v^i}, \mathcal{D}_j \wedge \frac{\partial}{\partial v^j}] + [\Lambda_I, \Lambda_I] + 2[\mathcal{D}_i \wedge \frac{\partial}{\partial v^i}, \Lambda_F] + \frac{2}{m} [\mathcal{D}_i \wedge \frac{\partial}{\partial v^i}, \Lambda_I] + 2[\Lambda_I, \Lambda_F] = 0. \quad (52)$$

Λ_I and Λ_F denote the second and third terms, respectively, in Λ (49). Using

$$[\mathcal{D}_i, \frac{\partial}{\partial v^i}] = 0,$$

we can reduce (52) to

$$\frac{1}{m^2} [\mathcal{D}_i, \mathcal{D}_j] \wedge \frac{\partial}{\partial v^j} \wedge \frac{\partial}{\partial v^i} + [\Lambda_I, \Lambda_I] + 2[\mathcal{D}_i, \Lambda_F] \wedge \frac{\partial}{\partial v^i} + \frac{2}{m} [\mathcal{D}_i, \Lambda_I] \wedge \frac{\partial}{\partial v^i} + 2[\Lambda_I, \Lambda_F] = 0. \quad (53)$$

Collecting our results we find the following conditions on the functions $\mathcal{F}_{ij}$, $\mathcal{A}_{ia}$ and C_{ab} :

$$\begin{aligned}\mathcal{D}_i C_{ab} &= -C_{aj} \frac{\partial \mathcal{A}_b^j}{\partial I_f} + C_{bf} \frac{\partial \mathcal{A}_a^f}{\partial I_f}, \\ \mathcal{D}_{[i} \mathcal{A}_{j]a} &= C_{ab} \frac{\partial \mathcal{F}^{ij}}{\partial I_b}, \\ \mathcal{D}_{[i} \mathcal{F}_{jk]} &= 0,\end{aligned}\quad (54)$$

in addition to the condition (44) on C_{ab} . Of the three conditions in (54), the third is the most familiar as it resembles the Bianchi identity of gauge theories. Its connection to the Bianchi identity will be clarified in the examples which follow.

The conditions (54) more generally apply if we replace the spatial indices $i, j, \dots$, going from 1 to 3 by space-time indices $\mu, \nu, \dots$, going from 0 to 3. As usual 0 denotes time, with $\mathcal{F}^{00} = -\mathcal{F}^{0i}$ and $\mathcal{A}_a^0$ defined in (50). The more general result follows upon taking the time derivative of Poisson brackets (47). For example, the time derivative of $\{I_a, I_b\} = C_{ab}$ yields

$$\{\dot{I}_a, I_b\} + \{I_a, \dot{I}_b\} = \frac{\partial C_{ab}}{\partial I_f} \dot{I}_f + \frac{\partial C_{ab}}{\partial x^i} \dot{x}^i. \quad (55)$$

Then after substituting (45) and (47) into (55) one obtains an equation which is at most linear in velocity v^i . The linear terms however cancel due to the first equation in (54) and we are left with

$$\mathcal{D}_0 C_{ab} = -C_{aj} \frac{\partial \mathcal{A}_b^j}{\partial I_f} + C_{bf} \frac{\partial \mathcal{A}_a^f}{\partial I_f}. \quad (56)$$

Eq. (56) is the first equation in (54) with i replaced by 0. Similarly, by taking the time derivative of $m\{v^i, I_a\} = \mathcal{A}_a^i$ and $m^2\{v^i, v^j\} = \mathcal{F}^{ij}$ one obtains

$$\mathcal{D}_{[0}\mathcal{A}_{i]a} = C_{ab}\frac{\partial\mathcal{F}^{0i}}{\partial I_b} \quad (57)$$

and

$$\mathcal{D}_{[0}\mathcal{F}_{ij]} = 0, \quad (58)$$

respectively, thus proving the result.

In the above system the particle interacts with the “fields” $C_{ab}(x, I)$, $\mathcal{A}_a^\mu(x, I)$ and $\mathcal{F}^{\mu\nu}(x, I)$. Since these fields are functions on the internal space as well as on space-time, they resemble Kaluza-Klein fields. To reduce the theory to one where the coupling is to fields defined on space-time only, it will be necessary to make certain ansätze for the form of the functions $C_{ab}(x, I)$, $\mathcal{A}_a^\mu(x, I)$ and $\mathcal{F}^{\mu\nu}(x, I)$. These ansätze must be consistent with conditions (44) and (54). In the following we give examples of such ansätze which lead to some standard particle interactions.

Wong Equations

Here we let the Poisson bracket algebra for I_a correspond to a Lie algebra; i.e., $C_{ab} = c_{ab}{}^f I_f$, $c_{ab}{}^f$ being structure constants. As stated earlier, condition (44) is then identically satisfied. The standard coupling of a particle to a Yang-Mills field is recovered upon choosing

$$\mathcal{A}_a^\mu(x, I) = c_{ab}{}^f I_f A^{b\mu}(x), \quad \mu = 0, 1, 2, 3, \quad (59)$$

where $A^{b\mu}$ is independent of the internal variable I_a and is to be identified with the Yang-Mills potential. This choice identically satisfies the first equation in (54). Upon substituting (59) into the second equation of (54), we get

$$\frac{\partial\mathcal{F}^{\mu\nu}}{\partial I_a} = \frac{\partial A^{a\nu}}{\partial x_\mu} - \frac{\partial A^{a\mu}}{\partial x_\nu} + c_{bf}{}^a A^{b\mu} A^{f\nu}. \quad (60)$$

The right hand side of (60) is independent of the internal variables I_a and is to be identified with the Yang-Mills field strength $F^{a\mu\nu}(x)$. Therefore,

$$\mathcal{F}^{\mu\nu}(x, I) = F^{a\mu\nu}(x) I_a, \quad (61)$$

plus terms which are independent of the internal variables. Finally, the third equation in (54) corresponds to the usual Bianchi identities, while the equations of motion following from (50) are the usual Wong particle equations,

$$\begin{aligned} m\ddot{x}^\mu &= F^{a\mu\nu}(x) I_a \dot{x}_\nu, \\ \dot{I}_a &= -c_{ab}{}^f A^{b\mu}(x) I_f \dot{x}_\mu. \end{aligned} \quad (62)$$

These equations of motion are obtainable using the Hamiltonian

$$H = \frac{m}{2} \dot{v}^2 - A^{a0}(x) I_a. \quad (63)$$

The above system of equations can be generalized to the case where I_a does not span a Lie algebra. In this case (59) is replaced by

$$\mathcal{A}_a^\mu(x, I) = C_{ab} A^{b\mu}(x), \quad (64)$$

Eq. (64) satisfies the first condition in (54) provided C_{ab} satisfies (44) and it is independent of x . Upon substituting (64) into the second equation of (54), we get

$$C_{ab}(I) \left(\frac{\partial\mathcal{F}^{\mu\nu}}{\partial I_a} - \frac{\partial A^{a\nu}}{\partial x_\mu} + \frac{\partial A^{a\mu}}{\partial x_\nu} - \frac{\partial C_{bf}}{\partial I_a} A^{b\mu} A^{f\nu} \right) = 0. \quad (65)$$

Ignoring the I_a -independent terms, (65) is solved by

$$\mathcal{F}^{\mu\nu}(x, I) = \left(\frac{\partial A^{a\nu}}{\partial x_\mu} - \frac{\partial A^{a\mu}}{\partial x_\nu} \right) I_a + C_{bf}(I) A^{b\mu}(x) A^{f\nu}(x). \quad (66)$$

Thus $\mathcal{F}^{\mu\nu}$ is not linear in I if C_{ab} is not. The third equation in (54) is identically satisfied for $\mathcal{A}_a^\mu$ and $\mathcal{F}^{\mu\nu}$ of the form (64) and (66), and thus all conditions (54) hold for this case. The equations of motion which follow from (64) and (66) are a generalization of the Wong particle equations (62) (although their physical meaning is not clear when I_a does not span a Lie algebra). These equations of motion are also obtainable from the Hamiltonian (63).

Magnetic Moment Coupling

We next show how the standard Hamiltonian formulation for a spinning particle interacting with a magnetic field can be obtained from the above treatment. For this we take to I_a denote spin. Its Poisson brackets should define an $\mathfrak{su}(2)$ Lie algebra on $\mathbf{R}^3$,

$$\{F, G\} = \epsilon^{abc} I_c \frac{\partial F}{\partial I_a} \frac{\partial G}{\partial I_b}, \quad (67)$$

and thus

$$C_{ab} = \epsilon_{abf} I^f. \quad (68)$$

Further we set $\mathcal{A}_a^i = 0$, from which it follows using (48) that I_a has no velocity dependence. Since $\mathcal{D}_i C_{ab} = 0$, the first condition in (54) is satisfied, while the second one implies that $\mathcal{F}^{ij}$ is independent of I_a . If we then set

$$\mathcal{F}^{ij} = \epsilon^{ijk} B^k, \quad (69)$$

the third equation in (54) tells us that $\vec{B}$ has zero divergence and hence can be identified with a magnetic field. It remains to specify $\mathcal{A}_a^0$ and $\mathcal{F}^{0i}$. They must be chosen to be consistent with conditions (56-58). We do not identify $\mathcal{A}_a^0$ with an electromagnetic potential but rather take

$$\mathcal{A}_a^0 = -\mu \epsilon_{abc} I_b B_c, \quad (70)$$

where μ represents the magnetic moment. Then (56) is identically satisfied while (57) gives

$$\mu \epsilon_{abc} I_b \partial_i B_c = -\epsilon_{abc} I_b \frac{\partial \mathcal{F}^{0i}}{\partial I_c} \quad (71)$$

and thus

$$\mathcal{F}^{0i} = -\mu I_c \partial_i B_c, \quad (72)$$

plus some function of $I_a I_a$. Here we see that $\mathcal{F}^{0i}$ is not interpreted as an electric field. Condition (58) is satisfied only for static magnetic fields.

Upon substituting the above results into the equations of motion (45) and (50) we recover the standard dynamics for a spinning particle,

$$m\ddot{x}^i = \epsilon^{ijk}\dot{x}^j B^k + \mu \partial_i B^j I_j, \quad (73)$$

$$\dot{I}_a = \mu \epsilon_{abc} I_b B_c. \quad (74)$$

These equations of motion are obtainable from the Hamilton function

$$H = \frac{m}{2} \dot{v}^2 - \mu B_a I_a. \quad (75)$$

The above dynamical system can be generalized by replacing the $\mathfrak{su}(2)$ Poisson bracket algebra for the spin by an arbitrary algebra. Thus instead of (68) we can take $C_{ab} = C_{ab}(I)$, where we assume that condition (44) holds. Here we still take $\mathcal{A}_a^i = 0$, from which it again follows that the first equation in (54) is satisfied, the second one implies that $\mathcal{F}^{ij}$ is independent of I_a , and the third equation gives that $\vec{B}$ has zero divergence. For $\mathcal{A}_a^0$, we can generalize (70) to

$$\mathcal{A}_a^0 = \mu C_{ab}(I) B_b, \quad (76)$$

which satisfies (56). Conditions (57) and (58) are once again fulfilled by (72) with static magnetic fields $\vec{B}$. Now the equation of motion (73) for the spatial coordinates is still valid,

$$\dot{I}_a = -\mu C_{ab}(I) B_b.$$

These equations are obtainable from the Hamilton function (75).

Part II

The Geometry of Poisson manifolds, tangent bundles, and Feynman's problem

5 Review of Poisson manifolds

In order to put the discussions of the previous two sections at a more fundamental level, we now review the theory of Poisson manifolds. We begin by examining the Schouten bracket.

Schouten bracket

If Λ is a Poisson tensor defined on a manifold M , the components Λ^{ab} defining the Poisson brackets are not completely arbitrary due to the Jacobi identity $[\Lambda, \Lambda] = 0$, which is equivalent to

$$\Lambda^{ad} \frac{\partial \Lambda^{bc}}{\partial \xi^d} + \Lambda^{cd} \frac{\partial \Lambda^{ab}}{\partial \xi^d} + \Lambda^{bd} \frac{\partial \Lambda^{ca}}{\partial \xi^d} = 0. \quad (77)$$

This condition was used extensively in the previous two sections. Conversely, if we are given some functions Λ^{ab} , then in every coordinate system satisfying the above mentioned relations it is possible to define an associated Poisson structure. The simplest case occurs when all the functions Λ^{ab} are constant. Another simple example occurs when the functions Λ^{ab} are linear in the coordinates ξ^a , $\Lambda^{ab} = f^{ab}_c \xi^c$. Then eq. (77) gives the conditions which define the structure constants f^{ab}_c of a Lie algebra. An example is given by the brackets (67) which gives the $\mathfrak{su}(2)$ Lie algebra on $\mathbb{R}^3$. It defines a matrix whose rank is 2 except at the origin, where obviously the rank is zero.

It is fundamental to notice that the coordinate expression for the Jacobi identity eq. (77) is not linear and therefore if Λ_1 and Λ_2 are Poisson tensors the sum $\Lambda_1 + \Lambda_2$ may be not a Poisson structure. On the other hand, if this does occur we say that Λ_1 and Λ_2 are compatible and that the Poisson tensor $\Lambda = \Lambda_1 + \Lambda_2$ is decomposable, as for example in eq. (43). An interesting way of expressing this property is by means of a commutator $[\cdot, \cdot]$ introduced by Schouten [Sc53] and used by Lichnerowicz in this context [Li77]. In fact it was Lichnerowicz who showed that Jacobi's identity is given by $[\Lambda, \Lambda] = 0$. We recall that the Schouten bracket is the unique extension of the Lie bracket of vector fields to the exterior algebra of multivector fields, making it into a graded Lie algebra (the grading in this algebra given by the ordinary degree of the multivectors minus one). In fact, given a multivector V , the linear operator $[V, \cdot]$ defines a derivation on the exterior algebra of multivector fields on M . The degree of $[V, \cdot]$ equals the ordinary degree of V . So, for any triplet (U, V, W) of multivectors,

$$[U, V \wedge W] = [U, V] \wedge W + (-1)^{u(v+1)} V \wedge [U, W], \quad (78)$$

u and v being the ordinary degree of U and V respectively. In the particular case of the wedge product of vector fields,

$$[X \wedge Y, U \wedge W] = [X, U] \wedge Y \wedge W + X \wedge [Y, U] \wedge V + Y \wedge [X, W] \wedge U + X \wedge U \wedge [Y, W]. \quad (79)$$

Therefore if $V = X \wedge Y$ is a monomial bivector, then

$$[V, V] = 2X \wedge Y \wedge [X, Y]. \quad (80)$$

Now, it is clear that if Λ_1 and Λ_2 are Poisson structures, then $\Lambda_1 + \Lambda_2$ is a Poisson structure if and only if $[\Lambda_1 + \Lambda_2, \Lambda_1 + \Lambda_2] = 0$, i.e., iff $[\Lambda_1, \Lambda_2] = 0$. Provided this condition is met, any linear combination of Λ_1 and Λ_2 with real coefficients is also a Poisson structure.

Another remarkable property is that if (M, Λ) is a Poisson manifold, then the correspondence

$$F \mapsto X_F = -\Lambda(dF) = \Lambda^{ab} \frac{\partial F}{\partial \xi^a} \frac{\partial}{\partial \xi^b}$$

is a Lie algebra anti-homomorphism. Obviously it is $\mathbb{R}$ -linear and if F, G and H are arbitrary functions in M , then

$$\begin{aligned} [X_F, X_G]H &= X_F(X_G H) - X_G(X_F H) \\ &= \{X_G H, F\} - \{X_F H, G\} = \{\{H, G\}, F\} - \{\{H, F\}, G\}. \end{aligned}$$

Using Jacobi's identity it is easy to see that the preceding relation can be written as

$$\begin{aligned} [X_F, X_G]H &= \{\{H, G\}, F\} + \{\{F, H\}, G\} \\ &= -\{\{G, F\}, H\} = -\{H, \{F, G\}\} = -X_{\{F, G\}}H, \end{aligned}$$

from which we obtain $[X_F, X_G] = -X_{\{F, G\}}$.

Let us recall that if $(\mathcal{L}, [\cdot, \cdot])$ is a Lie algebra, then for each $x \in \mathcal{L}$ the map $\text{adx} : \mathcal{L} \rightarrow \mathcal{L}$ defined by $\text{adx}(y) = [x, y]$, is a derivation. These derivations adx are said to be inner derivations. However there can be derivations of $(\mathcal{L}, [\cdot, \cdot])$ that are not inner. For the case of Poisson manifolds, for any function F , the inner derivation $\text{ad}F$ is $-X_F$.

Poisson maps

Definition 2 A map $\phi: M_1 \rightarrow M_2$ between Poisson manifolds $(M_1, \{\cdot, \cdot\}_1)$ and $(M_2, \{\cdot, \cdot\}_2)$ is said to be a Poisson map when

$$\{F \circ \phi, G \circ \phi\}_1 = \{F, G\}_2 \circ \phi$$

for any couple of functions F, G in M_2 .

In particular it is easy to see that if $\phi: M_1 \rightarrow M_2$ is a diffeomorphism, then ϕ is a Poisson map if and only if $\phi_* \Lambda_1 = \Lambda_2$. The flow ϕ_t of a vector field X in M is made of Poisson maps iff $\mathcal{L}_X \Lambda = 0$, because of

$$\phi_{t*} \mathcal{L}_X \Lambda = \frac{d}{dt} \phi_{t*} \Lambda.$$

A vector field X will be said to be canonical if

$$\mathcal{L}_X \Lambda = 0. \quad (81)$$

Canonical vector fields are characterized as follows: A vector field X in M is canonical iff

$$X\{F, G\} = \{XF, G\} + \{F, XG\}, \quad (82)$$

for any pair of functions F, G in M . Notice that in this case the vector field X is a derivation of the Lie algebra structure defined on M by $\{\cdot, \cdot\}$. Eq. (82) follows from

$$(\mathcal{L}_X \Lambda)(dF \wedge dG) = X(\Lambda(dF \wedge dG)) - \Lambda(\mathcal{L}_X dF \wedge dG) + \Lambda(dF \wedge \mathcal{L}_X dG),$$

namely,

$$(\mathcal{L}_X \Lambda)(dF \wedge dG) = X\{F, G\} - \{XF, G\} - \{F, XG\}.$$

In particular, when X_H is a Hamiltonian vector field, it is canonical, because of the relation

$$\begin{aligned} (\mathcal{L}_{X_H} \Lambda)(dF \wedge dG) &= \mathcal{L}_{X_H} \{F, G\} - \Lambda(d(X_H F) \wedge dG) - \Lambda(dF \wedge d(X_H G)) \\ &= \{\{F, G\}, H\} - \{\{F, H\}, G\} - \{F, \{G, H\}\}, \end{aligned} \quad (83)$$

and then $\mathcal{L}_{X_H} \Lambda = 0$ as a consequence of the Jacobi identity. Obviously the same proof works for a closed 1-form θ , for which $\mathcal{L}_{X_\theta} \Lambda = 0$ with $X_\theta = -\Lambda(\theta)$. However, not every canonical vector field is a Hamiltonian one. As an example, let us consider the Poisson structure in $\mathbb{R}^3$, parametrized by x^1, x^2, x^3 , defined by

$$\{F, G\} = \frac{\partial F}{\partial x^2} \frac{\partial G}{\partial x^1} - \frac{\partial F}{\partial x^1} \frac{\partial G}{\partial x^2}.$$

The tensor Λ defining the Poisson structure is $\Lambda = \partial/\partial x^2 \wedge \partial/\partial x^1$. The vector field $Y = \partial/\partial x^3$ is canonical, but it cannot be Hamiltonian, for if a corresponding Hamiltonian H did exist, then $Y(F) = \{F, H\}$, for any function F . Taking $F = x^3$ we will find that x^3 being a Casimir function, leads to a contradiction: $1 = Y(x^3) = \{x^3, H\} = 0$.

We can also consider the more general case of an arbitrary 1-form α in M . If we denote by X_α the vector field: $-\Lambda(\alpha)$. Then taking into account that for any pair of functions $F, G \in \mathcal{F}(M)$,

$$(\mathcal{L}_{X_\alpha} \Lambda)(dF \wedge dG) = X_\alpha \{F, G\} - \{X_\alpha F, G\} - \{F, X_\alpha G\},$$

and

$$\Lambda(dF \wedge \alpha) = -X_\alpha F = -\alpha(X_F),$$

we see that

$$(\mathcal{L}_{X_\alpha} \Lambda)(dF \wedge dG) = X_\alpha \{F, G\} - X_G(X_\alpha F) + X_F(X_\alpha G).$$

Also if we recall that

$$\alpha([X_F, X_G]) = -\alpha(X_{\{F, G\}}) = -X_\alpha \{F, G\},$$

then

$$(\mathcal{L}_{X_\alpha} \Lambda)(dF \wedge dG) = -\alpha([X_F, X_G]) + X_F(\alpha(X_G)) - X_G(\alpha(X_F)) = (d\alpha)(X_F \wedge X_G).$$

Then we see that X_α is a derivation iff $d\alpha$ is closed along Hamiltonian vector fields. Thus for generic Poisson manifolds, there will be derivations that will not be locally Hamiltonian vector fields.

Definition 3 A submanifold S of M endowed with a Poisson structure is called a Poisson submanifold of M when $i: S \rightarrow M$ is a Poisson map.

If S is a Poisson submanifold, the Hamiltonian vector fields in M are tangent to S . In fact, let ϕ be a constraint function for S , i.e., it vanishes onto S , $\phi \circ i = 0$. Then, for every $F \in \mathcal{F}(M)$,

$$\{F, \phi\} \circ i = \{F \circ i, \phi \circ i\} = 0,$$

and hence $0 = \{F, \phi\} \circ i = -X_F(\phi) \circ i$, which means that the restriction of X_F onto S is tangent to S .

Conversely, if $i: S \rightarrow M$ is a submanifold such that the Hamiltonian vector fields in M are tangent to S , then we can define a Poisson structure in S in such a way that it is a Poisson submanifold of M . It suffices to define the Poisson bracket of $F, G \in \mathcal{F}(S)$, by

$$\{F, G\} = \{\bar{F}, \bar{G}\} \circ i,$$

where $\bar{F}, \bar{G}$ are arbitrary extensions of F and G respectively, i.e., $F = \bar{F} \circ i$, $G = \bar{G} \circ i$.

We remark that in the computation of $\{F, G\} = X_G F$ only the values of F along the integral curves of X_G , which lie in S by the tangency condition, are relevant. The Poisson bracket is then well defined, as it does not depend on the choice of the representative $\bar{G}$, nor does it depend on the representative of $\bar{F}$.

Regular Poisson manifolds and symplectic structures

The correspondence $F \mapsto X_F$ is equivalent to giving an $\mathbb{R}$ -linear map $\hat{\Lambda}(m) : T_m^* M \rightarrow T_m M$ which preserves the base. This map induces a map between the set of sections, from 1-forms to vector fields on M as follows: $\Lambda(\alpha, \beta) = \langle \beta, \hat{\Lambda}(\alpha) \rangle$, α and β being 1-forms on M . The rank of $\hat{\Lambda}(m)$ at the point m will be called the *rank of the Poisson structure* at the point m . Usually we will not distinguish Λ from $\hat{\Lambda}$.

Let us now assume that the Poisson structure is nondegenerate, namely, $\hat{\Lambda}^{-1}$ exists. It allows for the identification of $T_m M$ with $T_m^* M$ at every point $m \in M$. It allows for similar identifications for higher order contravariant and covariant tensors. The 2-form ω corresponding to Λ , i.e., $\omega = \Lambda^{-1}$, is given by

$$\omega(X, Y) = \Lambda(\hat{\Lambda}^{-1}(X), \hat{\Lambda}^{-1}(Y)), \quad (84)$$

and then the Poisson bracket becomes

$$\{F, G\} = \omega(\Lambda(dF), \Lambda(dG)). \quad (85)$$

The Jacobi identity for the Poisson structure implies that ω is closed. In fact, taking into account that for any triplet of functions $F, G, H \in \mathcal{F}(M)$,

$$X_F(\omega(X_G, X_H)) = X_F\{G, H\} = \{\{G, H\}, F\},$$

and that $\omega([X_F, X_G], X_H)$ can be rewritten as

$$\omega([X_F, X_G], X_H) = \omega(X_{\{G, F\}}, X_H) = \{\{G, F\}, H\},$$

(along with the corresponding expressions obtained by cyclicly permuting the indices), we will find that

$$d\omega(X_F, X_G, X_H) = 2(\{\{G, H\}, F\} + \{\{H, F\}, G\} + \{\{F, G\}, H\}).$$

Therefore, as a local basis for the space of vector fields on M can be built from Hamiltonian vector fields, the Jacobi identity for the Poisson bracket is equivalent to ω to be closed.

We recall that (M, ω) is a symplectic manifold if M is a differentiable manifold and ω is a nondegenerate 2-form which satisfies $d\omega = 0$ (that is, ω is a closed 2-form). Recall

that the nondegeneracy of ω is equivalent, by definition, to the property that the maps $\hat{\omega}_m T_m M \rightarrow T_m^* M$, given by:

$$(\hat{\omega}_m(v), v') = \omega_m(v, v'), \quad v, v' \in T_m M,$$

have maximal rank. They induce a mapping between sections which, with a slight abuse of notation, we will also write $\hat{\omega}$ or again ω . Then a non-degenerate Poisson structure Λ is equivalent to the symplectic structure ω so defined [Jo64]. Conversely, given a symplectic structure ω , the regularity of the map $\hat{\omega}$ may be used to define a skewsymmetric contravariant tensor Λ by means of

$$\Lambda(\alpha, \beta) = \omega(\hat{\omega}^{-1}(\alpha), \hat{\omega}^{-1}(\beta)),$$

where α and β are 1-forms on M . In particular, when $\alpha = dF$ and $\beta = dG$ we again find the expression

$$\{F, G\} = \Lambda(dF, dG) = \omega(\hat{\omega}^{-1}(dF), \hat{\omega}^{-1}(dG))$$

for the Poisson bracket defined by the symplectic structure, and likewise the expression for the Hamiltonian vector field X_G .

The point remarked here is that a symplectic manifold is nothing but a maximal constant rank Poisson manifold. That is, if $\{F, G\} = 0$ for any function G , then the function F must be locally constant. On the other hand, for the case of a degenerate Poisson structure on a n dimensional manifold M , there will exist (at least locally defined) nonconstant Casimir functions $\mathcal{C}$, such that $\{\mathcal{C}, G\} = 0$ for any $G \in \mathcal{F}(M)$. Actually, the image of Λ defines an involutive distribution, and therefore integrable. If the rank of Λ is r in a neighborhood of the point $m \in M$, then it is possible to choose adapted coordinates in a neighborhood of the point $m \in M$, $\{y^1, \dots, y^n\}$, in such a way that $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^r}\}$ generate the distribution while $\{y^{r+1}, \dots, y^n\}$ are constants along the maximal integral submanifolds of the distribution and therefore they give locally defined Casimir functions.

Finally, we also recall that in case of symplectic structures (M, ω) , the condition $\mathcal{L}_X \Lambda = 0$ is equivalent to $\mathcal{L}_X \omega = 0$. This is also equivalent to $i_X \omega$ being a closed 1-form. Hamiltonian vector fields correspond to exact 1-forms, $i_{X_H} \omega = dH$, and those X 's for which $i_X \omega$ is closed are said to be locally Hamiltonian.

6 The Lagrangian setting

If the carrier space of a symplectic manifold is the tangent bundle of some base manifold Q , the configuration space, we can ask whether ω is of the Lagrangian type. We refer to [Mo90] for a review of the Lagrangian formalism on a tangent bundle. Here we just recall a few facts.

The geometrical approach for time-independent Lagrangian systems makes use of a differentiable manifold Q as the configuration space of the mechanical system and its tangent bundle $\tau : TQ \rightarrow Q$, as the velocity phase space (for notation see e.g. [Cr83]). This vector bundle has a canonical (1,1)-tensor field S called the vertical endomorphism, which defines the natural almost tangent structure on TQ . Its action on a vector tangent to TQ corresponds to first projecting onto the base Q and then lifting it vertically to the original point. In terms of natural coordinates (q^i, v^i) of the tangent bundle TQ , S is given by

$$S = \frac{\partial}{\partial v^i} \otimes dq^i.$$

This vertical endomorphism S satisfies $S^2 = 0$, and moreover, it may also be proved that the Nijenhuis tensor $N_S = [S, S]$ vanishes.

We can also consider the action of S on 1-forms α in TQ by duality, $(S^*\alpha)(X) = \langle \alpha, S(X) \rangle$. In particular, $S^*(dq^i) = 0$ and $S^*(dv^i) = dq^i$.

We recall that 1-forms β in TQ whose contractions with vertical vectors on TQ are zero, or in an equivalent way, $S^*(\beta) = 0$, are called semibasic 1-forms. Their coordinate expressions are $\beta = f_i(q, v) dq^i$. Particular examples are basic forms, i.e., semibasic forms for which the Lie derivatives with respect to vertical vector fields are also null. In this last case the basic 1-form is the pull-back of a form in the base Q .

The tangent bundle structure is characterized not only by S but also by the Liouville vector field Δ in TQ generating dilations along the fibres. In natural coordinates of TQ we have the following expressions:

$$\Delta = v^i \frac{\partial}{\partial v^i}.$$

Given a Lagrangian function L in TQ the energy function is defined by $E_L = \Delta(L) - L$,

$$E_L = v^i \frac{\partial L}{\partial v^i} - L.$$

Another fundamental ingredient for the description of time evolution of a system is that of second order differential equations

$$\frac{d^2 q^i}{dt^2} = f^i \left(q^j, \frac{dq^j}{dt} \right)$$

or, equivalently, the corresponding first order system

$$\frac{dq^i}{dt} = v^i; \quad \frac{dv^i}{dt} = f^i(q, v). \quad (86)$$

This system determines the integral curves of the vector field Γ in TQ ,

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v) \frac{\partial}{\partial v^i}. \quad (87)$$

These vector fields are called second order differential equation fields, SODE for short, and they are particular instances of vector fields in TQ which are characterized by $S(\Gamma) = \Delta$.

We can define a derivation of degree zero i_S on tensors and forms on TQ by setting

$$i_S(f) = 0; \quad i_S(X) = S(X); \quad (i_S(\alpha))(X) = \alpha(S(X)), \quad (88)$$

for any function f , 1-form α , and vector field X in TQ . We extend it to higher degree tensors by requiring it to be a derivation of degree zero of the tensor algebra on TQ .

By restricting to forms, we recall that the graded commutator $[D_1, D_2] = D_1 D_2 - (-1)^{rs} D_2 D_1$, of a derivation D_1 of degree r and a derivation D_2 of degree s is a derivation of degree $r + s$. We also recall that a derivation of degree r of a graded algebra $\mathcal{A} = \bigoplus \mathcal{A}_k$ is a linear map $D : \mathcal{A} \rightarrow \mathcal{A}$ such that $D(a * b) = Da * b + (-1)^{kr} a * Db$, when a is homogeneous of degree k .

In the case we are considering,

$$d_S = [i_S, d] = i_S \circ d - d \circ i_S,$$

is a derivation of degree 1, while for each vector field X ,

$$\delta_X = [i_S, \mathcal{L}_X] = i_S \circ \mathcal{L}_X - \mathcal{L}_X \circ i_S$$

is a derivation of degree 0, and

$$[d, d_S] = d \circ d_S + d_S \circ d = d \circ (i_S \circ d - d \circ i_S) + (i_S \circ d - d \circ i_S) \circ d = 0.$$

Another remarkable property is that as a consequence of the vanishing of the Nijenhuis tensor of S , $N_S = 0$, we have $d_S \circ d_S = 0$.

In natural coordinates of the bundle TQ , the action of d_S on functions is given by

$$d_S F = \frac{\partial F}{\partial v^i} dx^i.$$

The 1-form $d_S F$ associated to the function F will be called the Cartan 1-form defined by F and will be denoted by θ_F . With any function L we can associate a 2-form

$$\omega_L = -dd_S L. \quad (89)$$

These 2-forms are said to be of the Lagrangian type and called the Cartan 2-form defined by L . We remark that

$$i_S \omega_L = 0. \quad (90)$$

In fact, it suffices to consider that

$$0 = (d_S \circ d_S)^2 f = (i_S \circ d \circ i_S \circ d + d \circ i_S \circ d \circ i_S - i_S \circ d \circ d \circ i_S - d \circ i_S \circ i_S \circ d) f, \quad (91)$$

and therefore

$$0 = i_S \circ d \circ i_S \circ df. \quad (92)$$

A Lagrangian function L is said to be regular when ω_L is of maximal rank. For dynamical systems described by regular Lagrangians, the dynamical evolution is given by the SODE Γ solution of

$$i_\Gamma \omega_L = dE_L,$$

or equivalently

$$\mathcal{L}_\Gamma \theta_L = dL. \quad (93)$$

Let us now recall the fundamental theorem on the characterization of 2-forms ω of the Lagrangian type ω_L [Mo90]:

Theorem 1 *Let ω be a closed form in a tangent bundle TQ . Then, if $i_S \omega = 0$, there will exist locally a function L in TQ such that $\omega = \omega_L$.*

Proof: If the coordinate expression for the closed form ω is given by

$$\omega = F_{ij} dq^i \wedge dq^j + G_{ij} dq^i \wedge dv^j + H_{ij} dv^i \wedge dv^j,$$

then the condition $i_S\omega = 0$ implies that $H_{ij} = 0$ and the symmetry condition $G_{ij} = G_{ji}$, and therefore

$$\omega = F_{ij} dq^i \wedge dq^j + G_{ij} dq^i \wedge dv^j.$$

This means that we can choose a semibasic 1-form $\theta = A_i(q, v) dq^i$ such that $\omega = -d\theta$. This is because given $\alpha = C_i(q, v) dq^i + B_i(q, v) dv^i$ such that $\omega = -d\alpha$, then $\partial B_i / \partial v^j = \partial B^j / \partial v^i$ and there will exist a function $f(q, v)$ such that $B_i = \partial f / \partial v^i$. Therefore it suffices to take $\theta = \alpha - df$ which is a semibasic 1-form $\theta = A_i(q, v) dq^i$.

Now, the symplectic 2-form will be

$$\omega = \frac{\partial A_i}{\partial v^j} dq^i \wedge dv^j + \frac{1}{2} \left(\frac{\partial A_i}{\partial q^j} - \frac{\partial A_j}{\partial q^i} \right) dq^i \wedge dq^j,$$

and then the symmetry for G_{ij} implies that $\partial A_i / \partial v^j - \partial A_j / \partial v^i = 0$. Thus there will exist a function $L(q, v)$ such that $A_i = \partial L / \partial v^i$ and then

$$\omega = -d \left(\frac{\partial L}{\partial v^i} dq^i \right),$$

i.e., the (local) Lagrangian L is such that $\omega = \omega_L$. $\square$

7 Feynman's problem revisited

We now examine Feynman's problem in the framework of tangent bundles. As was discussed in the previous section, this is the natural setting for Lagrangian mechanics. Thus it makes sense to try to describe all dynamical vector fields which are Hamiltonian with respect to some Poisson structure subject to certain fundamental conditions. Then, we can ask whether all of them are associated with an Euler-Lagrange SODE defined by a Lagrangian. Finally, we consider the local structure of such Lagrangian.

Localizability

We will be discussing systems whose classical phase space is the tangent bundle of a configuration space Q . The first condition that we will impose on Poisson tensors Λ defined on TQ is that they will describe localizable systems, i.e.,

$$\{q^i, q^j\} = 0. \quad (94)$$

This condition was used in the Feynman procedure as presented in §2, and later on was a crucial ingredient for the introduction of interactions eq. (46). From a physical point of view, this condition reflects that the classical limit of the quantum theory defines a configuration space whose points can be used to label the quantum states of the system. This notion is thus borrowed from quantum theory and another way of stating it is simply that position operators commute. Geometrically, this means that the algebra of observables on Q is an Abelian subalgebra of the full algebra $\mathcal{F}(TQ)$.

Definition 4 We will say that a Poisson tensor Λ on TQ is localizable if $\tau^*(\mathcal{F}(Q))$ is an Abelian subalgebra of $\mathcal{F}(TQ)$.

Notice that if Λ is localizable, then the Hamiltonian vector fields corresponding to coordinates on the configuration space Q commute. On the other hand it is clear that if a Poisson dynamical system Γ has a Lagrangian realization, the Poisson tensor Λ is localizable. Because of this, this condition is called "variational admissibility" in [No82] and was used to show the existence of a (local) Lagrangian description for certain Poisson dynamical systems.

In other words, if the Poisson tensor Λ is given by

$$\Lambda = a^{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j} + b^{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial v^j} + c^{ij} \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial v^j}, \quad (95)$$

localizability amounts to $a^{ij} = 0$. But this is equivalent to imposing the requirement that the Hamiltonian vector fields X_{q^i} corresponding to the coordinates on Q , be vertical. In fact,

$$X_{q^i} = b^{ij} \frac{\partial}{\partial v^j} + 2a^{ij} \frac{\partial}{\partial q^k}, \quad (96)$$

which means that X_{q^i} are vertical if and only if $a^{ij} = 0$. Moreover, the vector fields X_{q^i} commute among themselves, then

$$b^{ij} \frac{\partial b^{kl}}{\partial v^j} - b^{kj} \frac{\partial b^{il}}{\partial v^j} = 0.$$

Hamiltonian second order equations

Another assumption which is often implicitly assumed is that there are dynamical systems which are SODE's on TQ and which are Hamiltonian with respect to a given Poisson tensor Λ . For instance, in the derivation of Feynman's results in §2 and later on §3, the existence of a SODE which was Hamiltonian with respect to Λ was assumed. In general, this need not to be the case. In this regard, consider the example of $\mathbb{R}^2$, with coordinates $(q^1, q^2; v^1, v^2)$ and the Poisson tensor $\Lambda = \partial / \partial q^1 \wedge \partial / \partial v^1$. For this case, there is no SODE Γ such that, $\Gamma = -\Lambda(dH)$.

Definition 5 We will say that a Poisson tensor Λ in TQ is Newtonian if there exists a SODE Γ such that $\Gamma = X_H$ for some Hamiltonian function H on TQ , in other words,

$$\Gamma = -\Lambda(dH). \quad (97)$$

Notice that there is a weaker assumption on the Newtonian character of Λ , which is as follows: Suppose that there is a SODE Γ such that $\mathcal{L}_\Gamma \Lambda = 0$. Then we will say that Γ is canonical. If Λ is Newtonian then the latter condition automatically holds, but the converse is not necessarily true. In the previous example on $\mathbb{R}^2$ it is easy to check that the SODE $\Gamma = v^1 \partial / \partial q^1 + v^2 \partial / \partial q^2$ is canonical.

From eq. (82), Γ being canonical implies that

$$\Gamma\{F, G\} = \{\Gamma(F), G\} + \{F, \Gamma(G)\}, \quad (98)$$

i.e., the Poisson bracket is compatible with the dynamics defined by Γ , but this does not guarantee the existence of a Hamiltonian function for Γ . However if Λ is nondegenerate, it is sufficient that Γ be canonical to insure the existence of a (local) Hamiltonian function. Notice that if Γ has a regular Lagrangian description, then it automatically is a SODE and Λ is Newtonian.

We can state the result as follows:

Lemma 1 *If Γ is a canonical SODE in TQ for a localizable Poisson tensor Λ , then the bracket $\{q^i, v^j\} = b^{ij}$ is symmetric. In particular if Λ is Newtonian and localizable the same result holds.*

Proof: If Λ is localizable we have that $\{q^i, q^j\} = 0$. However because of eq. (98), we get

$$0 = \Gamma\{q^i, q^j\} = \{\Gamma(q^i), q^j\} + \{q^i, \Gamma(q^j)\} = -b^{ji} + b^{ij}.$$

□

Another important observation concerning Hamiltonian SODE's is the following:

Lemma 2 *If Γ is a Hamiltonian SODE and α is a non-null basic 1-form, then $\Lambda(\alpha) \neq 0$. In particular, there are no nonconstant basic Casimirs.*

Proof: Let $\alpha = \alpha_i dq^i$ a basic form. Then,

$$i_{\Lambda(\alpha)} dH = \Lambda(\alpha, dH) = i_{\Gamma} \alpha = \alpha_i v^i,$$

and consequently, if $\Lambda(\alpha) = 0$ then $\alpha = 0$.

In particular, if there is a basic Casimir ϕ , then $\Lambda(d\phi) = 0$ and therefore $d\phi = 0$. □

Conversely, it can be easily seen that if Γ is a canonical SODE for Λ and it has basic Casimirs, then Λ must be nonregular. Otherwise, a canonical SODE Γ would be Hamiltonian and no basic Casimir functions would exist.

Regular Poisson tensors

Let us recall the important role played by the condition $i_S \omega = 0$ in the geometry of closed 2-forms on TQ . As was indicated in Thm. 1, this condition is necessary and sufficient to insure the existence of a (local) Lagrangian such that $\omega = \omega_L$. Has this condition similar implications for Poisson tensors?

To answer this question we first make the following observation:

Lemma 3 *If Λ is localizable and there is a SODE Γ which is canonical (in particular if it is Newtonian), then $i_S \Lambda = 0$.*

Proof: Consider Λ as in eq. (95). Then, a simple computation shows that,

$$i_S \Lambda = 2a^{ii} \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial q^i} + b^{ii} \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial v^i}.$$

Thus, if Λ is localizable, $a^{ii} = 0$, and if Λ is canonical, (Lemma 1) b^{ii} is symmetric. □

Notice that the converse is not true. (To prove this result one can use the above example.)

Thus, the condition $i_S \Lambda = 0$ is too weak to insure the existence of a local Lagrangian since it does not first insure the regularity of Λ . On the other hand, we have seen that the condition that it is localizable plus Newtonian implies $i_S \Lambda = 0$. In fact, we can show that these conditions are sufficient to insure that Λ is regular.

Theorem 2 *If Λ is a localizable and Newtonian Poisson tensor on TQ , then Λ is regular, and hence it defines a symplectic structure on TQ .*

Proof: Notice that $\Lambda^n = \Lambda \wedge \cdots \wedge \Lambda$ is proportional to

$$\det(b^{ij}) \frac{\partial}{\partial q^1} \wedge \cdots \wedge \frac{\partial}{\partial q^n} \wedge \frac{\partial}{\partial v^1} \wedge \cdots \wedge \frac{\partial}{\partial v^n}.$$

Thus, Λ will be regular iff $\det(b^{ij}) \neq 0$.

We shall show it by proving that Λ defines a monomorphism when restricted to closed basic 1-forms. In fact, let $F \in \mathcal{F}(\mathcal{U})$, where $\mathcal{U}$ is an open set in Q , and $f = \tau^* F \in \mathcal{F}(\tau^{-1}\mathcal{U})$. Computing $\Lambda(df)$ gives

$$\Lambda(df) = b^{ij} \frac{\partial f}{\partial q^i} \frac{\partial}{\partial v^j}, \quad (99)$$

where we have used that Λ is localizable, and hence $a^{ij} = 0$. On the other hand, if Λ were not a monomorphism on closed basic 1-forms, this would imply the existence of a function ϕ on an open set $\mathcal{U}$ in Q such that $\Lambda(d\phi) = 0$, i.e., that ϕ is a (locally defined) basic Casimir. However, because of Lemma 2, this cannot occur if Λ is Newtonian. Thus $\det b^{ij} \neq 0$, because of eq. (99). □

It is interesting to point out that if Λ is nondegenerate, it does not necessarily follow that Λ is localizable and/or Newtonian. In fact, the Poisson tensor $\Lambda = \partial/\partial q^1 \wedge \partial/\partial q^2 + \partial/\partial v^1 \wedge \partial/\partial v^2$ is neither one nor the other.

Therefore, if there exists such a vector field the Poisson tensor is invertible. When writing it in block matrix form

$$(\Lambda) = \left(\begin{array}{c|c} 0 & b \\ \hline -b & c \end{array} \right), \quad (100)$$

we find the inverse matrix given by

$$(\omega) = \left(\begin{array}{c|c} b^{-1}cb^{-1} & -b^{-1} \\ \hline b^{-1} & 0 \end{array} \right), \quad (101)$$

and we have a closed nondegenerate 2-form ω by inverting the Poisson structure Λ as described in §5, eq. (84). We can write ω as

$$\omega = G_{ij} dq^i \wedge dv^j + F_{ij} dq^i \wedge dq^j, \quad (102)$$

where

$$G_{ij} = b_{ik} c^{kl} b_{lj}; \quad F_{ij} = -b_{ij},$$

and $b_{ik} b^{kj} = \delta_i^j$.

On the other hand, if $i_S \Lambda = 0$ and ω is the inverse of Λ , then $i_S \omega = 0$ and conversely. Consequently, by Thm 1, if there is a second order differential equation vector field Γ which is Hamiltonian with respect to a localizable Poisson structure Λ , then there will be (locally) a Lagrangian function L in TQ such that Γ is the dynamical vector field defined by L . In this sense, as indicated by Dyson, Feynman's approach does not go beyond the Lagrangian formalism and internal degrees of freedom should be considered for such a purpose. Notice that we have made no use of further conditions, such as $\{q^i, v^j\} = \delta^{ij}$ [Hu92]. These results also provide the general setting for the remarks in [No82]. Thus the impossibility of going beyond the Lagrangian formalism using Feynman's procedure on tangent bundles is a fundamental restriction. Nevertheless, the Lagrangian functions arising in this manner can be quite different from the standard 'mechanical' type (in contrast with the result in [Mo93]).

We will see however that this is not the case, if we impose an additional physical condition on Λ .

Dynamical independence of the Poisson brackets from the state of motion of test particles

From a physical point of view, it is reasonable to assume that the fields acting on a particle are external. This means that if we are using a test particle to probe the field, then the field itself will not depend on the state of motion of the test particle, i.e., it will not depend on the velocity of the particle. In this sense, we can assume that the Poisson tensor Λ will not depend on the velocities. In other words, translation along the fibres of TQ will be Poisson maps for Λ . Yet another way of stating this is: The coefficients Λ^{ij} of the Poisson tensor Λ in natural coordinates q^i, v^i are defined on the configuration space.

Definition 6 We will say that a Poisson tensor is velocity independent if $\mathcal{L}_{\partial/\partial v^i} \Lambda = 0$, for all $i = 1, \dots, n$, i.e., if translations along the velocities are Poisson maps.

We have seen that the condition of localizability and that Λ is Newtonian are not equivalent to that of the regularity of Λ . In spite of this, we can show the following:

Theorem 3 If Λ is a nondegenerate Poisson tensor such that $i_S \Lambda = 0$ and it is velocity independent, then Λ is Newtonian.

Proof: The matrix b^{ij} is nondegenerate because Λ is regular. We will denote the inverse of b^{ij} by b_{ij} . Because $i_S \Lambda = 0$, then b_{ij} is symmetric, and because Λ is velocity independent $b_{ij} = b_{ij}(q)$ is a basic function. Then, $H = \frac{1}{2} b_{ij}(q) v^i v^j$ is such that

$$X_H = -\Lambda(dH) = v^i \frac{\partial}{\partial q^i} + \left(\frac{1}{2} v^i v^j b^{lm} \frac{\partial b_{ij}}{\partial q^l} + 2c^{lm} b_{ij} v^i \right) \frac{\partial}{\partial v^m},$$

which is a Hamiltonian SODE. $\square$

Notice that if the Poisson tensor Λ is velocity independent, then the transformation from one reference frame to another, having uniform velocity with respect to the original frame, is canonical. If the Poisson tensor Λ is velocity independent, then the functions G_{ij} and F_{ij} depend only on the coordinates q^i . Using the results in the previous section (Thm. 1), we know that there exists a semibasic 1-form $\theta = A_i(q, v) dq^i$ such that $d\theta = \omega$. But then

$$G_{ij} = \frac{\partial A_i}{\partial v^j},$$

and,

$$F_{ij} = \frac{\partial A^i}{\partial q^j} - \frac{\partial A^j}{\partial q^i}.$$

Because G_{ij} depends on just the q 's, then $A_i(q, v) = G_{ij}(q) v^j + a_i(q)$. On the other hand (again from Thm. 1), we get that $A_i = \partial L / \partial v^i$. Then integrating once more, we get

$$L = \frac{1}{2} G_{ij}(q) v^i v^j + a_i(q) v^i - V(q),$$

which is a mechanical type Lagrangian.

We thus conclude:

Theorem 4 Let Λ be a localizable, Newtonian and velocity independent Poisson tensor on TQ , then Λ defines a family of Hamiltonian SODE's whose Lagrangians are of mechanical type differing on the potential term, and whose Cartan 2-form is the inverse of Λ .

8 Poisson dynamics on bundles and the inclusion of internal variables

So far we have proved that the generalization of Feynman's attempt at constructing localizable second order Hamiltonian dynamics on tangent bundles always leads (locally) to Lagrangian systems with a well defined structure. Nevertheless, in part I we have constructed examples of dynamics which is not of the Lagrangian type (at least at first sight). This was done by introducing internal degrees of freedom. We will discuss such a situation in what follows. For that purpose we will first need to understand some geometrical aspects of Poisson structures on bundles, which now follows.

Poisson structures on bundles: Localizability

We shall consider a bundle $E \xrightarrow{\pi} B$ over the base manifold B . We can assume for simplicity that E is a vector bundle, but this is not strictly necessary for what follows. If we are given a Poisson tensor Λ_B on the base space B , we can ask for the existence of a Poisson tensor Λ_E on the total space E such that $\pi_*(\zeta) \Lambda_E(\zeta) = \Lambda_B(\pi(\zeta))$, $\zeta \in E$. We will then say that Λ_E is π -related to Λ_B . This is also equivalent to saying that the map π is a Poisson map from E to B . In fact, upon introducing local coordinates b^i for B and ζ_a along the fibres of E , we can then write

$$\Lambda_E = \Lambda_B^{ij}(b) \frac{\partial}{\partial b^i} \wedge \frac{\partial}{\partial b^j} + A_a^i(b, \zeta) \frac{\partial}{\partial b^i} \wedge \frac{\partial}{\partial \zeta_a} + C_{ab}(b, \zeta) \frac{\partial}{\partial \zeta_a} \wedge \frac{\partial}{\partial \zeta_b}, \quad (103)$$

where the first term on the r.h.s. of eq. (103) is simply the local expression for Λ_B .

Equivalently, the fact that Λ_E is π -related to Λ_B implies that the map $\pi^*: \mathcal{F}(B) \rightarrow \mathcal{F}(E)$ induced by π is a Lie algebra homomorphism.

A particular case arises when $\Lambda_B = 0$, i.e., the base manifold carries a trivial Poisson bracket. In such a case, the possible Poisson tensors on E which are π -related to it have the general form

$$\Lambda_E = A_a^i(b, \zeta) \frac{\partial}{\partial b^i} \wedge \frac{\partial}{\partial \zeta_a} + C_{ab}(b, \zeta) \frac{\partial}{\partial \zeta_a} \wedge \frac{\partial}{\partial \zeta_b}. \quad (104)$$

For instance, if B is the configuration space Q for a physical system, and E is the corresponding tangent bundle, $E = TQ$, then Poisson tensors Λ_{TQ} π -related to $\Lambda_Q = 0$ are equivalent to the condition that Λ_{TQ} is localizable.

Another example that will concern us here is the introduction of internal variables for a system defined on a configuration space Q . Internal variables are modeled by defining a bundle $\rho: F \rightarrow Q$. Attached to each base point $m \in Q$ there is a fibre $\rho^{-1}(m)$ that can be a representation space M for a group of gauge symmetries. Again the Poisson bracket on Q is trivial, and a Poisson tensor Λ_F which is ρ -related to it will have the form

$$\Lambda_F = A_a^i(q, I) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial I_a} + C_{ab}(q, I) \frac{\partial}{\partial I_a} \wedge \frac{\partial}{\partial I_b}, \quad (105)$$

where I_a denote local coordinates along the fibres of F .

If the I_a 's are to be considered as internal variables, they should be localized at points in configuration space. This means that they must be "measurable" simultaneously with position operators, i.e.,

$$A_a^i = \{q^i, I_a\}_F = 0. \quad (106)$$

We will call such a localizability property for Poisson tensors on F “localizability on the bundle”, or just “localizability”, which we trust will not cause any confusion. In this case, eq. (105) simplifies to

$$\Lambda_I = C_{ab}(q, I) \frac{\partial}{\partial I_a} \wedge \frac{\partial}{\partial I_b}. \quad (107)$$

Notice that Λ_I is a Poisson tensor iff $[\Lambda_I, \Lambda_I] = 0$, which is equivalent to eq. (44). This condition is true for all points $q \in Q$. This then implies that on each fibre $\rho^{-1}(q)$, the Poisson tensor Λ_I induces a Poisson tensor structure which is obtained from Λ_I by fixing q . We can then conclude that a localizable Poisson structure on a bundle $F \rightarrow Q$ is equivalent to a bundle of Poisson manifolds, but the Poisson structure can change from fibre to fibre.

Definition 7 A localizable Poisson tensor on a fibre bundle $\rho: F \rightarrow Q$ is a bundle Poisson tensor, i.e., a smooth assignment to each fibre of a Poisson tensor Λ_I .

Notice that Hamiltonian vector fields on such a bundle are vertical vector fields, with their restriction to each fibre Hamiltonian in the corresponding Poisson structure. A special case of such localizable Poisson tensors on F occurs when the transition functions of the bundle F are Poisson maps. Then we could say that the bundle $F \rightarrow Q$ is Poisson and the standard fibre is a Poisson manifold modeling all the others.

Examples of this situation occur when we consider a principal fibre bundle P over Q with structure group G . If Q is simply a point, then P is the group itself, and the Poisson structure Λ_P is nothing but a Poisson structure on G . This case will be discussed in detail in §11.

Let us instead consider the coadjoint representation of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G . Then we can construct the associated bundle $\text{ad}^*P \rightarrow Q$, with standard fibre $\mathfrak{g}^*$. Recall that this bundle is obtained as the quotient of $P \times \mathfrak{g}^*$ by the natural action of G on it. On $\mathfrak{g}^*$ we consider the linear Poisson tensor,

$$\Lambda_{\mathfrak{g}} = c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad (108)$$

where x_i denote linear coordinates on $\mathfrak{g}^*$ and c_{ij}^k are the structure constants of $\mathfrak{g}$ on a basis E_i which is dual to the basis defined by x_i on $\mathfrak{g}^*$. We then define a Poisson tensor Λ on $P \times \mathfrak{g}^*$ as the direct sum of the zero tensor in P and $\Lambda_{\mathfrak{g}}$. This Poisson tensor is G -invariant. Then it will induce a localizable Poisson tensor on ad^*P . If the bundle $P \rightarrow Q$ were already carrying a G -invariant localizable Poisson tensor Λ_P , we could form $\Lambda_P + \Lambda_{\mathfrak{g}}$ to obtain a new localizable Poisson tensor on ad^*P .

Alternatively, we can proceed as follows: Let Λ_P be a G -invariant localizable Poisson tensor on P and $T: G \rightarrow V$ a linear representation of G on V . Then in the associated vector bundle $E_T = P \times V/G$, there is a localizable Poisson tensor induced from Λ_P . Some specific examples of this situation will be discussed in §11.

Extension of Poisson structures and internal variables

We note that if $E \rightarrow B$ is a bundle, then it could happen that there are no smooth sections. For instance, if $E \rightarrow B$ is a principal bundle, then there are no smooth, nor continuous, sections unless the bundle is trivial. In physical terms, this means that it is not possible to globally fix the state of the inner variables in a continuous way. In this sense,

Figure 1: Internal variables and configuration bundles.

there is no way to “switch off” the inner variables and to restrict our attention to a system without them. On the other hand, if $E \rightarrow B$ is a vector bundle, i.e., if the inner variables are defined in a vector space, then there are always smooth sections. In fact, there is the zero section which assigns to each inner variable the zero value, thereby switching off the internal variables.

Choosing a section $\sigma: B \rightarrow E$ of the projection map π provides a different link between a Poisson tensor Λ_E on E and a Poisson tensor Λ_B on B . We will say that Λ_E is an extension of Λ_B if the map σ is Poisson, or equivalently if the induced map $\sigma^*: \mathcal{F}(E) \rightarrow \mathcal{F}(B)$ is a Lie algebra homomorphism. This will mean in particular that if E is a vector bundle and σ is the zero section, then in bundle coordinates ξ^i and ζ^a , the Poisson brackets will have the form

$$\{\xi^i, \xi^j\}(\xi, 0) = \Lambda_B(\xi), \quad \{\xi^i, \zeta^a\}(\xi, 0) = 0, \quad \{\zeta^a, \zeta^b\}(\xi, 0) = 0.$$

Whenever this accretive condition on inner variables is required from physical principles, we will impose in addition that the Poisson tensor Λ_E will be an extension of Λ_B with respect to the zero section.

Feynman’s problem in tangent bundles with inner degrees of freedom

We are now prepared to try to determine all Poisson structures on a tangent bundle with internal degrees of freedom and satisfying some fundamental physical restrictions. We will follow a similar chain of reasoning as that in §7.

Our setting will be that of a system with configuration space Q and possessing internal degrees of freedom modeled on a bundle $\rho: F \rightarrow Q$. We can pull back the bundle $\tau_Q: TQ \rightarrow Q$ along the map ρ , to obtain a new bundle $E = \rho^*(TQ)$ over F . The pull-back bundle $\rho^*(TQ)$ is defined as the set of pairs (I, v) where $I \in F$, $v \in TQ$, and their base points agree, i.e., $\rho(I) = q = \tau_Q(v)$. There is a natural map $p_1: \rho^*(TQ) \rightarrow F$, sending each pair (I, v) into its first component $p_1(v, I) = v$. Then the fibre of $\rho^*(TQ)$ at the point $I \in F$ is simply the vector space $T_q Q$ with $q = \rho(I)$. Thus $p_1: \rho^*(TQ) \rightarrow F$ is a vector bundle with the fibres of $TQ \rightarrow Q$, $p_1^{-1}(I) = T_{\rho(I)}Q$ (Fig 1.). Notice that if we take the pull-back of the bundle $F \rightarrow Q$ along the map $\tau_Q: TQ \rightarrow Q$, then we obtain the bundle $\tau_Q^*(F)$ made up of pairs (v, I) with the same property as before, and projection map $p_2: \tau_Q^*(F) \rightarrow TQ$ given by $p_2(v, I) = v$. The total spaces $\rho^*(TQ)$ and $\tau_Q^*(F)$ are the same (this will be denoted by $TQ \bowtie F$), but the fibre structures are different. The fibre of $TQ \bowtie F$ as a bundle over TQ with projection map p_2 is given by $F_q = \rho^{-1}(q)$ (see Fig. 1).

We will be searching for Poisson tensors Λ on $TQ \bowtie F = \rho^*(TQ)$ that will be p_1 -related to a localizable Poisson tensor Λ_I on $F \rightarrow Q$ given by eq. (107). Such Poisson tensors will

Figure 2: The almost tangent structure on $TQ \bowtie F$.

be called localizable on $TQ \bowtie F$. This requirement is equivalent to Λ to having the form

$$\begin{aligned} \Lambda = & b^{ij}(q, v, I) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial v^j} + c^{ij}(q, v, I) \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial v^j} \\ & + A_a^i(q, v, I) \frac{\partial}{\partial v^i} \wedge \frac{\partial}{\partial I_a} + C_{ab}(q, I) \frac{\partial}{\partial I_a} \wedge \frac{\partial}{\partial I_b}. \end{aligned} \quad (109)$$

Notice that the fundamental Poisson brackets of the variables q^i, v^i and I_a can all depend on q, v and I , except for C_{ab} which depends only on the q 's and I 's.

We have discovered in the previous section that the existence of a Hamiltonian SODE for a Poisson tensor has important consequences. Something similar happens now. Even though the bundle $p_1: TQ \bowtie F \rightarrow F$ is not a tangent bundle, there is a natural tensor S on it that plays the same role as the tangent structure on TQ discussed in §6. Locally, this tensor S has the same form as the tangent structure on TQ , i.e.,

$$S = \frac{\partial}{\partial v^i} \otimes dq^i. \quad (110)$$

It is easy to show that S is well defined, either by direct computation or by noticing that there is a natural projection Π from TF to $E = \rho^*(TQ)$, given by

$$\Pi: (q^i, I^a; v^i, I^a) \mapsto (q^i, I^a; v^i), \quad (111)$$

and that the natural tangent structure S_F on TF is projectable along this map (see Fig. 2). The projected tensor results from S given by eq. (110).

We will be considering dynamical systems which are second order in the variables q^i and v^i and first order on the variables I_a . The corresponding vector fields will again be called second order differential equations (SODE's) on E , or also driven SODE's [Ib91]. These vector fields Γ are characterized by $S(\Gamma) = \Delta$, where Δ is the dilation field along the fibres of $\rho^*(TQ)$, i.e., Δ has the form $v^i \partial / \partial v^i$. The most general form of a driven SODE is given by

$$\Gamma = v^i \frac{\partial}{\partial q^i} + F^i(q, v, I) \frac{\partial}{\partial v^i} + f_a(q, v, I) \frac{\partial}{\partial I_a}, \quad (112)$$

and its integral curves satisfy the coupled system of differential equations:

$$\dot{q}^i = F^i(q, v; I), \quad (113)$$

$$\dot{I}_a = f_a(q, v; I). \quad (114)$$

Compare these equations with eqs. (45).

We will denote by $\mathcal{F}_{TQ \bowtie F}(Q)$ the functions on $E = TQ \bowtie F$ obtained by pull-back of functions on Q , $\mathcal{F}_{TQ \bowtie F}(Q) = (\rho \circ p_1)^* \mathcal{F}(Q)$, i.e., functions depending only on q 's. We will denote by $\mathcal{F}_{TQ \bowtie F}(F)$ the functions on E obtained by pull-back of functions on F by the map p_1 , i.e., functions on E depending on both q 's and I 's.

Proposition 1 *Let Λ be a localizable Poisson tensor on $TQ \bowtie F$ (eq. 109). If Γ is a Hamiltonian SODE, then there are no nonconstant locally defined Casimirs on $\mathcal{F}_{TQ \bowtie F}(Q)$. Moreover, if the Poisson tensor Λ_I is regular along the fibres of ρ , then there are no non-constant locally defined Casimirs on $\mathcal{F}_{TQ \bowtie F}(F)$ either.*

Proof: Let ϕ be a locally defined Casimir on $\mathcal{F}_{TQ \bowtie F}(Q)$. Then $\Gamma(\phi) = 0$ if Γ is a Hamiltonian SODE on E . But clearly this implies that $\partial\phi/\partial q^i = 0$, as in Lemma 2.

On the other hand, if ϕ is a locally defined Casimir on $\mathcal{F}_{TQ \bowtie F}(F)$, then $\Gamma(\phi) = 0$ again because Γ is Hamiltonian. Because of eq. (112), this implies that

$$0 = v^i \frac{\partial \phi}{\partial q^i} + f_a(q, v, I) \frac{\partial \phi}{\partial I_a}.$$

Since ϕ does not depend on v , $f_a(q, v, I) = f_a(q, I)v^i$ and

$$\frac{\partial \phi}{\partial q^i} + f_{ia}(q, I) \frac{\partial \phi}{\partial I_a} = 0. \quad (115)$$

On the other hand, if ϕ is a locally defined Casimir on $\mathcal{F}_{TQ \bowtie F}(F)$, then $\Lambda(d\phi) = 0$, and this implies that

$$\left(b^{ij} \frac{\partial \phi}{\partial q^j} - A_a^i \frac{\partial \phi}{\partial I_a} \right) \frac{\partial}{\partial v^i} + C_{ab} \frac{\partial \phi}{\partial I_a} \frac{\partial}{\partial I_b} = 0.$$

Then $C_{ab}(q, I) \partial\phi/\partial I_a = 0$. But if Λ_I is nondegenerate along the fibres of ρ , there cannot be nonconstant locally defined Casimirs along the fibres of ρ , i.e., $\partial\phi/\partial I_a = 0$. Hence because of eq. (115), $\partial\phi/\partial q^i = 0$. $\square$

As in the previous section we will say that a Poisson tensor on $TQ \bowtie F$ is Newtonian if it possesses a Hamiltonian SODE.

What about the regularity of Λ on $TQ \bowtie F$? To answer this question, we can repeat the discussion leading to Theorem 2, but first we notice that:

Lemma 4 *A localizable Poisson tensor Λ on $TQ \bowtie F$ of the form given by eq. (109) will be regular iff b^{ij} and C_{ab} are invertible.*

Proof: The matrix defined by Λ on the basis constructed from $\partial/\partial q^i$, $\partial/\partial I_a$ and $\partial/\partial v^i$ is:

$$\left(\begin{array}{c|c|c} 0 & 0 & b \\ 0 & C & A \\ -b^i & -A^i & c \end{array} \right),$$

which is invertible iff b and C are. $\square$

Theorem 5 *If Λ is a localizable Poisson tensor on $TQ \bowtie F$ such that the Poisson tensor Λ_I on the configuration bundle $F \rightarrow Q$ is regular along the fibres, and possessing a Hamiltonian SODE, then Λ is regular.*

Proof: It is easy to show that b^{ij} is invertible using locally defined Casimirs on $\mathcal{F}_{TQ \bowtie F}(Q)$. In fact, Λ must be a monomorphism acting on $\mathcal{F}_{TQ \bowtie F}(Q)$. (If not there would be locally defined Casimirs on $\mathcal{F}_{TQ \bowtie F}(Q)$. This is impossible because of Prop. 1.) On the other hand, it is easy to check that $\Lambda(d\phi)$ is again $b^{ij} \partial\phi/\partial q^i \partial/\partial v^j$, implying the invertibility of b^{ij} .

Finally, Λ is also a monomorphism on $\mathcal{F}_E(F)$, because if Λ_I is regular along the fibres there are no locally defined Casimirs on $\mathcal{F}(F)$ (Prop. 1 again). Then, computing $\Lambda(d\phi)$ we find

$$\Lambda(d\phi) = C^{ab} \frac{\partial\phi}{\partial I_a} \frac{\partial}{\partial I_b} + \left(b^{ij} \frac{\partial\phi}{\partial q^j} - A_a^i \frac{\partial\phi}{\partial I_a} \right) \frac{\partial}{\partial v^i},$$

then, C^{ab} must be invertible. $\square$

From this result we cannot conclude the existence of a Lagrangian function on $TQ \bowtie F$, because it is not a tangent bundle. However we could try to construct a singular Lagrangian L on TF , such that after some reduction the Cartan form ω_L induces a form ω on E that is the inverse of Λ . The necessary and sufficient conditions for this to happen were established in [lb91] in a situation slightly different from the one we are dealing with. We will then state it, adapting it to the present context, as follows:

Theorem 6 *Let $p_1: TQ \bowtie F \rightarrow TQ$ be the vector bundle obtained by pull-back from a bundle F over Q , and let $p_2: TQ \bowtie F \rightarrow F$ be the natural projection on the second factor. Let S be the natural $(1,1)$ -tensor field on $TQ \bowtie F$ given by eq. (110) that projects onto the natural tangent structure S_Q on TQ along the map Π defined by eq. (111). If ω is a symplectic form on $TQ \bowtie F$, then there will exist a (locally defined) function L on TF such that the Cartan 2-form ω_L will project onto ω along Π , if and only if the following two conditions are satisfied:*

- i. $i_S \omega = 0$.
- ii. $\omega(V_1, V_2) = 0$ for all vector fields vertical with respect to p_2 , i.e., vectors tangent to the tangent spaces $T_q Q \subset TQ \bowtie F$.

Proof: The necessity of conditions (i) and (ii) is almost obvious. Let L be a Lagrangian function on TF such that ω_L projects onto ω along Π . Then, $\ker \omega_L = \ker T\Pi$. From the fact that ω_L vanishes on vertical bivectors on TF , we get condition (ii). In general, because of eq. (90), ω_L is such that $i_{S_F} \omega_L = 0$, where S_F is the tangent structure on TF . We get S by projecting S_F along the map Π , and consequently condition (i) is satisfied.

Conversely, let us assume that there is a symplectic form ω on $TQ \bowtie F$ satisfying (i) and (ii). Let $\Omega = \Pi^* \omega$ be the pull-back of ω to TF . Then we will show that $i_{S_F} \Omega = 0$. In fact,

$$i_{S_F} \Omega(X, Y) = \Omega(S_F(X), Y) + \Omega(X, S_F(Y)),$$

for any two vector fields X and Y on TF . Taking $X = \partial/\partial q^i$ and $Y = \partial/\partial q^j$, we get

$$i_{S_F} \Omega \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right) = \Omega \left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial q^j} \right) + \Omega \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial v^j} \right)$$

$$= \omega \left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial q^j} \right) + \omega \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial v^j} \right) = i_S \omega \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right) = 0.$$

If we take now $X = \partial/\partial q^i$ and $Y = \partial/\partial I_a$, we get

$$\begin{aligned} i_{S_F} \Omega \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial I_a} \right) &= \Omega \left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial I_a} \right) + \Omega \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial I_a} \right) \\ &= \omega \left(\frac{\partial}{\partial v^i}, \frac{\partial}{\partial I_a} \right) = i_S \omega \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial I_a} \right) = 0. \end{aligned}$$

Finally consider $X = \partial/\partial I_a$ and $Y = \partial/\partial I_b$. Then,

$$i_{S_F} \Omega \left(\frac{\partial}{\partial I_a}, \frac{\partial}{\partial I_b} \right) = \Omega \left(\frac{\partial}{\partial I_a}, \frac{\partial}{\partial I_b} \right) + \Omega \left(\frac{\partial}{\partial I_a}, \frac{\partial}{\partial I_b} \right) = 0.$$

The evaluation of $i_{S_F} \Omega$ on all the remaining pairs of basis vectors vanishes because they always contain a vertical vector field. Then because of Thm. 1, there will exist a (locally) defined Lagrangian function L on TF such that $\Omega = \omega_L$, completing the proof. $\square$

Thus, we can finally state:

Theorem 7 *If Λ is a Newtonian localizable Poisson tensor on $TQ \bowtie F$ such that the Poisson tensor Λ_I on the configuration bundle $F \rightarrow Q$ is regular along the fibres, then there is a Lagrangian realization of it on TF , in the sense that there exists a locally defined Lagrangian function L on $F \rightarrow Q$, such that Λ is obtained by direct reduction from ω_L .*

In this way we have obtained another no-go theorem analogous to that of Feynman. Thus even with the inclusion of inner degrees of freedom we do not escape the Lagrangian formalism whenever the Poisson brackets on the inner degrees of freedom are regular. The latter need not be the case as was seen in sections 2 and 3. In fact, for particles in a Yang-Mills field or for spinning particles, the inner degrees of freedom (the isospin and spin, respectively) are defined in a space with degenerate Poisson brackets. However, in both situations, we can restrict the classical inner degrees of freedom to a subbundle of F made up of symplectic leaves on each fibre; for example, spheres for the spinning particle and coadjoint orbits for the Yang-Mills particle, with the restriction that Λ_I is regular. Then we will apply the previous theorem to this bundle obtaining a Lagrangian representation for the system. Notice that the Lagrangian will be singular and only locally defined if the topology of the fibres of the symplectic bundle is complicated enough.

Part III

Some applications: Feynman's problem and Lie groups

9 Standard Lagrangian realizations for external variables

Group theory was relevant for some of the examples discussed in Part I, because the internal variables were associated with some representation of a Lie group. In §10 we will discuss systems whose configuration space are Lie groups, in particular we will discuss in detail the dynamics of an isotropic rotator or the B-phase of superfluid ^3He .

Because of the results on Part II, we know that Feynman's procedure gives a negative answer in many instances; negative in the sense that there is always a Lagrangian description for dynamics compatible with the requirements described in Theorem 5. In this section we will discuss Feynman's procedure from a group theoretical setting, showing its relation to the construction of Lagrangian realizations of systems defined on group manifolds or on associated bundles to principal fibre bundles. This construction again includes the cases of Yang-Mills particles and spinning particles.

We will start by considering Feynman's procedure for the case where the configuration space is a Lie group G .

Feynman's Procedure on group manifolds

The classical velocity phase space of the system we will examine here is TG with the Lie group G as the configuration space of the system. We will be looking for all dynamics and Poisson brackets on TG which are localizable and Newtonian. We know already because of Thm. 2 that the relevant Poisson tensors must be regular and that there will exist Lagrangian descriptions for such systems. If the Poisson tensor is velocity independent, we know that the only possible nontrivial dynamics are due to interactions with a background gravitational field arising from some non constant metric tensor η on G , and with an electromagnetic field, defined on the group by a vector and scalar potentials A and ϕ . If we impose G -invariance on the dynamical system and on the Poisson tensor, then all fields must be G -invariant. We will show how Feynman's procedure can be applied to this situation. The results agree with those already obtained in Part II.

We first need to introduce some notation and facts about the geometry of Lie groups, their tangent bundles and SODE's on them.

Let L (R) denote the action of G on itself by left (right) translations, i.e., $L: G \times G \rightarrow G$, $L(g, h) = L_g(h) = gh$, $(R(g, h) = R_g(h) = hg)$ for all $g, h \in G$. A vector field X on G is called left (right) invariant if $TL_g X(h) = X(gh)$ ($TR_g X(h) = X(hg)$). If X is left (right) invariant then $X(g) = TL_g X(e)$ ($X(g) = TR_g X(e)$). The Lie algebra $\mathfrak{g}$ of G is the Lie subalgebra of left (right) invariant vector fields on G . These invariant vector fields can be identified with their values at the identity element of the group. Then fixing a basis E_i of $T_e G$ we define the corresponding basis of invariant vector fields $X_i^L(g) = TL_g E_i$, $X_i^R(g) = TR_g E_i$. From now on we will identify $\mathfrak{g}$ with $T_e G$ and we fix a basis E_i of $\mathfrak{g}$.

We next define left (right) invariant forms on G . We shall denote by θ_L^i (θ_R^i) a basis of left (right) invariant 1-forms on G dual to X_i^L (X_i^R). They obviously satisfy $(X_i^L, \theta_L^j) = \delta_i^j$, $((X_i^R, \theta_R^j) = \delta_i^j)$. The canonical Maurer-Cartan $\mathfrak{g}$ -valued 1-forms Θ_L and Θ_R on G can be written as

$$\Theta_L = E_i \otimes \theta_L^i, \quad \Theta_R = E_i \otimes \theta_R^i \quad (116)$$

The structure constants c_{ij}^k of the Lie algebra $\mathfrak{g}$ are defined from

$$[X_i^L, X_j^L] = c_{ij}^k X_k^L,$$

or from the analogous relations using right invariant vector fields. The vector bundle TG is trivial and can be identified with $G \times \mathfrak{g}$, using left or right translations as follows: $l: TG \rightarrow G \times \mathfrak{g}$, $l(g, \dot{g}) = (g, TL_g^{-1} \dot{g})$, $(r: TG \rightarrow G \times \mathfrak{g}$, $r(g, \dot{g}) = (g, TR_g^{-1} \dot{g})$), for any $\dot{g} \in T_g G$. We will denote by $v_L = TL_g^{-1}(\dot{g})$ ($v_R = TR_g^{-1}(\dot{g})$) and $v_L = v_L^i E_i$. Coordinates on TG obtained using the map l will be called body or convective coordinates, and coordinates defined by means of r will be called spatial coordinates. It is clear that body and spatial coordinates are related by $(g, v_L) = (g, \text{Ad}_g v_R)$. From now on, we will be using only left-invariant or body coordinates, as it is equivalent to the description in terms of right-invariant ones. We will then suppress the indexes "L".

It is clear that the canonical 1-tensor S_G on TG has the following form when written in body coordinates:

$$S_G = \frac{\partial}{\partial v^i} \otimes \theta^i,$$

and a SODE D will be written as

$$D = v^i X_i + f^i(g, v) \frac{\partial}{\partial v^i}.$$

If L is a Lagrangian function on TG which is left-invariant, i.e., $L = l^* L_g$, or $L(g, \dot{g}) = L_g(TL_g^{-1} \dot{g})$, for a function L_g defined on $\mathfrak{g}$, then the Cartan 1-form is

$$\theta_L = \frac{\partial L}{\partial v^i} \theta^i,$$

and the Cartan 2-form,

$$\omega_L = \frac{\partial^2 L}{\partial v^i \partial v^j} \theta^i \wedge dv^j - \frac{1}{2} c_{jk}^i \frac{\partial L}{\partial v^i} \theta^j \wedge \theta^k.$$

If it is regular, the Lagrangian SODE defined by L is given by

$$\Gamma = v^i X_i + W^{jk} v^j c_{jk}^i \frac{\partial L}{\partial v^i} \frac{\partial}{\partial v^i},$$

where as usual W^{ij} are the elements of the inverse matrix of W with $W_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j}$.

Free motion

By free motion we mean geodesic motion with respect to a metric η on G . Then $\eta = \eta_{ij}^L \theta_L^i \otimes \theta_L^j = \eta_{ij}^R \theta_R^i \otimes \theta_R^j$, where η_{ij}^L and η_{ij}^R are functions on G . If η is left (right) invariant, then the functions η_{ij}^L (η_{ij}^R) are constant. If the metric η is biinvariant then it satisfies:

$$\eta_{ir} c_{jk}^r + \eta_{kr} c_{ji}^r = 0.$$

Let L_η be the kinetic energy defined by the left-invariant metric η ,

$$L_\eta = \frac{1}{2} \eta_{ij}(g) \dot{g}^i \dot{g}^j = \frac{1}{2} \eta_{ij} v^i v^j . \quad (117)$$

The Cartan 1-form will be

$$\theta_\eta = \eta_{ij} v^i \theta^j ,$$

while the Cartan 2-form is

$$\omega_\eta = \eta_{ij} \theta^i \wedge dv^j - \frac{1}{2} \eta_{ij} v^i c_{kl}{}^j \theta^k \wedge \theta^l .$$

The energy of the system is the Lagrangian itself, $E_{L_\eta} = L_\eta$, and the Euler-Lagrange SODE Γ_η is obtained from

$$\Gamma_\eta = v^i X_i + v^k v^j c_{kj}{}^i \frac{\partial}{\partial v^i} . \quad (118)$$

Thus the equations of motion look like:

$$\frac{dg}{dt} = T L_g v; \quad \frac{dv^i}{dt} = c_{jk}{}^i v^j v^k , \quad (119)$$

which defines the geodesic flow on G associated with η . If the metric is biinvariant, then the equations of motion reduce to

$$\frac{dv}{dt} = 0 . \quad (120)$$

The Poisson tensor defined by ω_{L_η} is given by

$$\Lambda_\eta = \eta^{ij} \frac{\partial}{\partial v^i} \wedge X_j - c_i{}^k v^i \frac{\partial}{\partial v^j} \wedge \frac{\partial}{\partial v^k} ,$$

and $c_i{}^k = \eta_{il} \eta^{jm} \eta^{kn} c_{mn}{}^l$. Because the Hamiltonian vector field with Hamiltonian v^i is given by $\eta^{ij} X_j$, the fundamental Poisson brackets defined by Λ_η are

$$\{g^i, g^j\} = 0, \quad \{g^i, v^j\} = \eta^{ik} X_k(g^j), \quad \{v^i, v^j\} = c^i{}_k v^k , \quad (121)$$

where g^i represent a parametrization of the elements $g \in G$.

We will now discuss Feynman's procedure in this setting. For this we examine the particular case of $G = SU(2)$.

The isotropic rotator and ${}^3\text{He}$ superfluid

It is well known that $SU(2)$ is diffeomorphic to S^3 . Thus we will be considering Feynman's problem on the sphere S^3 . We will first examine free motion on S^3 , i.e., geodesic motion with respect to the biinvariant metric on $SU(2)$. Physically, this corresponds to a noninteracting isotropic rotator. After discussing the free system, we introduce interactions by modifying the Poisson structure and equations of motion in a similar manner to what was done in §2.

Let g denote an $SU(2)$ matrix, $g^{-1} = g^\dagger$, where $\dagger$ denotes hermitean conjugation. Matrices g in $SU(2)$ will be parametrized as

$$g = \begin{pmatrix} x_0 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_0 - ix_3 \end{pmatrix} = x_0 I + ix_k \sigma_k, \quad (122)$$

with $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$ and σ_k are the Pauli matrices. In spatial coordinates, free motion means that, eq. (120),

$$\frac{d}{dt}(\dot{g}g^\dagger) = 0 , \quad (123)$$

This is the equation of motion for a noninteracting isotropic rotator. These equations are invariant under left and right multiplication by $SU(2)$

$$g \rightarrow ug, \quad g \rightarrow gv, \quad u, v \in SU(2) . \quad (124)$$

The associated velocity phase space of the system is $TSU(2)$ which we identify with $SU(2) \times \mathfrak{su}(2)$ by means of right translations. We will denote the spatial (angular) velocities (formerly v_R) by ω_i , $i = 1, 2, 3$, this is $\dot{g}g^\dagger = w = \frac{i}{2} \omega_k \sigma_k$. From eq. (121), the Poisson brackets between $SU(2)$ matrix elements are zero, with the remaining brackets given by

$$\{g, \omega_i\} = -\frac{i}{2} \sigma_i g , \quad (125)$$

$$\{\omega_i, \omega_j\} = \epsilon_{ijk} \omega_k , \quad (126)$$

For this choice ω_i generates right translations on $SU(2)$. The equation of motion (123) is recovered using the Hamiltonian function

$$H_0 = -\frac{1}{2} \sum_{i=1}^3 \omega_i^2 = -\frac{1}{2} \omega^2 . \quad (127)$$

which is the standard free Hamiltonian for an isotropic rotator. (We have set the moment of inertia equal to one.)

The chiral transformations (124) can be canonically implemented for the above Poisson structure. For this we must specify the following action of u and v on the angular velocities:

$$\omega_i \sigma_i \rightarrow u \omega_i \sigma_i v . \quad (128)$$

Thus ω_i is invariant under the left action of $SU(2)$ parametrized by u , while right $SU(2)$ parametrized by v induces rotations. The Hamiltonian (127) is invariant with respect to both left and right $SU(2)$, and the transformations are associated with canonical symmetries.

Interactions on S^3

We next apply the analogue of Feynman's assumptions in order to introduce interactions on S^3 . As on $\mathbb{R}^3$, we only obtain the coupling to electromagnetism.

We consider equations of motion of the form

$$\dot{g}g^\dagger = \frac{i}{2} \sigma_i \omega_i , \quad (129)$$

$$\dot{\omega}_i = f_i(g, \omega) . \quad (130)$$

They are the analogues of eqs. (12) and (13). As always we demand that the equations of motion are Hamiltonian. For the Poisson brackets we again assume (125) and that the Poisson brackets between matrix elements of g are zero, implying localizability on S^3 . As

with the case on $\mathbf{R}^3$, we generalize the Poisson brackets of the velocities (or here, angular velocities) with themselves. We take

$$\{\omega_i, \omega_j\} = \epsilon_{ijk}(\omega_k + B_k). \quad (131)$$

Particle interactions are due to the functions f_i and B_k which were zero for the free system. We next determine f_i and B_k , assuming that they depend on the coordinates g and the velocities ω_i . (For simplicity we shall assume that they have no explicit dependence on the time.) The procedure is identical to that carried out on $\mathbf{R}^3$.

It is not hard to see that B_k is a function of only the spatial coordinates, as was the case for $\mathbf{R}^3$. For this we can expand the Jacobi identity

$$\{g, \{\omega_i, \omega_j\}\} + \{\omega_i, \{\omega_j, g\}\} + \{\omega_j, \{g, \omega_i\}\} = 0,$$

to find that $\{g, B_k\} = 0$, and hence B_k is independent of ω_i . As on $\mathbf{R}^3$, B_k is divergenceless. This follows from the Jacobi identity involving three angular velocities and can be expressed by

$$\{\omega_i, B_i\} = 0.$$

Next we take the time derivatives of Poisson brackets (125) and (131). From (125) we get

$$\{\dot{g}, \omega_i\} + \{g, \dot{\omega}_i\} = -\frac{i}{2}\sigma_i \dot{g}.$$

Upon substituting the equations of motion (129) and (130), we find that

$$\{g, f_i\} = \frac{i}{2}\epsilon_{ijk}\sigma_j B_k(g)g,$$

and as a result, the force can be at most linear in the angular velocity ω_i ,

$$f_i = \epsilon_{ijk}B_j(g)\omega_k + E_i(g). \quad (132)$$

Eq. (132) is the Lorentz force equation on S^3 and it defines the electric field $E_i(g)$.

Conditions on $E_i(g)$ result from taking the time derivative of the Poisson bracket (131). We obtain

$$\epsilon_{ijk}\{\dot{\omega}_i, \omega_j\} = \dot{\omega}_k + \dot{B}_k(g). \quad (133)$$

Next we substitute (132) and use the condition that the magnetic fields have no explicit time dependence. Implicit time dependence through g means that $\dot{B}_i$ does not vanish, but instead

$$\dot{B}_i + \{B_i, \omega_j\}\omega_j = 0.$$

From (133) we are then left with

$$\epsilon_{ijk}\{E_i, \omega_j\} - E_k = 0, \quad (134)$$

which is the statement that the electric field has zero curl in the presence of static magnetic fields.

The Hamilton function for this system is

$$H = -\frac{1}{2}\omega^2 + \phi(g). \quad (135)$$

In order to recover the equations of motion (129), (130) and (132) using this Hamiltonian we need to define the electric field E_i in terms of the scalar potential $\phi(g)$ as follows:

$$E_i(g) = \{\omega_i, \phi(g)\}. \quad (136)$$

Furthermore, using this definition we can obtain the condition (134) after applying the Jacobi identity

$$\{\omega_i, \{\omega_j, \phi\}\} + \{\omega_j, \{\phi, \omega_i\}\} + \{\phi, \{\omega_i, \omega_j\}\} = 0.$$

It is easy to generalize the above treatment of a particle moving on $SU(2)$ to the case of a particle on an arbitrary Lie group manifold. For this we need only replace the Pauli matrices by the appropriate group generators and ϵ_{ijk} by the structure constants. For instance we can apply the previous procedure to a closely related group $SO(3)$. Then, we will obtain as Poisson brackets the following:

$$\{R_{ij}, R_{kl}\} = 0, \quad \{\omega_i, R_{jk}\} = \epsilon_{ijl}R_{lk}, \quad \{\omega_i, \omega_j\} = \epsilon_{ijk}(\omega_k + B_k), \quad (137)$$

where we denote by $R = (R_{ij})$ the elements of $SO(3)$ with

$$(\dot{R}R^t)_{ij} = \epsilon_{ijk}\omega_k,$$

and $R_{ij} = \cos \theta \delta_{ij} + (1 - \cos \theta)n_i n_j + \sin \theta \epsilon_{ijk}n_k$, $n^2 = 1$. The dynamics has the form of eq. (132), with

$$H(R, \omega) = -\frac{1}{2}\omega^2 + V(R),$$

and $V(R) = V(\cos \theta) = C(\frac{1}{2} + 2\cos \theta)^2$. We obtain in this way the Leggett equation for the B-phase of the ^3He superfluid [Br78]. Thus we can construct a Lagrangian for such system on $TSO(3)$ as indicated in §7.

10 Standard formulation for internal variables

If the external configuration space is trivial, i.e., the system has only internal degrees of freedom, we can call it a pure isospin system. Such systems are generally associated with first order equations of motion as was the case in §3. The isospin variables I_a parametrizing the system will be defined on the Lie algebra $\mathfrak{g}$ of a compact semisimple Lie group G , typically $SU(n)$. We can identify $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$ using a biinvariant metric on G , the Killing-Cartan metric for instance. In any case, we can assume that the inner space is the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of the Lie group G . The standard Poisson structure for a pure isospin system will be the linear one defined by eq. (108). In order to apply to it the results on §8 we can think of it as a trivial bundle F with fibre $\mathfrak{g}^*$ over a single point $Q = \{\text{point}\}$. The Poisson tensor Λ_f on it will again be chosen as the natural linear Poisson tensor $\Lambda_{\mathfrak{g}}$, eq. (108), induced by the Lie group G . Notice that because TQ is a single point, the bundle $TQ \bowtie F$ is simply $\mathfrak{g}^*$, and the Poisson tensor Λ will be $\Lambda_{\mathfrak{g}}$. But this tensor satisfies all conditions from Thm. 5 except that it is in general nonregular. If we select a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$, then $\Lambda_{\mathcal{O}} = \Lambda_{\mathfrak{g}}|_{\mathcal{O}}$ becomes regular and the associated symplectic form is the well-known Kostant-Kirillov-Souriau form $\omega_{\mathcal{O}}$ on $\mathcal{O}$. Then, Thm. 5 applied to $\Lambda_{\mathcal{O}}$ in $\mathcal{O}$, shows that there must be a Lagrangian (possibly local) realization on $T\mathcal{O}$. This is evident because locally $\omega_{\mathcal{O}}$ must

be exact. Then we can choose any locally defined symplectic potential A such that $dA = \omega_0$ and define the Lagrangian $L(\xi, \dot{\xi}) = A_i(\xi)\dot{\xi}^i$, where ξ^i denote local coordinates on $\mathcal{O}$. This Lagrangian is locally defined and singular. For instance, for the standard formalism for spin variables S_i discussed in §3 and §4 with the standard bracket defined by eq. (67), the coadjoint orbits spanned by S_i are spheres S_r^2 of radius r . The Kostant-Kirillov-Souriau 2-form is a multiple of the area form on S^2 which is locally exact. This allows to define local Lagrangians for the spin dynamics eq. (24). In this particular example, a global (singular) Lagrangian realization can be constructed using the Hopf fibration $S^3 \rightarrow S^2$ (see for instance [Ba82]). In general then, the Lagrangians constructed in this way will be singular. It would be desirable to have a Lagrangian realization by a regular Lagrangian. Such a Lagrangian regularization exists and will be described in the following paragraph.

A Lagrangian realization of the natural Poisson Bracket on $\mathfrak{g}^*$

The space $\mathfrak{g}^*$ can be considered as the Poisson reduction of T^*G , the cotangent bundle of the group G . We will identify T^*G with $G \times \mathfrak{g}^*$ using left translations on the group, i.e., $l^*: T^*G \rightarrow G \times \mathfrak{g}^*$, $l^*(\alpha_g) = (g, TL_g^*\alpha_g)$, for all $\alpha_g \in T_g^*G$, and we will denote by $x = TL_g^*\alpha_g$. With respect to a basis θ^i of left-invariant 1-forms we will have $x = x_i\theta^i$. In these coordinates the Poisson bracket on $\mathfrak{g}^*$ is again, $\{x_i, x_j\} = c_{ij}^k x_k$.

On $T^*G \cong G \times \mathfrak{g}^*$ we consider the canonical 1-form $\theta_0 = x_i\theta^i$. The canonical symplectic 2-form $\omega_0 = -d\theta_0$, given by

$$\omega_0 = \theta^i \wedge dx_i - \frac{1}{2}x_i c_{jk}^i \theta^j \wedge \theta^k,$$

on the cotangent bundle T^*G is biinvariant with respect to the lifting of the action of G by left or right invariant translations. For instance considering a basis of left invariant vector fields $X_1, \dots, X_n$ on G , we can consider their cotangent lifts $X_1^*, \dots, X_n^*$ to T^*G by setting

$$\mathcal{L}_{X_j^*}(\pi^*f) = \pi^*(\mathcal{L}_{X_j}f), \quad \forall f \in \mathcal{F}(G),$$

with $\pi: T^*G \rightarrow G$ the cotangent bundle projection, and

$$\mathcal{L}_{X_j^*}\theta_0 = 0.$$

We notice that $\mathcal{L}_{X_j^*}\theta_0 = 0$ implies that $i_{X_j^*}d\theta_0 = -d\langle X_j^*, \theta_0 \rangle$ and $\langle X_j^*, \theta_0 \rangle$ depends only on X_j . Then the nondegeneracy of ω_0 uniquely defines X_j^* . The vector fields X_i^* are thus Hamiltonian with Hamiltonian $J_i = \langle X_i^*, \theta_0 \rangle$. Similar results occur for right-invariant vector fields. Because the infinitesimal generators of the left action of G on itself are right-invariant vector fields, and vice versa, we have that the map $J_R = J_i\theta_i^*: T^*G \rightarrow \mathfrak{g}^*$, represents the momentum map of the Hamiltonian action defined by right-translations on the group. Then it is easy to see that J_R , the momentum map corresponding to right translations, is defined by $J_R(\alpha_g) = TL_g^*\alpha_g$, where $\alpha_g \in T_g^*G$. Notice that J_R corresponds to the projection π_L of T^*G onto $\mathfrak{g}^*$ by the left action of G on T^*G .

Elements of $\mathfrak{g}$ define linear functions on $\mathfrak{g}^*$ and the elements X_i define the linear functions x_i . Thus the projection $J_R: T^*G \rightarrow \mathfrak{g}^*$ gives $J_R^*(x_i) = J_i$. The natural symplectic structure ω_0 on T^*G defines a Poisson structure Λ_0 which can be written as

$$\Lambda_0 = X_i^L \wedge \frac{\partial}{\partial x_i} + c_{jk}^i x_i \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}. \quad (138)$$

The most important fact about J_R is that it is a Poisson map with respect to the Poisson structure Λ_0 defined on T^*G by ω_0 and the Poisson structure $\Lambda_{\mathfrak{g}}$. This is easily seen from the particular expression of the Poisson Brackets on eq. (138) and eq. (108).

Thus, T^*G provides a symplectic realization for a pure isospin system, however, because T^*G is not a tangent bundle, it will not provide a Lagrangian realization. For that, we just need to choose a right invariant Riemannian metric η on G and define the kinetic energy Lagrangian eq. (117) on TG . The Cartan 2-form ω_{L_η} will then be the pull-back of ω_0 by the identification between TG and T^*G provided by the metric η . Then, the action of G on (TG, ω_{L_η}) will define a Lagrangian momentum map $J: TG \rightarrow \mathfrak{g}^*$, given by $J(g, v) = \eta_{ij}v^i\theta^j$, that will be Poisson. More generally, we can consider any Lagrangian function $L \in \mathcal{F}(TG)$ which is nonsingular, i.e., ω_L is symplectic. We consider the left or right action on G and define a lift to TG which a priori does not coincide with the tangent lift. We define

$$\mathcal{L}_{X_i^L}(\tau_G^*f) = \tau_G^*(\mathcal{L}_{X_i}f); \quad \mathcal{L}_{X_i^L}\theta_L = 0,$$

where $\tau_G: TG \rightarrow G$ is the tangent bundle projection, and similarly, for the right action.

It is clear that $P_i = \langle X_i^L, \theta_L \rangle$ allows us to define a projection $P: TG \rightarrow \mathfrak{g}^*$ which is a Lagrangian realization of the Poisson manifold $(\mathfrak{g}^*, \Lambda_{\mathfrak{g}})$.

At this point we may wonder if the Lagrangian vector field associated with L will be P -projectable onto $\mathfrak{g}^*$. Of course a necessary and sufficient condition for this is that $\{E_L, P_i\}_L$ must be P -projectable onto $\mathfrak{g}^*$ for any P_i . This will be the case if the Lagrangian function is invariant under the flow lift of right invariant translations. When this is the case the flow lift coincides with the Liouville lift. (We call Liouville lift the one defined by using θ_L).

What is the relation between the regular Lagrangian realization of $\mathfrak{g}^*$ on TG described above and the (local) singular Lagrangian realization on $T\mathcal{O}$? The answer is easily obtained if we extend the coadjoint orbit to the full group. That is, we consider $\mathcal{O}$ as an homogeneous space of G , $\mathcal{O} = G/G_{\mathcal{O}}$ where $G_{\mathcal{O}}$ is the isotropy group of an arbitrary element $\mu \in \mathcal{O}$. Then μ will define a left-invariant 1-form α_μ on G by the formula

$$\alpha_\mu(g) = TL_{g^{-1}}^*\mu, \quad \forall g \in G;$$

Namely, $\alpha_\mu = \mu_i\theta^i$. A simple computation shows that $d\alpha_\mu = \mu_i c_{jk}^i \theta^j \wedge \theta^k$ is a closed presymplectic 2-form on G which is G_μ -invariant, i.e., the 2-form $d\alpha_\mu$ is projectable to $\mathcal{O}$. Moreover, the characteristic distribution of $d\alpha_\mu$ is spanned by the elements on $\mathfrak{g}_\mu$, the Lie algebra of G_μ . Then, the projected 2-form on $\mathcal{O}$ will be symplectic and it coincides with the Kostant-Kirillov-Souriau symplectic form $\omega_{\mathcal{O}}$.

Consider now the linear Lagrangian on TG defined by the 1-form α_μ , i.e.,

$$L_\mu(g, v) = \mu_i v^i. \quad (139)$$

It is clear that $\theta_{L_\mu} = \tau_G^*\alpha_\mu$ and the Cartan 2-form is given by,

$$\omega_{L_\mu} = \tau_G^*d\alpha_\mu.$$

The Lagrangian L_μ is of course degenerate and the kernel K of its Cartan 2-form is given by $K = \mathfrak{g}_\mu \times \mathfrak{g}$, the factor $\mathfrak{g}$ corresponding to all vertical vectors on $TG \cong G \times \mathfrak{g}$ and $\mathfrak{g}_\mu$ corresponding to the kernel of $d\alpha_\mu$ on G . It is then clear that $TG/K = \mathcal{O}$, and in this sense, the singular Lagrangian L_μ , eq. (139), provides a singular Lagrangian realization for the coadjoint orbit $\mathcal{O}$.

Lagrangian realization of hydrodynamical systems of Kirchhoff type

We can apply the previous ideas to a system quite different to the spin dynamics considered above. The equations of motion for a rigid body in an ideal incompressible fluid at rest at infinity are given by Kirchhoff equations [Mi60] assuming the motion of the fluid to be potential,

$$\dot{\vec{p}} = \vec{p} \times \vec{w}; \quad \dot{\vec{M}} = \vec{M} \times \vec{w} + \vec{p} \times \vec{u}, \quad (140)$$

$\vec{M}$ and $\vec{p}$ are the total angular and linear momentum and $u_i = \partial H / \partial p_i$, $w_i = \partial H / \partial M_i$, where H is the energy of the system and is assumed to be positive and quadratic in both variables $\vec{M}$ and $\vec{p}$. This equations can be put in Hamiltonian form considering the variables $\vec{M}$, $\vec{p}$ lying in the dual of the Lie algebra of the Euclidean group $E(3)$. The Poisson tensor has the form

$$\Lambda_{E(3)} = \epsilon_{ijk} p_k \frac{\partial}{\partial M_i} \wedge \frac{\partial}{\partial p_j} + \epsilon_{ijk} M_k \frac{\partial}{\partial M_i} \wedge \frac{\partial}{\partial M_j}. \quad (141)$$

Notice that if we think of the dual of the Lie algebra of $E(3)$ as a trivial bundle $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, the tensor $\Lambda_{E(3)}$ is not localizable on the bundle because of the term corresponding to the bracket $\{p_i, M_j\} = \epsilon_{ijk} p_k$ (see eqs. (106) and (109)). Another system that can be reduced to a Poisson dynamics using the tensor (141) is the Legget equation of spin dynamics in the A-phase of the superfluid ^3He (compare with the B-phase discussed in section before). The construction discussed above permits to lift this dynamics to a Lagrangian setting on $TE(3)$. For this we shall use a metric η on $E(3)$ and the momentum map $J_R: TE(3) \rightarrow \mathbb{R}^6$ and the Lagrangian will be simply the kinetic energy L_η .

The ideas developed in the previous paragraphs can be extended to the following situation: Let $P(G, Q)$ be a principal fibre bundle over the base manifold Q with fibre the group G . We shall consider the Poisson bundle $\text{ad}^*P \rightarrow Q$ described in §8. This bundle can be obtained by symplectic reduction of the universal phase space for particles in a Yang-Mills field [We78], [St77]. The details of the computations have been exhaustively described elsewhere (see for instance [Gu84]), which we summarize below. Consider the cotangent bundle T^*P . In addition, consider the cotangent lift of the action of G on P to T^*P . The action is Hamiltonian with momentum map $J: T^*P \rightarrow \mathfrak{g}^*$. The quotient of T^*P by the action of G is diffeomorphic to the bundle $p: E = \pi^*(\text{ad}^*P) \rightarrow T^*Q$. The bundle E carries a natural Poisson structure which is localizable, Newtonian and velocity independent. A Lagrangian realization can be easily obtained using a regular Lagrangian function L on TP and lifting the action of G to TP using θ_L as indicated in the previous subsection. [See [Al94] for details on the tangent bundle setting.]

11 Nonstandard formulations

In this section we will analyze some solutions to the inverse problem of Poisson dynamics on group manifolds. We will rediscover Lie-Poisson structures on group manifolds that will be used to provide an alternative Hamiltonian description of the isotropic rotator. The equations of motion (123) are SODE's and consequently are associated with external variables. In general, solutions to the inverse problem will provide different models for variables, beside the linear Poisson structure used in the two previous section.

The inverse problem of Poisson dynamics on lower dimensional spaces

To deal with Poisson structures we consider a general approach which is very useful for lower dimensional problems. The strategy has been developed in general terms in [Gr93], [Ko85] and [Ca94], and consists of using the exterior differential calculus on forms, instead of the graded Lie bracket on multivector fields. This can be done by associating $(n-k)$ -forms with k -vector fields V on $\mathbb{R}^n$ (or on an arbitrary manifold) by setting

$$\Psi_V = i_V \Omega, \quad (142)$$

Ω being a volume form on $\mathbb{R}^n$. The map Ψ provides an isomorphism between k -vector fields and $(n-k)$ -forms. On monomial multivector fields we have,

$$(i_V \Omega)(Y_1, \dots, Y_{n-k}) = \Omega(X_1, \dots, X_k, Y_1, \dots, Y_{n-k}),$$

for $V = X_1 \wedge \dots \wedge X_k$. For a bivector field $\Lambda = \Lambda^{ij} \partial / \partial x^i \wedge \partial / \partial x^j$ and $\Omega = dx^1 \wedge \dots \wedge dx^n$, we have

$$\Psi_\Lambda = 2 \sum_{i < j} (-1)^{i+j} \Lambda^{ij} dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n,$$

where the symbol $\widehat{}$ stands for omission. We should notice that this isomorphism is not natural and depends on Ω .

By using the Schouten bracket (see §5) on multivector fields V_1 and V_2 , we find

$$i_{[V_1, V_2]} = -i_{V_1} i_{V_2} d - di_{V_2} \wedge V_1 + i_{V_1} di_{V_2} + i_{V_2} di_{V_1},$$

which generalizes the analogous relation,

$$i_{[X, Y]} = i_X i_Y d - di_{X \wedge Y} + i_X di_Y - i_Y di_X,$$

for vector fields X and Y .

A relevant proposition for our purposes is the following:

Proposition 2 [Gr93] *A bivector field Λ is a Poisson structure if and only if $2i_\Lambda d\Psi_\Lambda = d\Psi_{\Lambda \wedge \Lambda}$. Two Poisson structures Λ_1 and Λ_2 are compatible if and only if $d\Psi_{\Lambda_1 \wedge \Lambda_2} = i_{\Lambda_1} d\Psi_{\Lambda_2} + i_{\Lambda_2} d\Psi_{\Lambda_1}$.*

In low dimensions we have some immediate results:

On $\mathbb{R}^2$, with coordinates (x, y) , any bivector field $\Lambda = f(x, y) \partial / \partial x \wedge \partial / \partial y$ defines a Poisson structure because there $[\Lambda, \Lambda]$ is a skewsymmetric three tensor on a two-dimensional manifold, which therefore vanishes identically.

On $\mathbb{R}^3$, with coordinates (x^1, x^2, x^3) , we have $\Lambda = \Lambda^{ij} \partial / \partial x^i \wedge \partial / \partial x^j$ and with $\Omega = dx^1 \wedge dx^2 \wedge dx^3$, we get $\Psi_\Lambda = \epsilon_{ijk} \Lambda^{ij} dx^k$. The Jacobi identity gives $\Psi_\Lambda \wedge d\Psi_\Lambda = 0$, i.e., the 1-form Ψ_Λ must admit an integrating factor. Locally this implies that any Poisson structure is parametrized by two functions, f_0, f_1 , such that $\Psi_\Lambda = f_0 df_1$ and the associated Poisson bracket has the form

$$\{x^i, x^j\} = f_0 \epsilon^{ijk} \frac{\partial f_1}{\partial x^k}.$$

This result was used in §3 to write eq. (22). If we have a dynamical vector field Γ and we look for a Hamiltonian description, we need to solve the equation $\Lambda(dH) = -\Gamma$. For this problem

it is better to start with a volume in contravariant form, say $\tilde{\Omega} = g \partial/\partial x_1 \wedge \dots \wedge \partial/\partial x_n$. Then instead of $i_\Lambda \Omega = \Psi$, we have $i_\Psi \tilde{\Omega} = \Lambda$. If it is possible to select Ω such that $\mathcal{L}_\Gamma \Omega = 0$, we have $\Lambda(dH) = i_\Psi \wedge dH \tilde{\Omega} = \Gamma$ or $i_\Gamma \Omega = \Psi \wedge dH$. Therefore if Γ admits an invariant volume, a necessary and sufficient condition for Γ to admit a Hamiltonian description is that $i_\Gamma \Omega$ is the product of two 1-forms $\alpha \wedge \beta$ with $d\beta = 0$ and $\alpha \wedge d\alpha = 0$. Thus, if $\alpha = f_0 df_1$, the function f_1 must be a constant of the motion for Γ .

Because of $d\Psi_{\Lambda\Lambda} = 0$, we find that

$$i_{\Lambda_1} d(f_2 df_3) + i_{\Lambda_2} d(f_0 df_2) = 0,$$

reduces to

$$d(f_0 df_1 + f_2 df_3) \wedge (f_2 df_3 + f_0 df_1) = 0,$$

i.e.,

$$(f_0 df_2 - f_2 df_0) \wedge df_1 \wedge df_3 = 0.$$

where we have used our formula on compatible Poisson structures.

If the bracket is an admissible bracket for Γ , then f_1 and f_3 are two constants of the motion and we get a family of admissible brackets by taking f_0 and f_2 to also be constants of the motion.

In general,

$$(f_0 df_2 - f_2 df_0)(\Gamma) = 0$$

gives $d(\log |f_2| - \log |f_0|)(\Gamma) = 0$, i.e., the ratio of f_0 and f_2 is a constant of the motion for Γ . Thus, if f_1 and f_3 are two independent constants of the motion, we get

$$\Psi = f_0 df_1 + f_0 F(f_1, f_3) df_3,$$

i.e.,

$$\{x^i, x^j\} = f_0 \epsilon^{ijk} \left(\frac{\partial f_1}{\partial x^k} + F(f_1, f_3) \frac{\partial f_3}{\partial x^k} \right)$$

is the most general expression for Poisson brackets compatible with Γ .

Poisson structures on three-dimensional Lie groups

We now consider a Lie group G realized as a group of matrices. In this way if $g = (g_{ij})$, we have $dg = (dg_{ij})$ and the left and right Maurer-Cartan 1-forms eq. (116) valued 1-form are given by,

$$\Theta_L = g^{-1} \cdot dg; \quad \Theta_R = dg \cdot g^{-1},$$

respectively. If we consider a basis E_i for the Lie algebra $\mathfrak{g}$ of G , the associated left invariant vector fields $X_1^L, \dots, X_n^L$, and right invariant ones $X_1^R, \dots, X_n^R$, satisfy,

$$i_{X_j^L}(g^{-1} \cdot dg) = E_j, \quad \text{or} \quad i_{X_j^R}(dg \cdot g^{-1}) = E_j.$$

Any bivector field on G can be written as $\Lambda = \Lambda_L^{ij} X_i^L \wedge X_j^L = \Lambda_R^{ij} X_i^R \wedge X_j^R$, where $\Lambda_L^{ij} \in \mathcal{F}(G)$. By using a basis of 1-forms we can also define a volume $\Omega_L = \theta_1^L \wedge \theta_2^L \wedge \dots \wedge \theta_n^L$ or $\Omega_R = \theta_1^R \wedge \theta_2^R \wedge \dots \wedge \theta_n^R$ which coincide for unimodular groups. Then we can associate a $(n-2)$ -form Ψ_Λ with any bivector field Λ by using eq. (142).

We shall now restrict our considerations to 3-dimensional Lie groups. We consider $\Omega_L = \theta_1^L \wedge \theta_2^L \wedge \theta_3^L$ and $\Lambda = \Lambda_L^{ij} X_i^L \wedge X_j^L$ to find the 1-form,

$$\Psi_\Lambda = \epsilon_{ijk} \Lambda_L^{ij} \theta_L^k = f_k \theta_L^k = \alpha,$$

with $f_k = \epsilon_{ijk} \Lambda_L^{ij}$. The Jacobi identity once more gives $\alpha \wedge d\alpha = 0$, i.e.,

$$f_j df_k \wedge \theta_L^j \wedge \theta_L^k = \frac{1}{2} f_k f_m c_{ij}{}^k \theta_L^m \wedge \theta_L^i \wedge \theta_L^j.$$

On the other hand, it is possible to write the differential of any function f in the form $df = (\mathcal{L}_{X_i^L} f) \theta_L^i$. Then by using this expression we get

$$f_j \mathcal{L}_{X_m^L} f_k \theta_L^m \wedge \theta_L^j \wedge \theta_L^k + \frac{1}{2} f_k f_m c_{ij}{}^k \theta_L^m \wedge \theta_L^i \wedge \theta_L^j = 0.$$

Therefore the equation for (f_1, f_2, f_3) reads as:

$$f_i \mathcal{L}_{X_m^L} f_j - f_j \mathcal{L}_{X_m^L} f_i = c_{ij}{}^k f_k f_m, \quad (143)$$

and gives all Poisson structures on the 3-dimensional Lie group G .

To be more specific, we give some examples:

The group $SU(2)$. The group $SU(2)$ is unimodular, and thus $\Omega_L = \Omega_R$. If we look for a left-invariant 1-form $\alpha = f_j \theta_L^j$, $f_j \in \mathbb{R}$, then eq. (143) gives, $f_j = 0$ for all $j = 1, 2, 3$. Thus there are no Poisson brackets on $SU(2)$ which are left-invariant. Similarly, one shows that there are no right-invariant Poisson brackets.

The simplest non trivial case would be to consider

$$\Lambda = \Lambda_L^{ij} X_i^L \wedge X_j^L + \Lambda_R^{ij} X_i^R \wedge X_j^R.$$

Because of the unimodularity of $SU(2)$ and carrying on the computation of $\alpha \wedge d\alpha$, we find a solution for Λ_L^{ij} and Λ_R^{ij} as

$$\Lambda_L^{ij} = -\Lambda_R^{ij} \in \mathbb{R}.$$

Therefore,

$$\Lambda = X_i^L \wedge X_j^L - X_i^R \wedge X_j^R \quad (144)$$

is a Poisson structure on $SU(2)$. This bracket defines a Lie-Poisson bracket on $SU(2)$ [Sk82] (see [Ta89] for a review).

It may be useful to find the precise Poisson structure associated with Λ when we realize $SU(2)$ as a group of 2×2 complex matrices, eq(122), with the condition $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. If we consider the left and right invariant vector fields $X_1^L, X_1^R, X_2^L, X_2^R$, associated with

$$E_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \text{ we find}$$

$$\begin{aligned} \Lambda_{SU(2)} &= (x_2^2 + x_1^2) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_0} + x_3 x_1 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} + x_0 x_2 \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_2} \\ &\quad + x_0 x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + x_0 x_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \end{aligned} \quad (145)$$

$$= \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \wedge \left(x_3 \frac{\partial}{\partial x_0} - x_0 \frac{\partial}{\partial x_3} \right) + (x_2^2 + x_1^2) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_0}, \quad (146)$$

along with

$$\Psi_{\Lambda_{SU(2)}} = (x_2^2 + x_1^2) \frac{x_2 dx_1 - x_1 dx_2}{x_2^2 + x_1^2} \wedge d\left(\frac{1}{2}(x_0^2 + x_1^2 + x_2^2 + x_3^2)\right).$$

The group $SL(2, \mathbb{R})$. It is possible repeat the above procedure for $SL(2, \mathbb{R})$. In this case we let our group matrices g have the form,

$$g = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad (147)$$

with the condition $x_1 x_4 - x_2 x_3 = 1$. We find,

$$\Lambda_{SL(2, \mathbb{R})} = x_2 x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1} + 2x_2 x_3 \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_1} + x_1 x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_2 x_4 \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_2} + x_3 x_4 \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_3},$$

with

$$\Psi_{\Lambda_{SL(2, \mathbb{R})}} = (x_2^2 + x_3^2) \frac{x_3 dx_2 - x_2 dx_3}{x_2^2 + x_3^2} \wedge d(x_1 x_4 - x_2 x_3).$$

Let us now consider the inverse problem for Γ on a three dimensional group. We consider $\Omega_L = \theta_L^1 \wedge \theta_L^2 \wedge \theta_L^3$, $i_{\Lambda} \Omega_L = \alpha$ and look for solutions of $\mathcal{L}_{\Gamma} \Lambda = 0$. By setting

$$\alpha = f_i \theta_L^i, \quad \Gamma = g^i X_i^L,$$

we get

- i) $f_i g^i = 0$, i.e., $i_{\Gamma} \alpha = 0$,
- ii) $f_k \text{div} \Gamma = -g^i (\mathcal{L}_{X_k^L} f_i - \mathcal{L}_{X_i^L} f_k + f_j c_{ik}^j)$

along with the conditions on functions f_j 's given by eq. (143).

When $\text{div} \Gamma = 0$ we obtain some particular solutions by using constants of the motion for Γ , say F with $\mathcal{L}_{\Gamma} F = 0$. Then we set $\alpha = dF$ and find $i_{\Lambda} \Omega_L = dF$. From $\text{div} \Gamma = 0$, we get $\mathcal{L}_{\Gamma} \Lambda_L = 0$, with Γ in the image of Λ_L because Γ is tangent to the foliation induced by F .

Alternative Hamiltonian description of the isotropic rotator

We have seen that the solution of the inverse problem for Poisson dynamics on Lie groups leads to families of Poisson brackets, the simplest ones being Lie-Poisson brackets on the group. Thus we can relax the localizability condition in the classical treatment of Feynman's problem to assume that the Poisson tensor in the total space is π -related to a Lie-Poisson tensor on the configuration group manifold. We shall explore such assumption in the simple situation of the group $SU(2)$, but most of the ideas can be extended to arbitrary Lie-Poisson structures. The main result we will obtain is that the Hamiltonian description of a particle moving on $S^3 \cong SU(2)$ is not unique [Ma93], [St94]. We end up with a Hamiltonian formulation which is canonically inequivalent to the free system given in the §9, yet it yields the same equations of motion, i.e., (123).

We begin by examining the spatial coordinates. As our spatial manifold is S^3 we can once again utilize $SU(2)$ matrices g as coordinates eq. (122). Then we introduce the weak assumption of nonlocalizability which means that the Poisson brackets between matrix elements take the quadratic relations eq. (146). It will be convenient in what follows to change

the notation and to use tensor product notation for the Poisson brackets on $SU(2)$. Then the Poisson brackets defined by the Poisson tensor eq. (146), reads as [Sk82]

$$\{g_1, g_2\} = [r_{12}, g_1 g_2]. \quad (148)$$

The indices 1 and 2 refer to two separate vector spaces on which the $SU(2)$ matrices act. The matrix r_{12} , known as the classical r -matrix, acts nontrivially on both vector spaces 1 and 2, while g_1 and g_2 are defined by the tensor products:

$$g_1 = g \otimes 1 \quad g_2 = 1 \otimes g.$$

As usual the Poisson bracket should be antisymmetric and satisfy the Jacobi identity. Antisymmetry of (148) requires that

$$[r_{12} + r_{21}, g_1 g_2] = 0, \quad (149)$$

or $r_{12} + r_{21}$ is an adjoint invariant. The Jacobi identity involving three $SU(2)$ elements g_1 , g_2 and g_3 leads to certain quadratic relations for the classical r matrix known as the modified classical Yang-Baxter relations. They are:

$$[r_{23}, r_{31}] + [r_{31}, r_{12}] + [r_{12}, r_{23}] = \text{adjoint invariant}. \quad (150)$$

Solutions to (149) and (150) are well known. If we express g in terms of its defining matrix representation then r can be written in terms of 4×4 matrices. A one parameter family of such matrices satisfying (149) and (150) is

$$r = \frac{i\lambda}{4} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & 4 & -1 & \\ & & & 1 \end{pmatrix}, \quad (151)$$

where we assume the parameter λ to be real. Only in the limit $\lambda \rightarrow 0$ do we recover the ordinary condition of localizability.

As the dimension of the phase space should be six, we must introduce three more variables in addition to the spatial coordinates g . We denote the new variables by S_3 and $S_{\pm}$, where S_3 is real and $S_{\pm}$ is complex. From them we shall later define the analogue of the angular velocity ω_i of section §9. For the Poisson brackets of S_i with themselves we take

$$\{S_{\pm}, S_3\} = \pm \frac{i\lambda}{2} S_{\pm} S_3, \quad \{S_+, S_-\} = -i\lambda \left((S_3)^2 - (S_3)^{-2} \right), \quad (152)$$

where S_+ is the complex conjugate of S_- . These brackets were utilized previously in an alternative description of spin, eq. (37). They were seen to satisfy the Jacobi identity, and to have the classical Casimir function

$$f_1 = S_+ S_- + (S_3)^2 + (S_3)^{-2}. \quad (153)$$

Thus $\{S_i, f_1\} = 0$. Moreover the Poisson brackets (152) were shown to correspond to the classical limit of the $U_q(\mathfrak{sl}(2))$ Hopf algebra.

The Poisson brackets (152) can be re-expressed so that they resemble the brackets (148) for g . For this define the 2×2 lower triangular matrices $\ell^{(-)}$,

$$\ell^{(-)} = \begin{pmatrix} S_3 & 0 \\ S_- & S_3^{-1} \end{pmatrix}. \quad (154)$$

The set of such matrices forms the $SB(2, \mathbb{C})$ group. Furthermore, if we denote their conjugate inverses by $\ell^{(+)} = \ell^{(-)\dagger}$, then (152) can be written according to

$$\{\ell_1^{(\pm)}, \ell_2^{(\pm)}\} = [r_{21}, \ell_1^{(\pm)} \ell_2^{(\pm)}], \quad (155)$$

$$\{\ell_1^{(+)}, \ell_2^{(-)}\} = [r_{21}, \ell_1^{(+)} \ell_2^{(-)}]. \quad (156)$$

Eqs. (155) and (156) satisfy the requirement of antisymmetry and the Jacobi identity for the same reasons as does (148). In terms of the matrix $\ell^{(-)}$ the classical Casimir function can be written $f_1 = \text{Tr } \ell^{(-)} \ell^{(-)\dagger}$.

It remains to specify the Poisson bracket relations for S_i with the spatial coordinates g . This choice must of course be consistent with the Jacobi identity and the Poisson brackets (148) and (152). One such choice is:

$$\begin{aligned} \{S_3, g\} &= -\frac{i\lambda}{4} S_3 \sigma_3 g, \\ \{S_{\pm}, g\} &= -i\lambda \left(\frac{1}{4} S_{\pm} \sigma_3 + S_3^{-1} \sigma_{\mp} \right) g, \end{aligned} \quad (157)$$

where $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. These relations can be re-expressed in terms of the matrices $\ell^{(\pm)}$ as follows:

$$\begin{aligned} \{\ell_1^{(-)}, g_2\} &= -\ell_1^{(-)} r_{12} g_2, \\ \{\ell_1^{(+)}, g_2\} &= \ell_1^{(+)} r_{21} g_2. \end{aligned} \quad (158)$$

Next we introduce the Hamilton function which we denote by $H^{(\lambda)}$. The equations of motion for the free system (123) state that the angular velocities are constants of the motion, so they should have zero Poisson brackets with $H^{(\lambda)}$. We will construct the angular velocities from S_i alone. Then $H^{(\lambda)}$ should have zero Poisson bracket with S_i , and the Hamiltonian can therefore depend only on the Casimir function (153). We take

$$H^{(\lambda)} = -\frac{1}{2\lambda^2} (f_1 - 2). \quad (159)$$

We will justify this choice later.

It then remains to compute the equation of motion for g . From (157) and (159) we get

$$-2i\dot{g}g^\dagger = -2i\{g, H^{(\lambda)}\} g^\dagger = \Omega_3 \sigma_3 + \Omega_- \sigma_+ + \Omega_+ \sigma_-. \quad (160)$$

Ω_i are angular velocities, analogous to ω_i of section §9. Here they are defined according to

$$\Omega_3 = -\frac{1}{2\lambda} \left((S_3)^2 - (S_3)^{-2} + S_- S_+ \right),$$

$$\Omega_{\pm} = \Omega_1 \pm i\Omega_2 = -\frac{1}{\lambda} S_{\mp} S_3^{-1}, \quad (161)$$

or

$$\Omega_i = -\frac{1}{2\lambda} \text{Tr } \ell^{(-)} \sigma_i \ell^{(-)\dagger}.$$

Since we also have that

$$\dot{\Omega}_i = \{\Omega_i, H^{(\lambda)}\} = 0,$$

we have recovered the equation of motion (123) of the isotropic rigid rotator.

The above formulation of the isotropic rotator is a one parameter deformation of the standard formulation given in section §9. Thus, it is possible to recover the standard formulation as a limiting case. The relevant parameter here is λ and the limiting value for the parameter is $\lambda = 0$. In this regard, we have already seen that the Poisson bracket (148) of matrix elements of g with themselves vanishes when $\lambda \rightarrow 0$. The remaining Poisson brackets reduce to the canonical ones [i.e., (125) and (126)] when $\lambda \rightarrow 0$ provided that we make the variables S_i functions of the angular velocity ω_i of section §9, as well as the parameter λ ,

$$S_i = S_i^{(\lambda)}(\omega).$$

For this, we need to assume the following behavior for small λ

$$\begin{aligned} S_3 &= 1 - \frac{\lambda}{2} \omega_3 + \frac{\lambda^2}{8} (\omega_3)^2 + O(\lambda^3), \\ S_{\pm} &= -\lambda \omega_{\mp} + O(\lambda^3). \end{aligned} \quad (162)$$

Then (125) and (126) result from (152) and (157), respectively, upon taking the limit $\lambda \rightarrow 0$. Because the variables S_i are functions of angular velocities ω_i and the parameter λ , then so are angular velocities Ω_i functions of ω_i and λ :

$$\Omega_i = \Omega_i^{(\lambda)}(\omega).$$

From (161) and (162) it follows that the two angular velocities coincide in the limit $\lambda \rightarrow 0$,

$$\Omega_i^{(\lambda \rightarrow 0)}(\omega) = \omega_i.$$

Finally we need to show that the Hamiltonians (127) and (159) of the two different formulations of the rotator agree in the $\lambda \rightarrow 0$ limit. For this we see that the classical Casimir function f_1 has the following small λ behaviour

$$f_1 \rightarrow 2 + \lambda^2 \omega_i \omega_i + O(\lambda^3),$$

from which it follows that

$$H^{(\lambda \rightarrow 0)} = H_0,$$

H_0 given by (127). This of course was the reason for choosing $H^{(\lambda)}$ of the form given in (159).

For $\lambda \neq 0$, the treatment given here is canonically inequivalent to that given in section §9. Here, for instance, we see that the Hamiltonian (159) is not proportional to the square of the angular velocity [unlike (127)], nor does the Poisson bracket algebra of the angular velocity Ω_i correspond to an $SU(2)$ algebra [unlike (126)]. Perhaps the most distinguishing feature

of the formulation presented here concerns the symmetries. In the standard formulation we saw that chiral transformations were canonical symmetries. This is not the case in the alternative formulation. In fact, chiral transformations can not be canonically implemented for the Poisson structure given by (148), (152) and (157). For example, under the right action of $SU(2)$,

$$g \rightarrow gv, \quad \ell^{(\pm)} \rightarrow \ell^{(\pm)}, \quad v \in SU(2), \quad (163)$$

the brackets (148) undergo the following transformation:

$$\{g_1, g_2\} \rightarrow \{g_1 v_1, g_2 v_2\} \neq [r_{12}, g_1 v_1 g_2 v_2]. \quad (164)$$

Instead of being canonical the right action (163) is Lie-Poisson. Poisson Lie group transformations are classical analogues of quantum group transformations. The former are defined so that group multiplication is a Poisson map. Thus with regard to transformation (163) we want that $\{v_1, v_2\} \neq 0$, $v_1 = v \otimes 1$, $v_2 = 1 \otimes v$. If one chooses the following Poisson bracket for matrix elements of v with themselves

$$\{v_1, v_2\} = [r_{12}, v_1 v_2], \quad (165)$$

(and zero with g and $\ell^{(\pm)}$) then the inequality in (164) can be replaced by an equality. In fact, from (165) it follows that all Poisson brackets (148), (152) and (157) are unchanged by this transformation. Then right multiplication is a Poisson map and (163) is a Poisson Lie group transformation. Further, since $\ell^{(\pm)}$ are unchanged by this transformation, and the Hamiltonian (159) depends only on these variables, then this transformation leaves the dynamics invariant. We then say that (163) is a Poisson Lie group symmetry.

There exists another Poisson Lie group symmetry and it is the analogue of the left $SU(2)$ canonical symmetry of the standard formalism. Unlike right $SU(2)$ given by (163) it has a nontrivial action on $\ell^{(\pm)}$ (as well as g). In order to specify this action it is convenient to introduce another set of variables $d^{(\pm)}$, known as classical double variables [Dr83]. These variables are defined by

$$d^{(\pm)} = \ell^{(\pm)} g. \quad (166)$$

It follows that $d^{(\pm)}$ is an element of $SL(2, \mathbb{C})$, (166) being an Iwasawa decomposition. In terms of these variables the right action of $SU(2)$ is given by $d^{(\pm)} \rightarrow d^{(\pm)} v$. We now define the left action of $SU(2)$ by

$$d^{(\pm)} \rightarrow u d^{(\pm)}, \quad u \in SU(2). \quad (167)$$

The Poisson brackets for $d^{(\pm)}$ can be constructed from those of g and $\ell^{(\pm)}$. One finds

$$\begin{aligned} \{d_1^{(\pm)}, d_2^{(\pm)}\} &= -d_1^{(\pm)} d_2^{(\pm)} r_{12} + r_{21} d_1^{(\pm)} d_2^{(\pm)}, \\ \{d_1^{(-)}, d_2^{(+)}\} &= -d_1^{(-)} d_2^{(+)} r_{12} - r_{12} d_1^{(-)} d_2^{(+)}. \end{aligned} \quad (168)$$

These relations are invariant under (167) upon insisting that $u \in SU(2)$ has the following Poisson bracket with itself

$$\{u_1, u_2\} = [r_{12}, u_1 u_2],$$

and zero Poisson bracket with $d^{(\pm)}$. Then $SU(2)$ left multiplication is a Poisson map and (167) is a Poisson Lie group transformation. Further, since the classical Hamiltonian can be written

$$H^{(\lambda)} = -\frac{1}{2\lambda^2} \left(\text{Tr } d^{(-)} d^{(+)-1} - 2 \right),$$

we can use the cyclic property of trace to show that it is unchanged under (167), and hence $SU(2)$ left multiplication is a Lie Poisson symmetry.

The Hamiltonian formalism presented here of free motion on $SU(2)$ can be quantized and the result is free motion on a noncommuting space, namely the quantum group $SU_q(2)$ [St94]. Furthermore, upon quantization, the two Lie Poisson symmetries turn into quantum group symmetries.

An intriguing question concerns whether or not there exists an analogous alternative Hamiltonian description for the interacting system given in §9, eqs. (129-136). It would involve introducing electromagnetic fields, which when quantized would be defined on a noncommutative space, namely $SU_q(2)$.

12 Conclusions and outlook

The classical Feynman procedure has been reexamined in the context of the geometry of Poisson manifolds and Lagrangian systems. A generalization of the classical procedure has been discussed and the conditions describing a family of systems was given. Among the systems are the Wong equations and the magnetic moment coupling for spinning particles, as well as systems related to Lie groups like the isotropic rotator and Legget equations for the phases A and B of a ^3He superfluid.

The close connection between this problem and the inverse problem for Poisson dynamics and the inverse problem of the calculus of particles variations was noted. The physical hypotheses introduced in the classical analysis of Feynman's problem were carefully described in this setting. Newtonian, localizable Poisson tensors provide the geometrical description of the basic assumptions on Feynman's problem. From our analysis of such tensors, we can make the following conclusions:

- The standard Feynman procedure will not produce dynamics beyond Lagrangian dynamics unless the mild regularity conditions discussed in the paper are violated.
- The form of the (possibly singular) Lagrangian for the system is fixed by the conditions of localizability, existence of a Hamiltonian second order differential equation (Newtonian character of the Poisson brackets), and velocity independence together with a regularity condition.
- Standard Lagrangian realizations can be recovered for pure isospin systems and for systems on Lie groups.
- Standard Lagrangian realizations for systems on Lie groups can be easily extended to systems with internal and external degrees of freedom. This is done by using reduction methods for universal phase spaces of Yang-Mills particles and the equivalent tangent bundle description.
- Non-standard Poisson descriptions exist if the localizability assumption is weakened in the context of Lie-Poisson geometry and an explicit construction is shown.

Finally, we indicate other avenues of development which have not yet been explored:

- The analogue of Feynman's procedure for physical systems with anticommuting degrees of freedom.
- The existence of symplectic and Lagrangian realizations for non-standard solutions to Feynman's problem.
- The uniqueness of such realizations if they exist.

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