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Deriving the Standard Model from the
Simplest two-point K-cycle

by

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Abstract

Basing on a differential algebra over the simplest two-point K-cycle and graded Lie algebras of homomorphisms of finite projective modules we derive the classical action of the standard model. This construction uses both the general framework of non-commutative geometry developed by A. Connes and ideas of the Mainz-Marseille approach to model building. We get a prediction of the Weinberg angle and constraints between the fermion masses and the masses of the W - and Higgs-bosons on tree-level, which differ from the relations obtained by Kastler and Schücker for the quaternionic K-cycle.

1 Introduction

This paper is the continuation of two earlier papers ([13], [14]) by R. Matthes, G. Rudolph and R. Wulkenhaar. In the first of them we presented an analysis of the structure of a differential algebra $\Lambda_{\mathcal{A}}^*$ associated to a K-cycle $(\mathcal{A}, h, D)$ over the simplest two-point algebra $\mathcal{A} = C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$. In the second one we constructed for a given finite projective right $\mathcal{A}$ -module with Hermitian structure $\mathcal{E}$ the graded Lie algebra $\mathcal{H} = \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Lambda_{\mathcal{A}}^*)$. We showed that a certain graded Lie subalgebra $\mathcal{H}_0$ of $\mathcal{H}$ provides a rigorous mathematical link between Connes' theory and the Mainz–Marseille model building scheme. The general definitions of connections, curvatures, gauge groups and bosonic and fermionic actions were given, basing on the general theory by A. Connes ([3], [4]), see also [16]. The K-cycle mentioned above together with a finite projective module $\mathcal{E} = e\mathcal{A}^2$ was used by Connes in [3], [2] and by Connes and Lott in [4] to derive the Salam–Weinberg model in a unified form. Within this scheme the above K-cycle cannot be used to obtain the full standard model. This is why Connes and Lott proposed a K-cycle over the algebra $C_{\mathbb{R}}^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{H})$, a detailed exposition of these ideas was presented by Kastler in [9] and [10], see [8] for an earlier version. One obtains a prediction of the Weinberg angle and certain constraints between the masses of the fermions and the masses of the W-, Z-, and Higgs-bosons [11].

Another way of obtaining the standard model by non-commutative geometry is the Mainz–Marseille model ([6], [5]), which uses the graded Lie algebra $\Lambda^*(X) \otimes \text{spl}(2, 1)$ of matrix-valued differential forms, see also [15]. We have shown in [14] that the assumptions of this approach are natural within Connes' framework. For the present paper the construction of the fermionic sector of the standard model given in [5] is of particular interest. This construction makes use of the theory of representations of the graded Lie algebra $\text{spl}(2, 1)$ in a finite dimensional vector space.

Our strategy is the following: In section 2 we give a review of some results obtained in [13] and [14]. In section 3 we construct an isomorphism $\mathbf{i}$ of graded Lie algebras from $\mathcal{H}_0$, which we constructed in [14] within Connes' theory, onto its image, for which the fermionic sector in [5] serves as a guiding line. Next, in section 4, we give an embedding of a subspace of $\mathbf{i}(\mathcal{H}_0)$ into the algebra of bounded operators on the Hilbert space of fermions. This allows to construct the fermionic action in section 5, the bosonic action of the Salam–Weinberg model in section 6, and the chromodynamics sector in section 7. After a Wick rotation to Minkowski space and a reparametrization this action coincides with the classical action of the standard model, where some free parameters of the standard model are fixed, see section 8.

Thus, our construction rests – as the construction of Connes, Lott and Kastler – upon a K-cycle. However, we take a K-cycle over the algebra $\mathcal{A} = C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$, while they start with a K-cycle over the much bigger algebras $C_{\mathbb{R}}^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{H})$ and $C_{\mathbb{R}}^\infty(X) \otimes (\mathbb{C} \oplus M_3\mathbb{C})$. To compensate this difference we use two finite projective $\mathcal{A}$ -modules as additional input, one module for the electroweak sector and the other one for the chromodynamics sector. In the construction of Connes, Lott and Kastler the module is identical with the algebra of their K-cycle, such that in their version the calculus of finite projective modules is not necessary. In both versions the differential on the

differential algebras associated to the K-cycles are composed of the classical exterior differential and a matrix-differential, which in both versions contains fermionic mass parameters. In their construction the just mentioned differential algebra is of primary importance for the physical model, while in our version the differential algebra plays an auxiliary role for building the graded Lie algebra $\mathcal{H}_0$. Moreover, the Hilbert space of our K-cycle is only auxiliary, we have to add our physical Hilbert space by hand.

As it was shown in [14] one can derive from $\mathcal{H}_0$ the graded Lie algebra $\Lambda^*(X) \otimes spl(2, 1)$, which is the starting point for the construction of the standard model in the Mainz–Marseille approach. The essential difference between $\mathcal{H}_0$ and $\Lambda^*(X) \otimes spl(2, 1)$ is that $\mathcal{H}_0$ carries the fermionic mass parameters of the differential algebra associated to the K-cycle. It are these fermionic mass parameters which relate the masses of the fermions with the masses of the W -, Z - and Higgs-bosons in the model of Connes, Lott and Kastler as well as in our model. Since the fermionic mass parameters are absent in $\Lambda^*(X) \otimes spl(2, 1)$, there is no relation between fermion masses and boson masses in the Mainz–Marseille model.

Both in our model and in the Mainz–Marseille approach one needs certain representations of the graded Lie algebras $\mathcal{H}_0$ respectively $\Lambda^*(X) \otimes spl(2, 1)$, which are generalizations of classical $spl(2, 1)$ -representations [15]. The representations of $\Lambda^*(X) \otimes spl(2, 1)$ generalize reducible indecomposable representations of $spl(2, 1)$, which give the possibility to describe mixing between fermion generations [5]. For our model we take generalizations of the simplest irreducible representations of $spl(2, 1)$, because the fermion generations are intrinsically contained in $\mathcal{H}_0$. With these representations there enters a great number of additional parameters the models due to not canonically determined free normalization constants of $sl(2, \mathbb{C}) \oplus gl(1, \mathbb{C})$ -subrepresentations of $spl(2, 1)$. In the Mainz–Marseille construction these parameters are fitted to the fermion masses and the Kobayashi–Maskawa matrix. In our model there is a subtle interplay between these normalization parameters and the intrinsic fermionic mass parameters of the differential calculus. In the final Lagrangian there occur only such combinations of these many parameters that effectively there is only one additional free parameter in our model compared with the model of Connes, Lott and Kastler.

We obtain $\sin^2 \theta_W = \frac{3}{8}$ for the Weinberg angle θ_W , the relation $\sqrt{2}m_W < m_t \leq \sqrt{\frac{8}{3}}m_W$ between the masses m_W of the W -boson and m_t of the top quark and $m_H \leq 2.43m_W$ for the mass m_H of the Higgs boson. These relations differ from the relations derived by Kastler and Schücker [11] from the standard approach using a K-cycle over the algebra $C_{\mathbb{R}}^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{H})$. The above uncertainty is due to the one additional parameter.

2 Review of Earlier Results

In this section we give a review of some results, which were obtained in [13] and [14] and which we need for what follows. For technical reasons let for the moment X be a compact four dimensional Riemannian spin manifold. We denote by $L^2(X, S)$ the Hilbert space of square integrable sections of the spinor bundle over X , by F a finite

dimensional Hilbert space, by C the Clifford bundle of the cotangent space over X and by C^k the set of those sections of C , whose values at each point $x \in X$ belong to the subspace spanned by products of less than or equal k elements of T_x^*X of the same parity. The simplest two-point K-cycle $(\mathcal{A}, h, D)$ consists of the algebra $\mathcal{A} = C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$ acting on the Hilbert space $h = L^2(X, S) \otimes (F \oplus F)$ and a generalized Dirac operator $D = (D^{cl} \otimes \text{id}_{F \oplus F}) \oplus (\gamma^5 \otimes \mathcal{M})$. Here, D^{cl} is the classical Dirac operator and γ^5 the grading operator on $L^2(X, S)$. Moreover, $\mathcal{M} = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}$, where M is an endomorphism of F .

Now we are going to specify the parameters. We take F and M as

$$F = \mathbb{C}^3 \oplus \mathbb{C}^3, \quad M = \begin{pmatrix} -m_l & 0 \\ 0 & -m_q \end{pmatrix}, \quad (1)$$

where m_l and m_q are real diagonal 3×3 -matrices with non-negative entries. In principle, we could take for m_l and m_q arbitrary complex 3×3 -matrices, which can be written as $u_1 \delta u_2$, with u_1, u_2 unitary and δ diagonal and positive. However, the unitary matrices u_1, u_2 can be absorbed by unitary transformations of the physical fields.

One shows that the algebra $\pi(\Omega^*)$ obtained from an involutive representation π of the universal differential algebra Ω^* over $\mathcal{A}$ has the structure

$$\begin{aligned} \pi(\Omega^*) &= \bigoplus_{k=0}^{\infty} \pi(\Omega^k), \\ \pi(\Omega^k) &= \begin{bmatrix} \bigoplus_{r=0}^m C^{k-2r} \otimes \mathbb{C} M_1^r & ; & \bigoplus_{r=0}^m C^{k-2r-1} \gamma^5 \otimes \mathbb{C} M_2^r \\ \bigoplus_{r=0}^m C^{k-2r-1} \gamma^5 \otimes \mathbb{C} M_3^r & ; & \bigoplus_{r=0}^m C^{k-2r} \otimes \mathbb{C} M_4^r \end{bmatrix}, \end{aligned} \quad (2)$$

where

$$M_1^r := (MM^*)^r, \quad M_2^r := M(M^*M)^r, \quad M_3^r := M^*(MM^*)^r, \quad M_4^r := (M^*M)^r. \quad (3)$$

In particular, we can identify $\mathcal{A}$ with $\pi(\Omega^0)$. We denote by $(m+1)$ the number of linear independent elements $(MM^*)^r$. In our case M and M^* commute with each other and generically we have $m=6$. We define $L^n := C^n / C^{n-2}$, $L^* := \bigoplus_{k=0}^4 L^k$, and put $L^n = \{0\}$ for $n < 0$. There is a graded algebra $\Lambda_{\mathcal{A}}^*$ associated with $\pi(\Omega^*)$ defined as follows:

$$\Lambda_{\mathcal{A}}^* := \bigoplus_{k=0}^{\infty} \Lambda_{\mathcal{A}}^k, \quad \Lambda_{\mathcal{A}}^k \equiv \sigma_k \circ \pi(\Omega^k) := \begin{cases} \pi(\Omega^k) / \pi(\Omega^{k-2}) & \text{for } k \geq 2, \\ \pi(\Omega^k) & \text{for } k = 0, 1, \end{cases} \quad (4)$$

with multiplication

$$\Lambda_{\mathcal{A}}^k \times \Lambda_{\mathcal{A}}^n \ni (\lambda^k, \tilde{\lambda}^n) \mapsto \lambda^k \bullet \tilde{\lambda}^n := \sigma_{k+n}(\tau^k \tilde{\tau}^n) \in \Lambda_{\mathcal{A}}^{k+n}, \quad (5)$$

where $\tau^k \in \pi(\Omega^k)$, $\tilde{\tau}^n \in \pi(\Omega^n)$, such that $\sigma_k(\tau^k) = \lambda^k$, $\sigma_l(\tilde{\tau}^n) = \tilde{\lambda}^n$. One can show that elements $\lambda^k \in \Lambda_{\mathcal{A}}^k$ have the form

$$\lambda^k = \begin{pmatrix} \sum_{r=0}^m \alpha_1^{k-2r} \otimes \mathbb{C} M_1^r & ; & \sum_{r=0}^m \alpha_2^{k-2r-1} \gamma^5 \otimes \mathbb{C} M_2^r \\ \sum_{r=0}^m \alpha_3^{k-2r-1} \gamma^5 \otimes \mathbb{C} M_3^r & ; & \sum_{r=0}^m \alpha_4^{k-2r} \otimes \mathbb{C} M_4^r \end{pmatrix}, \quad \alpha_q^n \in L^n. \quad (6)$$

Let ι_k denote the isomorphism of L^k onto $\Lambda^k(X)$, where $\Lambda^k(X)$ is the space of complex valued k -forms on X . We define the following operations:

$$\begin{aligned} [\gamma^5 \otimes \mathcal{M}, \lambda^k]_g &:= (\gamma^5 \otimes \mathcal{M}) \bullet \lambda^k - (-1)^k \lambda^k \bullet (\gamma^5 \otimes \mathcal{M}), \\ \mathbf{d}\alpha^k &:= \iota_{k+1}^{-1} \circ d \circ \iota_k(\alpha^k), \mathbf{d}^* := \gamma^5 \mathbf{d} \gamma^5, \\ \mathbf{D}\lambda^k &:= pr_{k+1} \circ ((\mathbf{d} - \mathbf{d}^*) \otimes \text{id}_F \otimes \mathbb{1}_{2 \times 2})(\lambda^k), \end{aligned} \quad (7)$$

for $\alpha^k \in L^k$, $\lambda^k \in \Lambda_{\mathcal{A}}^k$, where pr_{k+1} denotes the projection from $\Lambda_{\mathcal{A}}^{k+1} \oplus \Lambda_{\mathcal{A}}^{k-1}$ onto $\Lambda_{\mathcal{A}}^{k+1}$. One shows that

$$\hat{d} := \mathbf{D} - i[\gamma^5 \otimes \mathcal{M}, \cdot]_g \quad (8)$$

is a graded differential on $\Lambda_{\mathcal{A}}^*$, and one can write down explicit multiplication and differential rules for elements of $\Lambda_{\mathcal{A}}^*$, see [13].

In [14] we considered finite projective right $\mathcal{A}$ -modules with Hermitian structure $\mathcal{E} = e\mathcal{A}^p$, where $e \in \text{End}_{\mathcal{A}}(\mathcal{E})$, fulfilling $e^2 = e$. For the above defined algebra $\mathcal{A}$ one can pass by similarity transformations to equivalent modules, such that $e = e^*$. We consider the vector spaces $\mathcal{H}^k := \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Lambda_{\mathcal{A}}^k)$. Elements $\varrho^k \in \mathcal{H}^k$ can be identified with $p \times p$ -matrices of elements of $\Lambda_{\mathcal{A}}^k$, fulfilling $e\varrho^k e = \varrho^k$:

$$\varrho^k = \begin{pmatrix} \varrho_{11}^k & \cdots & \varrho_{1j}^k & \cdots & \varrho_{1p}^k \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \varrho_{i1}^k & \cdots & \varrho_{ij}^k & \cdots & \varrho_{ip}^k \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \varrho_{p1}^k & \cdots & \varrho_{pj}^k & \cdots & \varrho_{pp}^k \end{pmatrix}, \quad \varrho_{ij}^k \in \Lambda_{\mathcal{A}}^k, \quad i, j = 1, \dots, p. \quad (9)$$

The space $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}^k$ is an algebra with multiplication $\bullet$ given by the tensor product of the multiplication $\bullet$ in $\Lambda_{\mathcal{A}}^*$ and multiplication of $p \times p$ -matrices. On $\mathcal{H}$ we have a canonical derivation $D_{\mathcal{H}}$ given by

$$D_{\mathcal{H}}\varrho = e\hat{d}(\varrho)e, \quad \varrho \in \mathcal{H}, \quad (10)$$

where $\hat{d}$ means the componentwise action on matrix elements belonging to $\Lambda_{\mathcal{A}}^*$. With respect to the graded commutator

$$[\varrho^k, \tilde{\varrho}^n]_g := \varrho^k \bullet \tilde{\varrho}^n - (-1)^{kn} \tilde{\varrho}^n \bullet \varrho^k, \quad \varrho^k \in \mathcal{H}^k, \quad \tilde{\varrho}^n \in \mathcal{H}^n, \quad (11)$$

$\mathcal{H}$ is a graded Lie algebra with graded derivation $D_{\mathcal{H}}$.

We define on $\Lambda_{\mathcal{A}}^*$ a linear map $T_{\Lambda} : \Lambda_{\mathcal{A}}^* \rightarrow L^*$ by

$$T_{\Lambda} \left(\begin{pmatrix} \sum_{i=0}^m \alpha_1^{k-2i} \otimes M_1^i & ; & \sum_{i=0}^m \alpha_2^{k-2i-1} \gamma^5 \otimes M_2^i \\ \sum_{i=0}^m \alpha_3^{k-2i-1} \gamma^5 \otimes M_3^i & ; & \sum_{i=0}^m \alpha_4^{k-2i} \otimes M_4^i \end{pmatrix} \right) := \sum_{i=0}^m (\alpha_1^{k-2i} + \alpha_4^{k-2i}), \quad (12)$$

see (6). Moreover, we define a special element $\Gamma_{\Lambda} = \begin{pmatrix} 1 \otimes \text{id}_F & 0 \\ 0 & -1 \otimes \text{id}_F \end{pmatrix} \in \mathcal{A}$, where 1 denotes the unit element of $C^{\infty}(X)$. We have proved in [14] that there exists a graded Lie subalgebra $\mathcal{H}_0$ of $\mathcal{H}$ defined by

$$\mathcal{H}_0 = \bigoplus_{k=0}^{\infty} \mathcal{H}_0^k, \quad \mathcal{H}_0^k = \{ \varrho^k \in \mathcal{H}^k, \sum_{i=1}^p T_{\Lambda}(\Gamma_{\Lambda} \varrho_{ii}^k) = 0 \} \quad (13)$$

(in the notation of (9)). Moreover, $D_{\mathcal{H}}$ is a graded derivation on $\mathcal{H}_0$.

Any connection ∇ on $\mathcal{E}$ has the form

$$\nabla = e\hat{d} + \rho, \quad \rho = -\rho^* \in \mathcal{H}^1. \quad (14)$$

The situation found in physics suggests to consider special connections, namely such that $\rho \in \mathcal{H}_0^1$. This implies that the gauge group is restricted to

$$\mathcal{U} = \{ u \in \text{Aut}_{\mathcal{A}}(\mathcal{E}), \quad uu^* = u^*u = e, \quad u\mathbf{d}u^* \in \mathcal{H}_0^1 \}, \quad (15)$$

see [14], where $\mathbf{d}u^*$ means the action of the differential $\mathbf{d}$ on the L^* -component of $u^* \in \mathcal{U}$. Gauge transformations of the connection ∇ are given by $u\nabla u^*$.

3 An Isomorphism of Graded Lie Algebras

Here we fix the module $\mathcal{E}$ by taking $p = 2$ and $e \doteq \begin{pmatrix} e' & 0 \\ 0 & \mathbb{1}_{\mathcal{A}} \end{pmatrix}$, where $\mathbb{1}_{\mathcal{A}} = \begin{pmatrix} 1 \otimes \text{id}_F & 0 \\ 0 & 1 \otimes \text{id}_F \end{pmatrix} \in \mathcal{A}$ is the identity of $\mathcal{A}$ and $e' = \begin{pmatrix} 1 \otimes \text{id}_F & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$. Then from (6), (9), (12), (13) and the discussion in [14] we get that elements $\varrho^k \in \mathcal{H}_0^k$ have the form

$$\varrho^k = \sum_{r=0}^m \left(\begin{array}{cc|cc} \frac{1}{2}(\alpha_0^{k-2r} + \alpha_3^{k-2r}) \otimes M_1^r & 0 & \alpha_2^{k-2r} \otimes M_1^r & \alpha_4^{k-2r-1} \gamma^5 \otimes M_2^r \\ 0 & 0 & 0 & 0 \\ \hline \alpha_+^{k-2r} \otimes M_1^r & 0 & \frac{1}{2}(\alpha_0^{k-2r} - \alpha_3^{k-2r}) \otimes M_1^r & \alpha_5^{k-2r-1} \gamma^5 \otimes M_2^r \\ \alpha_6^{k-2r-1} \gamma^5 \otimes M_3^r & 0 & \alpha_7^{k-2r-1} \gamma^5 \otimes M_3^r & \alpha_0^{k-2r} \otimes M_4^r \end{array} \right), \quad (16)$$

where $\alpha_f^n \in L^n$, $f = 0, +, -, 3, 4, 5, 6, 7$. Due to (1) and (3) we have $M_s^r \in M_3\mathbb{C} \otimes M_2\mathbb{C}$, which means $\varrho^k \in L^* \otimes M_4\mathbb{C} \otimes M_3\mathbb{C} \otimes M_2\mathbb{C}$. In (16) we considered ϱ^k as a 4×4 -matrix with $L^* \otimes M_3\mathbb{C} \otimes M_2\mathbb{C}$ -valued entries. Of course, ϱ^k can also be treated as a 2×2 -matrix with $L^* \otimes M_4\mathbb{C} \otimes M_3\mathbb{C}$ -valued entries. With regard to this view, ϱ^k has the form

$$\varrho^k = \begin{pmatrix} \varrho_l^k & 0 \\ 0 & \varrho_q^k \end{pmatrix}, \quad (17)$$

$$\varrho_l^k = \sum_{r=0}^m \left(\begin{array}{cc|cc} \frac{1}{2}(\alpha_0^{k-2r} + \alpha_3^{k-2r}) \otimes l_1^r & 0 & \alpha_-^{k-2r} \otimes l_1^r & \alpha_4^{k-2r-1} \gamma^5 \otimes l_2^r \\ 0 & 0 & 0 & 0 \\ \alpha_+^{k-2r} \otimes l_1^r & 0 & \frac{1}{2}(\alpha_0^{k-2r} - \alpha_3^{k-2r}) \otimes l_1^r & \alpha_5^{k-2r-1} \gamma^5 \otimes l_2^r \\ \alpha_6^{k-2r-1} \gamma^5 \otimes l_3^r & 0 & \alpha_7^{k-2r-1} \gamma^5 \otimes l_3^r & \alpha_0^{k-2r} \otimes l_4^r \end{array} \right),$$

$$\varrho_q^k = \sum_{r=0}^m \left(\begin{array}{cc|cc} \frac{1}{2}(\alpha_0^{k-2r} + \alpha_3^{k-2r}) \otimes q_1^r & 0 & \alpha_-^{k-2r} \otimes q_1^r & \alpha_4^{k-2r-1} \gamma^5 \otimes q_2^r \\ 0 & 0 & 0 & 0 \\ \alpha_+^{k-2r} \otimes q_1^r & 0 & \frac{1}{2}(\alpha_0^{k-2r} - \alpha_3^{k-2r}) \otimes q_1^r & \alpha_5^{k-2r-1} \gamma^5 \otimes q_2^r \\ \alpha_6^{k-2r-1} \gamma^5 \otimes q_3^r & 0 & \alpha_7^{k-2r-1} \gamma^5 \otimes q_3^r & \alpha_0^{k-2r} \otimes q_4^r \end{array} \right),$$

where

$$\begin{aligned} l_1^r &= (m_l m_l^*)^r, \quad l_2^r = -m_l (m_l^* m_l)^r, \quad l_3^r = -m_l^* (m_l m_l^*)^r, \quad l_4^r = (m_l^* m_l)^r, \\ q_1^r &= (m_q m_q^*)^r, \quad q_2^r = -m_q (m_q^* m_q)^r, \quad q_3^r = -m_q^* (m_q m_q^*)^r, \quad q_4^r = (m_q^* m_q)^r. \end{aligned} \quad (18)$$

We introduce an isomorphism $\mathbf{i}$ of graded Lie algebras, which generalizes certain representations ([15], [5]) of the graded Lie algebra $spl(2, 1)$ to the graded Lie algebra $\mathcal{H}_0$:

$$\mathbf{i}(\varrho^k) = \begin{pmatrix} \mathbf{i}(\varrho_l^k) & 0 \\ 0 & \mathbf{i}(\varrho_q^k) \otimes \mathbb{1}_{3 \times 3} \end{pmatrix}, \quad (19)$$

$$\mathbf{i}(\varrho_l^k) = \sum_{r=0}^m \left(\begin{array}{ccc} \frac{1}{2}(\alpha_0^{k-2r} + \alpha_3^{k-2r}) \otimes l_1^r & \alpha_-^{k-2r} \otimes l_1^r & \alpha_4^{k-2r-1} \gamma^5 \otimes \epsilon l_2^r \\ \alpha_+^{k-2r} \otimes l_1^r & \frac{1}{2}(\alpha_0^{k-2r} - \alpha_3^{k-2r}) \otimes l_1^r & \alpha_5^{k-2r-1} \gamma^5 \otimes \epsilon l_2^r \\ \alpha_6^{k-2r-1} \gamma^5 \otimes \epsilon^{-1} l_3^r & \alpha_7^{k-2r-1} \gamma^5 \otimes \epsilon^{-1} l_3^r & \alpha_0^{k-2r} \otimes l_4^r \end{array} \right), \quad (20)$$

$$\mathbf{i}(\varrho_q^k) = \sum_{r=0}^m \left(\begin{array}{ccc|c} \left(\frac{1}{2} \alpha_3^{k-2r} - \frac{1}{6} \alpha_0^{k-2r} \right) \otimes q_1^r & \alpha_-^{k-2r} \otimes q_1^r & \alpha_7^{k-2r-1} \gamma^5 \otimes \beta \sqrt{\frac{2}{3}} q_2^r & \alpha_4^{k-2r-1} \gamma^5 \otimes \sqrt{\frac{1}{3}} q_2^r \chi \gamma \\ \alpha_+^{k-2r} \otimes q_1^r & \left(-\frac{1}{2} \alpha_3^{k-2r} - \frac{1}{6} \alpha_0^{k-2r} \right) \otimes q_1^r & -\alpha_6^{k-2r-1} \gamma^5 \otimes \beta \sqrt{\frac{2}{3}} q_2^r & \alpha_5^{k-2r-1} \gamma^5 \otimes \sqrt{\frac{1}{3}} q_2^r \chi \gamma \\ -\alpha_5^{k-2r-1} \gamma^5 \otimes \beta^{-1} \sqrt{\frac{2}{3}} q_3^r & \alpha_4^{k-2r-1} \gamma^5 \otimes \beta^{-1} \sqrt{\frac{2}{3}} q_3^r & -\frac{2}{3} \alpha_0^{k-2r} \otimes q_4^r & 0 \\ \alpha_6^{k-2r-1} \gamma^5 \otimes (\chi \gamma)^{-1} \sqrt{\frac{1}{3}} q_3^r & \alpha_7^{k-2r-1} \gamma^5 \otimes (\chi \gamma)^{-1} \sqrt{\frac{1}{3}} q_3^r & 0 & \frac{1}{3} \alpha_0^{k-2r} \otimes \chi^{-1} q_4^r \chi \end{array} \right). \quad (21)$$

The isomorphism $\mathbf{i}$ fulfils $\mathbf{i}([\varrho^k, \tilde{\varrho}^n]_g) = \mathbf{i}(\varrho^k)\mathbf{i}(\tilde{\varrho}^n) - (-1)^{kn}\mathbf{i}(\tilde{\varrho}^n)\mathbf{i}(\varrho^k)$, for $\varrho^k \in \mathcal{H}_0^k$ and $\tilde{\varrho}^n \in \mathcal{H}_0^n$, where the multiplication $\mathbf{i}(\varrho^k)\mathbf{i}(\tilde{\varrho}^n)$ is the natural combination of the multiplication $\wedge$ in L^* and matrix multiplication.

In the above formulae ϵ and β are invertible diagonal 3×3 -matrices which, therefore, commute with m_l, m_l^*, m_q, m_q^* . For the invertible 3×3 -matrices γ and χ we have to demand $(\chi\gamma)^{-1}m_q^*m_q\chi\gamma = \chi^{-1}m_q^*m_q\chi$, which is achieved by taking $\chi\gamma\chi^{-1}$ diagonal. The matrix χ need not to be unitary. In principle, we could in an analogous way introduce matrices such as χ in the third row and column in (20) and (21), too. However, this can be reabsorbed by unitary transformations of the physical fields. Thus, the freedom in the choice of the isomorphism $\mathbf{i}$, modulo unitary transformations, introduced a lot of additional free parameters in (20) and (21). But, we shall see at the end of section 5 that all these parameters are (not uniquely) fixed by the physical model. The genuine free parameters are the six eigenvalues of M , see (1).

The motivation to consider just the isomorphism (20), (21) comes from our paper [14], where we constructed a partial homomorphism of $\mathcal{H}_0$ onto $\Lambda^*(X) \otimes spl(2, 1)$, and the paper [5], where similar looking representations of $spl(2, 1)$ were used to write down a fermionic Lagrangian for the standard model.

4 Bounded Operators on the Hilbert Space of Fermions

Here we are going to associate to each $\mathbf{i}(\varrho^k) \in \mathbf{i}(\mathcal{H}_0^k)$, $k = 0, 1, 2$, a bounded operator on the Hilbert space

$$H = \{L^2(X, S) \otimes \mathbb{C}^3 \otimes \mathbb{C}^3\} \oplus \{L^2(X, S) \otimes (\mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4) \otimes \mathbb{C}^3\}. \quad (22)$$

First, because of $L^0 \equiv C^0$ and $L^1 \equiv C^1$ we can regard elements of $\mathbf{i}(\mathcal{H}_0^0)$ and $\mathbf{i}(\mathcal{H}_0^1)$ in a natural way as bounded operators on H , see (20) and (21), where the matrices l_s^r and q_s^r act on the last $\mathbb{C}^3$ -components. Next, we associate to the local basis $[\gamma^\mu] \wedge [\gamma^\nu]$ in L^2 the operator $\mathbf{c}([\gamma^\mu] \wedge [\gamma^\nu]) := \gamma^\mu \cdot \gamma^\nu \in B(L^2(X, S))$, for $1 \leq \mu < \nu \leq 4$. By linear extension we obtain a vector space isomorphism $\mathbf{c}$ of L^2 onto its image. The isomorphism $\mathbf{c}$ induces the vector space isomorphism $\tilde{\mathbf{c}} : \mathbf{i}(\mathcal{H}_0^2) \rightarrow \tilde{\mathbf{c}} \circ \mathbf{i}(\mathcal{H}_0^2) \subset B(H)$:

$$\tilde{\mathbf{c}} \circ \mathbf{i}(\varrho^2) := \begin{pmatrix} (\mathbf{c} \otimes \text{id}_{\mathbb{C}^3 \otimes \mathbb{C}^3})(\mathbf{i}(\varrho_l^2)) & 0 \\ 0 & \{(\mathbf{c} \otimes \text{id}_{\mathbb{C}^4 \otimes \mathbb{C}^3})(\mathbf{i}(\varrho_q^2))\} \otimes \mathbb{1}_{3 \times 3} \end{pmatrix}, \quad (23)$$

for $\varrho^2 \in \mathcal{H}_0^2$. Next, we define a vector subspace $\tilde{\mathcal{T}}^2 \subset B(H)$ as

$$\tilde{\mathcal{T}}^2 = \left\{ \sum_{\alpha} (\mathbf{i}(\varrho_{\alpha}^1) \cdot \mathbf{i}(\tilde{\varrho}_{\alpha}^1) + \mathbf{i}(\tilde{\varrho}_{\alpha}^1) \cdot \mathbf{i}(\varrho_{\alpha}^1)) , \quad \varrho_{\alpha}^1, \tilde{\varrho}_{\alpha}^1 \in \mathcal{H}_0^1, \text{ finite sum } \right\}, \quad (24)$$

where $\cdot$ denotes the multiplication in $B(H)$, and put

$$\mathcal{T}^2 = \tilde{\mathbf{c}} \circ \mathbf{i}(\mathcal{H}_0^2) + \tilde{\mathcal{T}}^2. \quad (25)$$

One easily convinces oneself that another characterization of this space is

$$\begin{aligned}\mathcal{T}^2 &= \tilde{\mathbf{c}} \circ \mathbf{i}(\mathcal{H}_0^2) \oplus \mathbf{i}(\mathcal{H}_0^0) \oplus \Delta, \\ \Delta &= \{ \mathbf{i}(\text{diag}(\alpha_3^0, 0, -\alpha_3^0, 0) \otimes \text{id}_F) \cdot \mathbf{i}(\text{diag}(\tilde{\alpha}_3^0, 0, -\tilde{\alpha}_3^0, 0) \otimes \text{id}_F) \\ &\quad + \mathbf{i}(\text{diag}(\alpha_0^0, 0, \alpha_0^0, 2\alpha_0^0) \otimes \text{id}_F) \cdot \mathbf{i}(\text{diag}(\tilde{\alpha}_0^0, 0, \tilde{\alpha}_0^0, 2\tilde{\alpha}_0^0) \otimes \text{id}_F), \\ &\quad \alpha_3^0, \tilde{\alpha}_3^0, \alpha_0^0, \tilde{\alpha}_0^0 \in C^0 \}.\end{aligned}\tag{26}$$

Let $\mathbf{p}_2$ be the projection of $\mathcal{T}^2 = \tilde{\mathbf{c}} \circ \mathbf{i}(\mathcal{H}_0^2) \oplus \mathbf{i}(\mathcal{H}_0^0) \oplus \Delta$ onto its first component $\tilde{\mathbf{c}} \circ \mathbf{i}(\mathcal{H}_0^2)$. Then we have

$$\tilde{\mathbf{c}}^{-1} \circ \mathbf{p}_2(\mathbf{i}(\varrho^1) \cdot \mathbf{i}(\tilde{\varrho}^1) + \mathbf{i}(\tilde{\varrho}^1) \cdot \mathbf{i}(\varrho^1)) = \mathbf{i}([\varrho^1, \tilde{\varrho}^1]_g),\tag{27}$$

which is in some sense an analogy to (5).

On $B(H)$ there is a natural scalar product given by the Dixmier trace Tr_ω :

$$\langle b, \tilde{b} \rangle_{B(H)} = \text{Tr}_\omega(b^* \tilde{b} |D|^{-4}), \quad b, \tilde{b} \in B(H),\tag{28}$$

where D is the generalized Dirac operator of the K-cycle and $*$ denotes the involution in $B(H)$. In the case considered here the Dixmier trace can be expressed by a combination of the usual trace over the matrix structure including the trace in the Clifford algebra and integration over the manifold X :

$$\langle b, \tilde{b} \rangle_{B(H)} = \frac{1}{32\pi^2} \int_X v_g \text{tr}(b^* \tilde{b}),\tag{29}$$

where v_g is the canonical volume form on X and the factor $\frac{1}{32\pi^2}$ is taken from [16]. We denote the restriction of the scalar product (28) to $\mathcal{T}^2 \subset B(H)$ by $\langle, \rangle_{\mathcal{T}^2}$.

The direct sum decomposition $\mathcal{T}^2 = \tilde{\mathbf{c}} \circ \mathbf{i}(\mathcal{H}_0^2) \oplus \ker \mathbf{p}_2$ is not an orthogonal decomposition with respect to the above defined scalar product on $\mathcal{T}^2$. Let $\mathbf{s}_2$ be the orthogonal projection from $\mathcal{T}^2$ onto the orthogonal complement of the subspace $\ker \mathbf{p}_2$ (after a completion with respect to the scalar product). Then we define the canonical embedding $\mathbf{e} : \mathcal{H}_0^2 \rightarrow B(H)$ by

$$\mathbf{e}(\varrho^2) = \mathbf{s}_2 \circ \tilde{\mathbf{c}} \circ \mathbf{i}(\varrho^2), \quad \varrho^2 \in \mathcal{H}^2.\tag{30}$$

Let us define

$$\tilde{l}_s^2 := l_s^2 - \frac{\text{tr } l_s^2}{3} \mathbb{1}_{3 \times 3}, \quad \tilde{q}_s^2 := q_s^2 - \frac{\text{tr } q_s^2}{3} \mathbb{1}_{3 \times 3}, \quad s = 1, 4.\tag{31}$$

In the same way as in [13] one can show that $\mathbf{e}(\varrho^2)$ is given by (20) and (21) if we replace

$$\alpha_f^2 \mapsto \mathbf{c}(\alpha_f^2), \quad f = 0, +, -, 3, \quad l_s^2 \mapsto \tilde{l}_s^2, \quad q_s^2 \mapsto \tilde{q}_s^2, \quad s = 1, 4.\tag{32}$$

Thus, the operators on H associated to $\mathbf{i}(\varrho^k) \in \mathbf{i}(\mathcal{H}_0^k)$, $k = 0, 1, 2$, are $\mathbf{i}(\varrho^k)$ themselves for $k = 0, 1$ and $\mathbf{e}(\varrho^k)$ for $k = 2$. Of course, this construction can be extended to $\mathbf{i}(\mathcal{H}_0^k)$, $k > 2$, but we do not need this for model building.

5 The Fermionic Action

The connection form ρ is a skew-adjoint element of $\mathcal{H}_0^1$, see (14). Therefore, we have

$$\rho = \begin{pmatrix} (\frac{1}{2}A^3 + \frac{1}{2}A^0) \otimes \text{id}_F & 0 & A^- \otimes \text{id}_F & -i\Phi^1 \gamma^5 \otimes M \\ 0 & 0 & 0 & 0 \\ A^+ \otimes \text{id}_F & 0 & (-\frac{1}{2}A^3 + \frac{1}{2}A^0) \otimes \text{id}_F & -i\Phi^2 \gamma^5 \otimes M \\ -i\bar{\Phi}^1 \gamma^5 \otimes M^* & 0 & -i\bar{\Phi}^2 \gamma^5 \otimes M^* & A^0 \otimes \text{id}_F \end{pmatrix}, \quad (33)$$

where $A^0 = -(A^0)^*$, $A^3 = -(A^3)^*$, $A^+ = -(A^-)^* \in L^1$ and $\Phi^1, \Phi^2 \in L^0$. Applying the isomorphism $\mathbf{i}$ and abbreviating $\mathbb{1} = \mathbb{1}_{3 \times 3}$, we get from (18), (20) and (21)

$$\mathbf{i}(\rho_l) = \begin{pmatrix} (\frac{1}{2}A^3 + \frac{1}{2}A^0) \otimes \mathbb{1} & A^- \otimes \mathbb{1} & i\Phi^1 \gamma^5 \otimes \epsilon m_l \\ A^+ \otimes \mathbb{1} & (-\frac{1}{2}A^3 + \frac{1}{2}A^0) \otimes \mathbb{1} & i\Phi^2 \gamma^5 \otimes \epsilon m_l \\ i\bar{\Phi}^1 \gamma^5 \otimes \epsilon^{-1} m_l^* & i\bar{\Phi}^2 \gamma^5 \otimes \epsilon^{-1} m_l^* & A^0 \otimes \mathbb{1} \end{pmatrix}, \quad (34)$$

$$\mathbf{i}(\rho_q) = \begin{pmatrix} (\frac{1}{2}A^3 - \frac{1}{6}A^0) \otimes \mathbb{1} & A^- \otimes \mathbb{1} & i\bar{\Phi}^2 \gamma^5 \otimes \sqrt{\frac{2}{3}} \beta m_q & i\Phi^1 \gamma^5 \otimes \sqrt{\frac{1}{3}} m_q \chi \gamma \\ A^+ \otimes \mathbb{1} & (-\frac{1}{2}A^3 - \frac{1}{6}A^0) \otimes \mathbb{1} & -i\bar{\Phi}^1 \gamma^5 \otimes \sqrt{\frac{2}{3}} \beta m_q & i\Phi^2 \gamma^5 \otimes \sqrt{\frac{1}{3}} m_q \chi \gamma \\ -i\Phi^2 \gamma^5 \otimes \sqrt{\frac{2}{3}} \beta^{-1} m_q^* & i\Phi^1 \gamma^5 \otimes \sqrt{\frac{2}{3}} \beta^{-1} m_q^* & -\frac{2}{3}A^0 \otimes \mathbb{1} & 0 \\ i\bar{\Phi}^1 \gamma^5 \otimes \sqrt{\frac{1}{3}} (\chi \gamma)^{-1} m_q^* & i\bar{\Phi}^2 \gamma^5 \otimes \sqrt{\frac{1}{3}} (\chi \gamma)^{-1} m_q^* & 0 & \frac{1}{3}A^0 \otimes \mathbb{1} \end{pmatrix}. \quad (35)$$

The connection ∇ can be extended to an operator $\nabla : \mathcal{E} \otimes_{\mathcal{A}} h \rightarrow \mathcal{E} \otimes_{\mathcal{A}} h$,

$$\begin{aligned} \nabla(\xi \otimes_{\mathcal{A}} \psi) &= (i\nabla \xi) \otimes_{\mathcal{A}} \psi + \xi \otimes_{\mathcal{A}} D\psi = e\{((D \otimes \mathbb{1}_{2 \times 2})\xi - \xi D) + i\rho \xi\} \otimes_{\mathcal{A}} \psi + \xi \otimes_{\mathcal{A}} D\psi \\ &= (e(D \otimes \mathbb{1}_{2 \times 2})e + i\rho)(\xi \otimes_{\mathcal{A}} \psi) = (D^{\text{cl}} + \mu + i\rho)(\xi \otimes_{\mathcal{A}} \psi), \\ \mu &= e(\gamma^5 \otimes \mathcal{M} \otimes \mathbb{1}_{2 \times 2})e, \end{aligned} \quad (36)$$

for $\xi \in \mathcal{E}$, $\psi \in h$, see [4]. We have used $\hat{d}a = -i[D, a]$ for $a \in \mathcal{A}$, see [13], and $L^n \equiv C^n$ for $n = 0, 1$. We have $\mu \in \mathcal{H}_0^1$, hence we can apply the isomorphism $\mathbf{i}$, and we find

$$\mathbf{i}(\mu_l) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\gamma^5 \otimes \epsilon m_l \\ 0 & -\gamma^5 \otimes \epsilon^{-1} m_l^* & 0 \end{pmatrix}, \quad (37)$$

$$\mathbf{i}(\mu_q) = \begin{pmatrix} 0 & 0 & -\gamma^5 \otimes \sqrt{\frac{2}{3}} \beta m_q & 0 \\ 0 & 0 & 0 & -\gamma^5 \otimes \sqrt{\frac{1}{3}} m_q \chi \gamma \\ \gamma^5 \otimes \sqrt{\frac{2}{3}} \frac{1}{\beta} m_q^* & 0 & 0 & 0 \\ 0 & -\gamma^5 \otimes \sqrt{\frac{1}{3}} (\chi \gamma)^{-1} m_q^* & 0 & 0 \end{pmatrix}.$$

The situation found in nature demands to use a pseudo-Riemannian manifold X_M instead of the Riemannian manifold X . We are interested in the case that X_M is the Minkowski space. We convert the results obtained so far for the Euclidian manifold X by a Wick rotation to Minkowski space. If we denote by $L^2(X_M, S)$ the space of square integrable sections of the spinor bundle over X_M then instead of H , see (22), we have to take the space

$$H_M = \{L^2(X_M, S) \otimes \mathbb{C}^3 \oplus \mathbb{C}^3\} \oplus \{L^2(X_M, S) \otimes (\mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4) \otimes \mathbb{C}^3\}. \quad (38)$$

On H_M we have the invariant product

$$\langle \Psi, \tilde{\Psi} \rangle_{H_M} := \int_{X_M} v_M \Psi^* \gamma^0 \tilde{\Psi}, \quad \Psi, \tilde{\Psi} \in H_M, \quad (39)$$

where v_M is the canonical volume form on X_M . Due to (36) the natural fermionic action is

$$S_F = \frac{1}{2} \langle \Psi, (D^{c\ell} + \mathbf{i}(\mu + i\rho_M))\Psi \rangle_{H_M} + h.c., \quad \Psi \in H_M, \quad (40)$$

where ρ_M is the connection form ρ rotated to Minkowski space and $h.c.$ denotes the Hermitian conjugate of the preceding term. We take

$$\Psi = (\Psi_l, \Psi_q)^T, \quad \Psi_l = (v_L, e_L, e_R)^T, \quad \Psi_q = (u_L, d_L, u_R, d_R)^T, \quad (41)$$

where $e_L, v_L, e_R \in L^2(X_M, S) \otimes \mathbb{C}^3$ and $u_L, d_L, u_R, d_R \in L^2(X, S) \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, with

$$(\gamma^5 \otimes \text{id})f_L = -f_L, \quad (\gamma^5 \otimes \text{id})f_R = f_R, \quad f = e, v, u, d. \quad (42)$$

We obtain, denoting the Wick rotated fields $A^{0,+,-,3}$ and $\Phi^{1,2}$ by the same symbols,

$$\begin{aligned} S_F &= \int_{X_M} v_M (\mathcal{L}_l + \mathcal{L}_l' + \mathcal{L}_q + \mathcal{L}_q'), \\ \mathcal{L}_l &= \begin{pmatrix} v_L^*; e_L^* \end{pmatrix} \left\{ \gamma^0 \begin{pmatrix} D^{c\ell} + i(\frac{1}{2}A^3 + \frac{1}{2}A^0) & A^- \\ A^+ & D^{c\ell} + i(-\frac{1}{2}A^3 + \frac{1}{2}A^0) \end{pmatrix} \otimes \mathbb{1} \right\} \begin{pmatrix} v_L \\ e_L \end{pmatrix} \\ &\quad + e_R^* \{ \gamma^0 (D^{c\ell} + iA^0) \otimes \mathbb{1} \} e_R, \\ \mathcal{L}_l' &= -\{ e_R^* \{ \gamma^0 \begin{pmatrix} \bar{\Phi}^1; \bar{\Phi}^2 + 1 \end{pmatrix} \otimes \frac{1}{2}(\epsilon^* - \epsilon^{-1})m_l^* \} \begin{pmatrix} v_L \\ e_L \end{pmatrix} + h.c. \}, \\ \mathcal{L}_q &= \begin{pmatrix} u_L^*; d_L^* \end{pmatrix} \left\{ \gamma^0 \begin{pmatrix} D^{c\ell} + i(\frac{1}{2}A^3 - \frac{1}{6}A^0) & A^- \\ A^+ & D^{c\ell} + i(-\frac{1}{2}A^3 - \frac{1}{6}A^0) \end{pmatrix} \otimes \mathbb{1}_c \otimes \mathbb{1} \right\} \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\ &\quad + u_R^* \{ \gamma^0 (D^{c\ell} - \frac{2}{3}iA^0) \otimes \mathbb{1}_c \otimes \mathbb{1} \} u_R + d_R^* \{ \gamma^0 (D^{c\ell} + \frac{1}{3}iA^0) \otimes \mathbb{1}_c \otimes \mathbb{1} \} d_R, \\ \mathcal{L}_q' &= -\{ u_R^* \{ \gamma^0 \begin{pmatrix} \Phi^2 + 1; -\Phi^1 \end{pmatrix} \otimes \mathbb{1}_c \otimes \frac{1}{2}\sqrt{\frac{2}{3}}(\beta^* + \beta^{-1})m_q^* \} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + h.c. \} \\ &\quad -\{ d_R^* \{ \gamma^0 \begin{pmatrix} \bar{\Phi}^1; \bar{\Phi}^2 + 1 \end{pmatrix} \otimes \mathbb{1}_c \otimes \frac{1}{2}\sqrt{\frac{1}{3}}((\chi\gamma)^* - (\chi\gamma)^{-1})m_q^* \} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + h.c. \}, \end{aligned} \quad (43)$$

where $\mathbb{1}_c = \mathbb{1}_{3 \times 3}$ is the identity acting on the colour space of the quarks, see section 7. Comparing this with the classical fermionic action [7], we read off the mass matrices of the fermions

$$m_e = \frac{1}{2}(\epsilon^* - \epsilon^{-1})m_l^*, \quad m_u = \frac{1}{2}\sqrt{\frac{2}{3}}(\beta^* + \beta^{-1})m_q^*, \quad m'_d = \frac{1}{2}\sqrt{\frac{1}{3}}((\chi\gamma)^* - (\chi\gamma)^{-1})m_q^*, \quad (44)$$

where $e \equiv (e, \mu, \tau)^T$, $u \equiv (u, c, t)^T$, $d \equiv (d, s, b)^T$. The matrices m_e, m_u are diagonal. We can assume that their diagonal matrix elements are positive, otherwise this is achieved by unitary transformations $e_R \mapsto u_e e_R$, $u_R \mapsto u_u u_R$ with diagonal unitary matrices u_e, u_u . However, the matrix m'_d is – in general – an arbitrary non-diagonal 3×3 -matrix. There exist unitary matrices u_1, u_2 , such that $m'_d = u_1 m_d u_2^*$ for a diagonal matrix m_d with positive diagonal matrix elements. The matrix u_1 can be absorbed by means of a unitary transformation $d_R \mapsto u_1 d_R$. But the matrix u_2^* cannot be absorbed by a unitary transformation $\begin{pmatrix} u_L \\ d_L \end{pmatrix} \mapsto \begin{pmatrix} u_2 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, because this would make the matrix m_u non-diagonal. All we can do are transformations $\begin{pmatrix} u_L \\ d_L \end{pmatrix} \mapsto \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, $u_R \mapsto \delta_1 u_R$ and $d_R \mapsto \delta_2 d_R$ by diagonal unitary matrices δ_1, δ_2 , such that m_u remains invariant and $u_2^* \mapsto \delta_1^* u_2^* \delta_2 =: V$. The matrix V is the famous Kobayashi–Maskawa matrix, which for an appropriate choice of δ_1, δ_2 can be brought into the standard form parametrized by three rotation angles and one phase. Thus, we can rewrite $\mathcal{L}'_l$ and $\mathcal{L}'_q$ as

$$\begin{aligned} \mathcal{L}'_l &= -\{e_R^* \left\{ \gamma^0 \begin{pmatrix} \bar{\Phi}^1 & \bar{\Phi}^2 + 1 \end{pmatrix} \otimes m_e \right\} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + h.c.\}, \\ \mathcal{L}'_q &= -\{u_R^* \left\{ \gamma^0 \begin{pmatrix} \Phi^2 + 1 & -\Phi^1 \end{pmatrix} \otimes m_u \otimes \mathbb{1}_c \right\} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + h.c.\} \\ &\quad -\{d_R^* \left\{ \gamma^0 \begin{pmatrix} \bar{\Phi}^1 & \bar{\Phi}^2 + 1 \end{pmatrix} \otimes m_d V \otimes \mathbb{1}_c \right\} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + h.c.\}. \end{aligned} \quad (45)$$

Now we count the free parameters on which the physical masses in our model depend. We can regard (44) as equations for computing the fermion masses and the Kobayashi–Maskawa matrix for given 3×3 -matrices $\beta, \gamma, \epsilon, \chi, m_l, m_q$. But if we take the point of view that the fermion masses and the Kobayashi–Maskawa matrix (13 parameters) are given by experiment and if we keep the diagonal positive matrices m_l and m_q as additional free parameters then (44), with $m'_d = m_d V$, are equations to determine $\beta, \gamma, \epsilon, \chi$. This gives a system of quadratic equations for the matrix elements of $\beta, \gamma, \epsilon, \chi$. We take any of the (possibly complex) solutions of this system to fix the matrices $\beta, \gamma, \epsilon, \chi$. Then, there remain only the six free parameters in the diagonal of m_l and m_q .

We remark that the presented construction of the fermionic action of the standard model yields immediately the correct hypercharges of the fermions – namely the coefficients in front of $-\frac{1}{2}A^0$ in (34) and (35). This is possible, because we use the isomorphism $\mathfrak{i}$ of graded Lie algebras, which allows embeddings into the space of bounded operators on the fermionic Hilbert space different from the fundamental embedding.

For matrix algebras there exist – besides the trivial representation – only the fundamental representation and, hence, only the fundamental embedding. Therefore, in the derivation of standard model elaborated by Kastler [9] one must additionally consider the chromodynamics algebra and impose a generalized Poincaré duality condition ([9], [11]) in order to obtain the correct hypercharges.

6 The Bosonic Action

We recall [14] that the curvature θ of the connection ∇ is given by

$$\theta = D_{\mathcal{H}}\rho + \frac{1}{2}[\rho, \rho]_g + \theta_0 = \mathbf{D}\rho - i[\mu, \rho]_g + \frac{1}{2}[\rho, \rho]_g + \theta_0, \quad (46)$$

where

$$\theta_0 = ed(e) \bullet \hat{d}(e)e = \text{diag}(MM^*, 0, 0, 0). \quad (47)$$

The problem is that $\theta_0 \notin \mathcal{H}_0$, such that it is not possible to apply the isomorphism $\mathbf{i}$. We propose to replace here the isomorphism $\mathbf{i}$ by a linear mapping $\mathbf{i}'$ defined as follows. Looking at (16) we see that θ_0 is given by putting $\alpha_3^0 = \alpha_0^0 = 1$ and all other $\alpha_f^{0,1,2} = 0$, and then projecting away the last row and column. Thus, to the element $\tilde{\theta}_0 \in \mathcal{H}_0^2$ given by $\alpha_3^0 = \alpha_0^0 = 1$ we apply the isomorphism $\mathbf{i}$, giving $\mathbf{i}(\tilde{\theta}_{0,l}) = \text{diag}(l_1^1, 0, l_4^1)$ and $\mathbf{i}(\tilde{\theta}_{0,q}) = \text{diag}(\frac{1}{3}q_1^1, -\frac{2}{3}q_1^1, -\frac{2}{3}q_4^1, \frac{1}{3}q_4^1)$, see (20) and (21). Now we apply a reasonable projection:

$$\mathbf{i}'(\theta_{0,l}) := \text{diag}(l_1^1, 0, 0), \quad \mathbf{i}'(\theta_{0,q}) := \text{diag}(\frac{1}{3}q_1^1, -\frac{2}{3}q_1^1, -\frac{2}{3}q_4^1, 0). \quad (48)$$

The choice of $\mathbf{i}'(\theta_{0,l})$ is plausible, but I have no deeper explanation for $\mathbf{i}'(\theta_{0,q})$, except that we need this below. We recall that $l_1^1 = l_4^1 = m_l m_l^* \equiv |m_l|^2$ and $q_1^1 = q_4^1 = m_q m_q^* \equiv |m_q|^2$. Then, using (34), (35) and (37), the transported curvature is given by

$$\mathbf{i}(\theta_{l/q}) = \mathbf{D}(\mathbf{i}(\rho_{l/q})) - i[\mathbf{i}(\mu_{l/q}), \mathbf{i}(\rho_{l/q})]_g + \frac{1}{2}[\mathbf{i}(\rho_{l/q}), \mathbf{i}(\rho_{l/q})]_g + \mathbf{i}'(\theta_{0,l/q}), \quad (49)$$

$$\mathbf{i}(\theta_l) = \left(\begin{array}{c|c|c} \left(\frac{1}{2}\mathbf{d}(A^3 + A^0) + A^- \wedge A^+ \right) \otimes \mathbb{1} - (|\Phi^1|^2 - 1) \otimes |m_l|^2 & (\mathbf{d}A^- + A^3 \wedge A^-) \otimes \mathbb{1} - \Phi^1(\bar{\Phi}^2 + 1) \otimes |m_l|^2 & i(\mathbf{d}\Phi^1 + \frac{1}{2}(A^3 - A^0)\Phi^1 + A^-(\Phi^2 + 1))\gamma^5 \otimes \epsilon m_l \\ \hline (\mathbf{d}A^+ + A^+ \wedge A^3) \otimes \mathbb{1} - (\Phi^2 + 1)\bar{\Phi}^1 \otimes |m_l|^2 & (\frac{1}{2}\mathbf{d}(A^0 - A^3) + A^+ \wedge A^-) \otimes \mathbb{1} - (|\Phi^2 + 1|^2 - 1) \otimes |m_l|^2 & i(-\frac{1}{2}(A^3 + A^0)(\Phi^2 + 1) + \mathbf{d}\Phi^2 + A^+\Phi^1)\gamma^5 \otimes \epsilon m_l \\ \hline i(\mathbf{d}\bar{\Phi}^1 - \frac{1}{2}\bar{\Phi}^1(A^3 - A^0) - (\bar{\Phi}^2 + 1)A^+)\gamma^5 \otimes \epsilon^{-1}m_l^* & i(\frac{1}{2}(\bar{\Phi}^2 + 1)(A^3 + A^0) + \mathbf{d}\bar{\Phi}^2 - \bar{\Phi}^1 A^-)\gamma^5 \otimes \epsilon^{-1}m_l^* & \mathbf{d}A^0 \otimes \mathbb{1} - (|\Phi^1|^2 + |\Phi^2 + 1|^2 - 1) \otimes |m_l|^2 \end{array} \right),$$

$$i(\theta_q) = \begin{pmatrix} \begin{aligned} &(\mathbf{d}(\frac{1}{2}A^3 - \frac{1}{6}A^0) \\ &+ A^- \wedge A^+) \otimes \mathbb{1} \\ &+ (\frac{2}{3}|\Phi^2 + 1|^2 - \frac{1}{3} \\ &- \frac{1}{3}|\Phi^1|^2) \otimes |m_q|^2 \end{aligned} & \begin{aligned} &(\mathbf{d}A^- + A^3 \wedge A^-) \\ &\otimes \mathbb{1} \\ &- \Phi^1(\Phi^2 + 1) \\ &\otimes |m_q|^2 \end{aligned} & \begin{aligned} &i(\mathbf{d}\bar{\Phi}^2 - \bar{\Phi}^1 A^- \\ &+ \frac{1}{2}(\bar{\Phi}^2 + 1) \times \\ &(A^3 + A^0))\gamma^5 \\ &\otimes \sqrt{\frac{2}{3}} \beta m_q \end{aligned} & \begin{aligned} &i(\mathbf{d}\Phi^1 + \\ &\frac{1}{2}(A^3 - A^0)\Phi^1 + \\ &+ A^-(\Phi^2 + 1))\gamma^5 \\ &\otimes \sqrt{\frac{1}{3}} m_q \chi \gamma \end{aligned} \\ \hline \begin{aligned} &(\mathbf{d}A^+ + A^+ \wedge A^3) \\ &\otimes \mathbb{1} \\ &- (\Phi^2 + 1)\bar{\Phi}^1 \\ &\otimes |m_q|^2 \end{aligned} & \begin{aligned} &(\mathbf{d}(-\frac{1}{2}A^3 - \frac{1}{6}A^0) \\ &+ A^+ \wedge A^-) \otimes \mathbb{1} \\ &- (\frac{1}{3}|\Phi^2 + 1|^2 + \frac{1}{3} \\ &- \frac{2}{3}|\Phi^1|^2) \otimes |m_q|^2 \end{aligned} & \begin{aligned} &-i(\mathbf{d}\bar{\Phi}^1 - \\ &(\bar{\Phi}^2 + 1)A^+ - \\ &\frac{1}{2}\bar{\Phi}^1(A^3 - A^0))\gamma^5 \\ &\otimes \sqrt{\frac{2}{3}} \beta m_q \end{aligned} & \begin{aligned} &i(\mathbf{d}\Phi^2 + A^+ \Phi^1 \\ &- \frac{1}{2}(A^3 + A^0) \times \\ &(\Phi^2 + 1))\gamma^5 \\ &\otimes \sqrt{\frac{1}{3}} m_q \chi \gamma \end{aligned} \\ \hline \begin{aligned} &-i(\mathbf{d}\Phi^2 + A^+ \Phi^1 - \\ &\frac{1}{2}(A^3 + A^0) \times \\ &(\Phi^2 + 1))\gamma^5 \\ &\otimes \sqrt{\frac{2}{3}} \beta^{-1} m_q^* \end{aligned} & \begin{aligned} &i(\mathbf{d}\Phi^1 + \\ &A^-(\Phi^2 + 1) + \\ &\frac{1}{2}(A^3 - A^0)\Phi^1)\gamma^5 \\ &\otimes \sqrt{\frac{2}{3}} \beta^{-1} m_q^* \end{aligned} & \begin{aligned} &-\frac{2}{3}\mathbf{d}A^0 \otimes \mathbb{1} \\ &+ \frac{2}{3}(|\Phi^1|^2 \\ &+ |\Phi^2 + 1|^2 - 2) \\ &\otimes |m_q|^2 \end{aligned} & 0 \\ \hline \begin{aligned} &i(\mathbf{d}\bar{\Phi}^1 - \\ &(\bar{\Phi}^2 + 1)A^+ - \\ &\frac{1}{2}\bar{\Phi}^1(A^3 - A^0))\gamma^5 \\ &\otimes \sqrt{\frac{1}{3}} (\chi \gamma)^{-1} m_q^* \end{aligned} & \begin{aligned} &i(\mathbf{d}\bar{\Phi}^2 - \bar{\Phi}^1 A^- \\ &+ \frac{1}{2}(\bar{\Phi}^2 + 1) \times \\ &(A^3 + A^0))\gamma^5 \\ &\otimes \sqrt{\frac{1}{3}} (\chi \gamma)^{-1} m_q^* \end{aligned} & 0 & \begin{aligned} &\frac{1}{3}\mathbf{d}A^0 \otimes \mathbb{1} \\ &- \frac{1}{3}(|\Phi^1|^2 \\ &+ |\Phi^2 + 1|^2 - 1) \\ &\otimes \chi^{-1} |m_q|^2 \chi \end{aligned} \end{pmatrix}.$$

Now we take the embedding $\mathbf{e}$ of θ into $B(H)$, which means to perform the replacements (32) in the above matrices, see section 4. We introduce the abbreviations $|m|^2 := mm^*$, $|m|^4 := (mm^*)^2$ and $|m|^{-2} := (mm^*)^{-1}$, for a 3×3 -matrix m . Thus, we get the Euclidean bosonic action – for an appropriate choice of the constants – as

$$S_B = \frac{2\pi^2}{3g_0^2} \text{Tr}_\omega \{(\mathbf{e}(\theta))^* \mathbf{e}(\theta)\} = \frac{1}{48g_0^2} \int_X v_g \text{tr} \{(\mathbf{e}(\theta))^* \mathbf{e}(\theta)\} = \int_X v_g (\mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0), \quad (50)$$

where

$$\mathcal{L}_2 = -\frac{1}{2g_0^2} \delta^{\mu\kappa} \delta^{\nu\lambda} \{ \text{tr}(F_{\mu\nu} F_{\kappa\lambda}) + \frac{5}{6} F'_{\mu\nu} F'_{\kappa\lambda} \},$$

$$F_{\mu\nu} = \begin{pmatrix} \frac{1}{2} \partial_{[\mu} A_{\nu]}^3 + A_{[\mu}^- A_{\nu]}^+ & \partial_{[\mu} A_{\nu]}^- + A_{[\mu}^3 A_{\nu]}^- \\ \partial_{[\mu} A_{\nu]}^+ + A_{[\mu}^+ A_{\nu]}^3 & -\frac{1}{2} \partial_{[\mu} A_{\nu]}^3 - A_{[\mu}^- A_{\nu]}^+ \end{pmatrix}, \quad F'_{\mu\nu} = \partial_{[\mu} A_{\nu]}^0,$$

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{12g_0^2} \delta^{\mu\nu} ((D_\mu \Phi^1) D_\nu \Phi^1 + \overline{(D_\mu (\Phi^2 + 1))} D_\nu (\Phi^2 + 1)) \times \\ &\quad \text{tr}((|\epsilon|^2 + |\epsilon|^{-2}) |m_l|^2 + \{2(|\beta|^2 + |\beta|^{-2}) + (|\chi \gamma|^2 + |\chi \gamma|^{-2})\} |m_q|^2), \\ &\quad \begin{pmatrix} D_\mu \Phi^1 \\ D_\mu (\Phi^2 + 1) \end{pmatrix} = \begin{pmatrix} \partial_\mu \Phi^1 \\ \partial_\mu \Phi^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(A_\mu^3 - A_\mu^0) & A_\mu^- \\ A_\mu^+ & -\frac{1}{2}(A_\mu^3 + A_\mu^0) \end{pmatrix} \begin{pmatrix} \Phi^1 \\ \Phi^2 + 1 \end{pmatrix}, \end{aligned}$$

$$\mathcal{L}_0 = \frac{1}{12g_0^2} \{ (|\Phi^1|^2 + |\Phi^2 + 1|^2 - 1)^2 \text{tr}(2|\tilde{m}_l|^4 + \frac{10}{3}|\tilde{m}_q|^4) + \text{tr}(|\tilde{m}_l|^4 + 3|\tilde{m}_q|^4) \},$$

with $X_{[\mu} Y_{\nu]} \equiv X_\mu Y_\nu - X_\nu Y_\mu$. We have used $\text{tr}_C(1) = 4$, $\text{tr}_C(\gamma^{\mu'} \cdot \gamma^{\nu'}) = 4 \delta^{\mu' \nu'}$ and $\text{tr}_C(\gamma^\mu \cdot \gamma^\nu \cdot \gamma^\kappa \cdot \gamma^\lambda) = 4(\delta^{\nu\kappa} \delta^{\mu\lambda} - \delta^{\mu\kappa} \delta^{\nu\lambda})$ for $\mu \neq \nu$, $\kappa \neq \lambda$, where tr_C denotes

the trace in the space of sections of the Clifford bundle C . Next we perform some reparametrisations. We put

$$A_\mu^3 = ig_0 W_\mu^3, A_\mu^- = \frac{i}{2} g_0 (W_\mu^1 - iW_\mu^2), A_\mu^+ = \frac{i}{2} g_0 (W_\mu^1 + iW_\mu^2), A_\mu^0 = -\sqrt{\frac{3}{5}} ig_0 W_\mu^0, \\ \Phi^{1,2} = \frac{g_0}{\sqrt{\frac{1}{6} \text{tr}((|\epsilon|^2 + |\epsilon|^{-2})|m_l|^2 + \{2(|\beta|^2 + |\beta|^{-2}) + (|\chi\gamma|^2 + |\chi\gamma|^{-2})\}|m_q|^2)}} \phi^{1,2}. \quad (51)$$

This gives ($I.T. \equiv$ interaction terms, $C.C. \equiv$ cosmological constant)

$$\mathcal{L}_2 = \frac{1}{4} \delta^{\mu\kappa} \delta^{\nu\lambda} \left\{ \sum_{a=1}^3 F_{\mu\nu}^a F_{\kappa\lambda}^a + F_{\mu\nu}^0 F_{\kappa\lambda}^0 \right\}, \quad (52)$$

$$\mathcal{L}_1 = \frac{1}{2} \delta^{\mu\nu} \left\{ \sum_{j=1}^2 (\partial_\mu \phi_j)(\partial_\nu \phi_j) + m_W^2 (W_\mu^1 W_\nu^1 + W_\mu^2 W_\nu^2) + m_Z^2 Z_\mu Z_\nu \right\} + I.T., \quad (53)$$

$$\mathcal{L}_0 = \frac{1}{2} m_H^2 (\text{Re } \phi^2)^2 + I.T. + C.C., \quad (54)$$

where

$$F_{\mu\nu}^a = \partial_{[\mu} W_{\nu]}^a - g_0 \sum_{b,c=1}^3 \varepsilon_{bc}^a W_\mu^b W_\nu^c, \quad F_{\mu\nu}^0 = \partial_{[\mu} W_{\nu]}^0, \\ Z = \cos \theta_W W^3 - \sin \theta_W W^0, \\ P = \sin \theta_W W^3 + \cos \theta_W W^0, \quad \sin^2 \theta_W = \frac{3}{8},$$

$$m_W = \frac{1}{2} \sqrt{\frac{1}{6} \text{tr}((|\epsilon|^2 + |\epsilon|^{-2})|m_l|^2 + \{2(|\beta|^2 + |\beta|^{-2}) + (|\chi\gamma|^2 + |\chi\gamma|^{-2})\}|m_q|^2)}, \quad (55)$$

$$m_Z = m_W / \cos \theta_W,$$

$$m_H = \frac{1}{m_W} \sqrt{\text{tr}(\frac{5}{9} |\tilde{m}_q|^4 + \frac{1}{3} |\tilde{m}_l|^4)}. \quad (56)$$

We can transform this action to Minkowski space by a Wick rotation of the physical fields, replacing $\delta^{\mu\nu} \mapsto -g^{\mu\nu}$ and introducing a global minus sign in the action. Then this action coincides with the classical bosonic action of the Salam–Weinberg model, see [7]. Inserting the reparametrizations (51) into (43), we get precisely the fermionic action of the Salam–Weinberg model [7]. According to [14], the gauge group associated to the module $\mathcal{E}$ discussed in the beginning of section 3 is isomorphic to $C_{\mathbb{R}}^\infty(X_M) \otimes (SU(2) \times U(1))$, where $C_{\mathbb{R}}^\infty(X_M)$ denotes the algebra of real smooth functions on the Minkowski space. The action is by construction invariant under gauge transformations.

7 The Chromodynamics Sector

The chromodynamics sector can be obtained from the module $\mathcal{E}_c = e_c \mathcal{A}^3$, with $e_c = \text{diag}(e', e', e')$. The analysis of this case [14] shows that

$$\mathcal{H}_{0,c}^k = \bigoplus_{t=0}^m L^{k-2t} \otimes sl(3, \mathbb{C}) \otimes M_1^t, \quad (57)$$

after omitting the rows and columns consisting of zeros only. Thus, elements $\varrho^k \in \mathcal{H}_{0,c}^k$ are of the form $\varrho^k = \sum_{r=0}^m G^{k-2r} \otimes M_1^r$, where $G^{k-2r} \in L^{k-2r} \otimes sl(3, \mathbb{C})$. We split $\mathcal{H}_{0,c}^k$ into its l_1^r - and q_1^r -part, we take the trivial representation $\mathbf{i}_c(\varrho_l^k) \equiv 0$ of the l_1^r -part and the fundamental representation (tensorized with $\mathbb{1}_{4 \times 4}$) of the q_1^r -part:

$$\mathbf{i}_c(\varrho_q^k) = \sum_{r=0}^m \text{diag}(G^{k-2r}, G^{k-2r}, G^{k-2r}, G^{k-2r}) \otimes q_1^r. \quad (58)$$

Obviously, elements $\mathbf{i}_c(\varrho^k)$ are bounded operators on H for $k = 0, 1$. In analogy to the procedure in section 4 we construct the space $\mathcal{T}_c^2 \subset B(H)$, which turns out to be

$$\mathcal{T}_c^2 = \tilde{\mathbf{c}}_c \circ \mathbf{i}_c(\mathcal{H}_{0,c}^2) + \mathbf{i}_c(C^0 \otimes gl(3, \mathbb{C}) \otimes \text{id}_F), \quad (59)$$

where $\tilde{\mathbf{c}}_c$ is defined in analogy to (23) and where we extended $\mathbf{i}_c|_{\mathcal{H}_{0,c}^0}$ naturally to $C^0 \otimes gl(3, \mathbb{C}) \otimes \text{id}_F$.

From (57) we find for the chromodynamics connection form $\rho_c = G \otimes \text{id}_F$, $G = -G^* \in L^1 \otimes su(3)$. After rotating to Minkowski space ($G \mapsto G_M$) we obtain the following contribution to the fermionic action:

$$\mathcal{L}_s = \left(\begin{array}{c} u_L^* \\ d_L^* \\ u_R^* \\ d_R^* \end{array} \right) \left\{ \gamma^0 \left(\begin{array}{cccc} iG_M & 0 & 0 & 0 \\ 0 & iG_M & 0 & 0 \\ 0 & 0 & iG_M & 0 \\ 0 & 0 & 0 & iG_M \end{array} \right) \otimes \mathbb{1} \right\} \left(\begin{array}{c} u_L \\ d_L \\ u_R \\ d_R \end{array} \right). \quad (60)$$

The curvature θ_c of the chromodynamics connection ∇_c is given by

$$\theta = d\rho_c + \frac{1}{2}[\rho_c, \rho_c]_g + \theta_{0,c}, \quad \theta_{0,c} = \mathbb{1}_c \otimes M_1^1. \quad (61)$$

These equations follow from the discussion in [14]. We put in local bases

$$G = \frac{i}{2} g_0 G_\mu^a \gamma^\mu \otimes \lambda_a, \quad (62)$$

where λ_a , $a = 1, \dots, 8$, are the Gell-Mann matrices, fulfilling $[\lambda_b, \lambda_c] = \sum_{a=1}^8 2if_{bc}^a \lambda_a$, $f_{bc}^a \in \mathbb{R}$, and $\text{tr}(\lambda_a \lambda_b) = 2\delta_{ab}$. Denoting by $\mathbf{e}_c(\theta_c)$ the embedding of θ_c into $B(H)$, we obtain for the Euclidian bosonic action of the chromodynamics sector

$$\begin{aligned} S_B^c &= \frac{2\pi^2}{3g_0^2} \text{Tr}_\omega \{(\mathbf{e}_c(\theta_c))^* \mathbf{e}_c(\theta_c)\} = \frac{1}{48g_0^2} \int_X v_g \text{tr} \{(\mathbf{e}_c(\theta_c))^* \mathbf{e}_c(\theta_c)\} = \int_X v_g \mathcal{L}_c, \\ \mathcal{L}_c &= \frac{1}{4} \sum_{a=1}^8 \delta^{\mu\kappa} \delta^{\nu\lambda} F_{c,\mu\nu}^a F_{c,\kappa\lambda}^a + \frac{1}{g_0^2} \text{tr}(|\tilde{m}_q|^4), \\ F_{c,\mu\nu}^a &= \partial_{[\mu} G_{\nu]}^a - g_0 \sum_{b,c=1}^8 f_{bc}^a G_\mu^b G_\nu^c. \end{aligned} \quad (63)$$

Rotation to Minkowski space ($G \mapsto G_M$, $\delta^{\mu\nu} \rightarrow -g^{\mu\nu}$, $v_g \mapsto v_M$ and global minus sign) transforms the first term in $\mathcal{L}_c$ into the bosonic action of chromodynamics [7], the second term in $\mathcal{L}_c$ contributes to the “cosmological constant”. It is interesting that the coupling constants of the weak and strong interaction coincide in this model (just as in [11]). According to [14], the gauge group associated to the module $\mathcal{E}_c$ is isomorphic to $C_{\mathbb{R}}^\infty(X_M) \otimes SU(3)$.

8 Remarks on Mass Relations

Lepton masses are small compared with the quark masses of the same generation. Comparing this fact with (44), there are at least two possible explanations: The eigenvalues of m_l could be indeed much smaller than the eigenvalues of m_q , or $m_l = m_q$ and ϵ is almost precisely equal to $\mathbb{1}$ while $\chi\gamma$ is not. Of course, the truth can lie anywhere between these extreme cases (the case $\|m_l\| > \|m_q\|$ seems to be too artificial¹).

i) $\|m_l\| \ll \|m_q\|$: We neglect terms containing m_l completely. Inserting (44) into (55) we get

$$\text{tr}(\frac{1}{2}|m_u|^2 + \frac{1}{2}|m_d|^2) > m_W^2 = \text{tr}(\frac{1}{2}|m_u|^2 + \frac{1}{2}|m_d|^2 - \frac{1}{12}|m_q|^2) \geq \text{tr}(\frac{3}{8}|m_u|^2 + \frac{1}{2}|m_d|^2). \quad (64)$$

Since the fermionic mass spectrum is dominated by the mass m_t of the top quark, which is one eigenvalue of m_u , we obtain

$$\sqrt{2}m_W < m_t \leq \sqrt{\frac{8}{3}}m_W. \quad (65)$$

The maximum for m_t is attained for $\beta = \mathbb{1}$. To estimate the mass m_H of the Higgs-boson we assume that one eigenvalue t^2 of $|m_q|^2$ is much larger than the other two, i.e. the differences in the mass spectrum are not only due to differences of β and γ . In this case we get from (31)

$$\text{tr}(|\tilde{m}_q|^4) \equiv \text{tr}\{(|m_q|^2 - \frac{1}{3}(\text{tr}(|m_q|^2))\mathbb{1})^2\} \rightarrow \frac{2}{3}t^4, \quad (66)$$

which gives with (55) and (56)

$$m_H = \sqrt{\frac{10}{27}} \frac{t^2}{m_W} = 8\sqrt{\frac{10}{3}} \frac{m_W}{2(|\beta_t|^2 + |\beta_t|^{-2}) + |(\chi\gamma)_t|^2 + |(\chi\gamma)_t|^{-2}} \leq \frac{4}{3}\sqrt{\frac{10}{3}}m_W, \quad (67)$$

where $|\beta_t|^2, |(\chi\gamma)_t|^2$ are the matrix elements sitting at the same site in $|\beta|^2$ and $|\chi\gamma|^2$ as t^2 in $|m_q|^2$. One can show that the relation $m_H \leq \frac{4}{3}\sqrt{\frac{10}{3}}m_W$ still holds if we drop the assumption that t^2 is much larger than the other eigenvalues of $|m_q|^2$.

ii) $|m_l|^2 = |m_q|^2$: In this case we have

$$m_W^2 = \frac{1}{2}\text{tr}(|m_e|^2 + |m_u|^2 + |m_d|^2), \quad m_t = \sqrt{2}m_W. \quad (68)$$

¹ $\|m\| \in \mathbb{R}$ is a norm of $m \in M_3\mathbb{C}$, for instance the operator norm

To estimate m_H we assume again that t^2 is much larger than the other eigenvalues of $|m_q|^2$. Then we have $\text{tr}(|\tilde{m}_q|^4) = \text{tr}(|\tilde{m}_l|^4) \rightarrow \frac{2}{3}t^4$ and

$$m_H = \frac{4\sqrt{3}}{9m_W}t^2 = \frac{32\sqrt{3}m_W}{3(|\epsilon_l|^2 + |\epsilon_l|^{-2} + 2(|\beta_l|^2 + |\beta_l|^{-2}) + |(\chi\gamma)_l|^2 + |(\chi\gamma)_l|^{-2})} \leq \frac{4\sqrt{3}}{3}m_W. \quad (69)$$

These mass relations differ from the relations $m_t = 2m_W$ and $m_H = 3.14m_W$ obtained in [11] for the quaternionic K-cycle by Kastler and Schücker. The values predicted for $\frac{m_t}{m_W}$ in both models differ by a factor of the order 1, whereas $\frac{m_t}{m_W} > 2$ seems to be favoured by experiment. For the quaternionic K-cycle this can be achieved by admitting more general scalar products, see [12]. We have not yet studied the influence of additional parameters in the scalar product for our approach.

The cases $\|m_l\|^2 \ll \|m_q\|^2$ and $|m_l|^2 = |m_q|^2$ have in common that only the three free parameters contained in m_q play a role. Moreover, if one of these parameters is much larger than the other two then only this large parameter is essential. This means: Although we introduced a plenty of free parameters during the construction, these parameters occur in the final Lagrangians only in such combinations that we end up with nine parameters for the fermion masses, four parameters of the Kobayashi–Maskawa matrix and one additional parameter, which determines the ratio between the masses of the Higgs–boson and the W –boson. For this last parameter we have only an upper limit. In addition, there is one undetermined coupling constant. We see that in comparison to the model by Connes, Lott and Kastler we get one additional parameter, which leads to the undesired feature that in our model the value for $\frac{m_H}{m_W}$ can be – in principle – arbitrarily small. If we assume $\beta = 1$ then we can conclude that $m_H = 2.43m_W$ for $\|m_l\| \ll \|m_q\|$ and $m_H = 2.30m_W$ for $m_l = m_q$ (if we neglect all fermions except the top–quark). But nevertheless, our approach is less predictive than the model by Kastler et. al. Moreover, the mass relations are tree–level constraints. They do not survive the classical quantization procedure, but there seems to be only a weak scale dependence [1].

Thus, the purpose of this paper is more to present a different construction of the classical action of the standard model, which uses the simplest possible (non–classical) algebra and puts the complexity into the module. In non–commutative geometry a K-cycle replaces the notion of a manifold and a finite projective module over the algebra of this K-cycle the notion of a vector bundle over this manifold. Therefore, our approach (a K-cycle together with two associated modules – one for the electroweak sector and the other one for the chromodynamics sector) corresponds to a classical manifold with two associated fibre bundles over that manifold – a principal fibre bundle with structure group $SU(2) \times U(1)$ for the electroweak sector and a principal fibre bundle with structure group $SU(3)$ for the chromodynamics sector.

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