The Electric Charge of a Dirac Monopole at Nonzero Temperature

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Abstract

We study the effect of nonzero temperature on the induced electric charge around a Dirac monopole. While at zero temperature the charge is known to be proportional to a CP violating $\theta$ parameter, we find that at high temperature the charge is proportional to $\sin \theta$. Other features of the charge at nonzero temperature are discussed. We also compute the induced charge at nonzero temperature around an Aharonov-Bohm flux string in $2 + 1$ dimensions and compare the result with an index theorem, and also with the electron-monopole problem in $3 + 1$ dimensions.

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Dirac [1] showed that the quantum mechanics of an electron in the presence of a fixed magnetic monopole was consistent only if the electron charge $e$ and the monopole magnetic charge $g$ satisfied the quantization condition

$$ q \equiv eg = \frac{1}{2} \times \text{integer}. \quad (1) $$

While Dirac described his monopole in terms of a singular vector potential, a non-singular description was later obtained [2] which gave a simpler derivation of (1). The non-singular formulation has the advantage of enabling a clearer discussion of the dynamics of a fermion interacting with a monopole [3, 4, 5]. A careful investigation of the fermion-monopole problem [4] revealed that the Hamiltonian of the system was not self-adjoint, leading thereby to non-real eigenvalues. By imposing appropriate boundary conditions on the wavefunctions, the Hamiltonian could be made self-adjoint, but at the price of introducing an undetermined angular $\theta$ parameter. The classical $CP$ symmetry of the problem is maintained only for the particular choices $\theta = 0$ or $\pi$ [4].

For a general $\theta$, it was found [5] that the monopole acquired an electric charge due to vacuum polarization. The electric charge was computed to be

$$ Q = -\frac{e\theta}{2\pi} 2q \quad (2) $$

in agreement with a general statement of Witten [6].

Our purpose here is to investigate how the relation (2) is modified at non-zero temperature. We consider a fixed Dirac monopole in an electron-positron plasma at temperature $\beta = 1/T$. That is, we ignore first the effects of photons which will be estimated later.

In terms of the eigenstates of the electron-monopole Hamiltonian, the thermal ensemble average of the induced charge is given by

$$ Q = -\frac{e}{2} \int d^3r \sum_{E,j,m} \tanh \frac{\beta|E|}{2} \text{sign}(E) \psi_{E,j,m}^\dagger \psi_{E,j,m}. \quad (3) $$

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In (3) $E$ is the energy, $j$ labels the total angular momentum and $m$ labels the $z-$component of angular momentum. The values of $j$ increase in steps of one [2, 3] starting from the lowest value $j_0 = |q| - 1/2$. For any $j \geq j_0$ there is a norm-preserving map [5] between the negative and the positive energy eigenstates so that only the lowest partial wave $j_0$ contributes to (3). Using the expressions for the wave functions given in [5] we obtain

$$Q = Q_b + Q_c,$$

where

$$Q_b = -\frac{eM}{2} (2q)(-2\cos \theta) \Theta(-\cos \theta) \text{sign(sin } \theta) \tanh \frac{\beta M |\sin \theta|}{2} \int_0^\infty dr \ e^{-2rM|\cos \theta|}$$

is the contribution from a bound state which exists if $\cos \theta < 1$, and

$$Q_c = -\frac{e}{2\pi} (2q)(-\frac{M}{2}) \int_0^\infty dr \int_{-\infty}^\infty \frac{dk}{\omega_k} \tanh \frac{\beta \omega_k}{2} \frac{2ik \sin \theta}{M \cos \theta - ik} e^{2ikr}$$

is the contribution from the continuum states. In the above, $M$ is the electron mass and $\omega_k \equiv \sqrt{k^2 + M^2}$. The $k$ integral in (6) may be evaluated by forming a contour integral in the upper half of the complex $k-$plane. For $\cos \theta < 1$ there will be a contribution to $Q_c$ from a pole at $k = -iM \cos \theta$ which cancels exactly $Q_b$, just as at zero temperature. Note that at non-zero temperature there is no branch cut in the complex $k-$plane. There is however an infinite string of temperature dependent poles along the imaginary axis due to the $	anh (\beta \omega_k/2)$ factor. Evaluating the contribution of these poles we obtain

$$Q = -\frac{ex}{\pi} (2q) \sin \theta \sum_{n=0}^\infty \frac{1}{(2n+1)^2 + x^2 + x \cos \theta \sqrt{(2n+1)^2 + x^2}},$$

where $x \equiv M/(\pi T)$. A number of features are apparent from (7): (a) As expected, the charge decreases as the temperature increases ($x \to 0$), (b) the expression is odd under $\theta \to -\theta$ just as at zero temperature (so from now on we will discuss only the
range $0 < \theta < \pi$), (c) at a fixed temperature the charge decreases as $\theta$ increases from 0 to $\pi/2$, (d) For $\theta$ between $\pi/2$ and $\pi$ the expression (7) has no obvious universal behaviour other than vanishing at $\theta = \pi$. Thus at nonzero temperature the charge vanishes at the $CP$ even values of $\theta = 0$ and $\pi$.

Consider the high temperature ($x \to 0$) limit of (7). One easily obtains

$$Q \to -\frac{eM}{T}(2q) \frac{\sin \theta}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + O\left(\frac{M}{T}\right)^2$$

$$= -\frac{eM}{T}(2q) \frac{\sin \theta}{8} + O\left(\frac{M}{T}\right)^2.$$  

(8)

The most striking feature of the high temperature limit is the proportionality of the charge to $\sin \theta$ rather than $\theta$ which is the case at zero temperature. Therefore at high temperature the charge is a maximum at $\theta = \pi/2$. The behaviour of the charge as a function of the $CP$ violation parameter $\theta$ for different temperatures is shown in the figure. It is interesting to note also from (6) that the charge density at large distances goes as

$$\rho(r) = O\left(\frac{e^{-r\sqrt{M^2+\pi^2T^2}}}{r^2}\right).$$  

(9)

Let us now show that in the limit of zero temperature ($x \to \infty$) the expression (7) reduces to (2). Split the sum in (7) as $\sum_{n=0}^{N} + \sum_{n=N}^{\infty}$ for some $N \gg 1$. Then for $x \gg N$, the finite sum contributes to $Q$ an amount which vanishes as $O(1/x)$, while the contribution of the sum $\sum_{n=N}^{\infty}$ may be estimated by writing it as an integral. Thus

$$\lim_{x \to \infty} Q \sim \lim_{x \to \infty} -\frac{e}{\pi}(2q) \lim_{x \to \infty} \int_{\Lambda}^{\infty} dy \frac{dy}{(2y)^2 + x^2 + x \cos \theta \sqrt{(2y)^2 + x^2}},$$  

(10)

where $\Lambda \sim N$. Making the change of variables $2y = x \sinh z$ we obtain

$$\lim_{x \to \infty} Q = -\frac{e}{2\pi}(2q) \sin \theta \left. \lim_{x \to \infty} \int_{\sinh^{-1} \frac{2x}{z}}^{\infty} \frac{dz}{\cosh z + \cos \theta} \right| \frac{\theta}{\sin \theta}$$

$$= -\frac{e}{2\pi}(2q) \frac{\sin \theta}{\sin \theta}$$

$$= -\frac{e\theta}{2\pi}(2q).$$  

(11)
which is Eq. (2).

There is one value of $\theta$ for which the expression (7) may be evaluated in closed form for any temperature. For $\theta = \pi/2$ we obtain

$$Q(\theta = \pi/2) = -\frac{e}{\pi} (2q) \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2 + x^2} = -\frac{e}{4} (2q) \tanh \frac{M}{2T} \quad (12)$$

which agrees at high temperature with (8) and at zero temperature with (2). It is known [7] that for $\theta = \pi/2$, the problem of induced charge around a monopole (with $2q = 1$) in 3+1 dimensions is mathematically equivalent to the problem of induced charge around an Aharonov-Bohm (AB) flux string with flux $F = 1/2$ in 2+1 dimensions. However the AB problem is exactly solvable for any flux not only at zero temperature, but also at nonzero temperature, as we now illustrate.

Using the wavefunctions for the electron-AB flux string system (see, for example [8]), we obtain the following expression for the induced charge at nonzero temperature

$$Q_{AB} = -\frac{eM \text{sign}(F)}{4\pi} \int d^2 r \int_0^\infty \frac{kdk}{\omega_k} \tanh \frac{\beta \omega_k}{2} \left( J^2_{-|F|}(kr) - J^2_{|F|}(kr) \right), \quad (13)$$

where $F = -\frac{e}{2\pi} \oint \vec{A} \cdot d\vec{l}$ is the flux ($|F| < 1$) in the string, $r$ is the radial distance from the string, and $J_\mu(kr)$ is the Bessel function. Note the following inverse-Mellin-transform representation for the hyperbolic tangent,

$$\tanh \frac{\beta \omega_k}{2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \left( 2^{1-s}(2^{1-s} - 1)\Gamma(s)\zeta(s) \right). \quad (14)$$

Define also the integral

$$A_\nu(t) \equiv \int_0^\infty \alpha^{\nu-1} e^{-t\alpha^2} d\alpha. \quad (15)$$

A scaling of (15) gives

$$\omega_k^{s-1} = \frac{A_{s+1}(\omega_k^2)}{A_{s+1}(1)}. \quad (16)$$
Using (14) and (16) in (13), the $\int dk$ and the $\int d^2 r$ integrals become the same as at zero temperature [8], giving thereby

$$Q_{AB} = -e M \text{sign}(F) \frac{2}{4\pi} \frac{2}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \ \beta^{-s}(2^{1-s} - 1)\Gamma(s)\zeta(s) \frac{2\pi|F|}{|M|^{s+1}}$$

$$= -e F M |M| \tanh \frac{\beta |M|}{2}.$$

That is,

$$Q_{AB} = -e F \frac{M}{2} \text{sign}(M) \tanh \frac{\beta |M|}{2}. \quad (17)$$

As stated, $Q_{AB}(F = 1/2)$ agrees with Eq. (12) for $2q = 1$. (In (17) the $\text{sign}(M)$ refers to the irreducible representations of two-component spinors in $2 + 1$ dimensions. So the mapping here is for $M > 0$). The calculation above for the AB case can be extended beyond $|F| < 1$ by taking into account the contribution from a discrete set of threshold states [8].

In $2 + 1$ dimensions, the induced charge around any static external magnetic flux $F$ is a topological invariant at zero temperature [9], $Q_{2+1}(T = 0) = -e F \frac{M}{2} \text{sign}(M)$. It has been argued [10] to be an invariant also at nonzero temperature by using the fact that the regulated index $\eta(s) \equiv \sum E \text{sign}(E)|E|^{-s}$ for the corresponding Dirac Hamiltonian is an invariant (depending only on the total flux rather than the details of the field). Indeed the induced charge due to a uniform external magnetic field in $QED_3$ is given by the right-hand-side of (17) and our explicit calculation of (17) for the AB case verifies the argument of [10].

We mention now some corrections to the result (7). The effect of gauge-field fluctuations (virtual and thermal photons) on the magnitude of the induced charge should be suppressed by powers of the fine structure constant (this is certainly true for $2q \gg 1$). From (9) we see that the induced charge is more localized around the monopole at nonzero temperature than at zero temperature. Far from the monopole this electric charge will be screened by plasma collective effects; at high temperature the Debye screening mass is $\sim eT$. On the other hand the static magnetic field of the
monopole remains unscreened (for a plasma in its normal state). If there is a non-zero chemical potential, then the relative contribution of the positive and negative energy eigenstates to the charge is different, so that in the electron-monopole problem the states with \( j \geq |q| + 1/2 \) will contribute. (For \( QED_3 \) the induced charge around a magnetic flux tube is no longer an invariant when the chemical potential is non-zero \([10]\)). It should be noted also that the fractional charge as computed from Eq.(3) is a thermal expectation value rather than a sharp eigenvalue which would be the case at zero temperature \([11]\).

We conclude by summarizing the main features of the induced charge around the monopole:

(i) The induced charge decreases with increasing temperature, going as \( \sim M/T \) at large temperatures.

(ii) The dependence of the charge on the \( \theta \) parameter is modified from that at zero temperature (see Figure). The charge vanishes at the \( CP \) even values of \( \theta = 0 \) and \( \pi \). At high temperature the charge is proportional to \( \sin \theta \).

(iii) The induced charge becomes more localized with an increase in temperature. At high temperatures it is localized to within the thermal Compton wavelength \( 1/T \).

(iv) At a non-zero temperature the charge vanishes in the limit of massless fermions, just as is the case for induced charge in \( QED_3 \) (cf Eq.(17) and \([10]\)).

An interesting open question is how the analysis of \([6]\), which holds more generally for extended monopoles \([12]\) in non-Abelian gauge theories with a \( CP \) violating vacuum angle, changes at non-zero temperature.

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References


Figure Caption

A plot of $S$ against $\theta$ for several values of $x = M/\pi T$. The charge is given by $Q = -\frac{c^{(2g)}}{\pi} S$. The five curves correspond to the values $x = 0.05$, $0.1$, $0.5$, $1$ and $10$, the curves with increasing amplitude corresponding to larger $x$ (lower temperatures). For the $x = 10$ curve (which is almost a straight line approaching the zero temperature relation (2)), we have not indicated the sharp drop which occurs near $\theta = \pi$. 

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