A General Analytic Formula for the Spectral Index of the Density Perturbations produced during Inflation

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Abstract

The standard calculation of the spectrum of density perturbations produced during inflation assumes that there is only one real dynamical degree of freedom during inflation. However, there is no reason to believe that this is actually the case. In this paper we derive general analytic formulae for the spectrum and spectral index of the density perturbations produced during inflation.
1 Introduction

During inflation [1] vacuum fluctuations on scales less than the Hubble radius in scalar fields with effective masses much less than the Hubble parameter\(^1\) are magnified into classical perturbations in the scalar fields on scales larger than the Hubble radius. These classical perturbations in the scalar fields can then change the effective number of e-folds of inflation and so lead to classical density perturbations after inflation. These density perturbations are thought to be responsible for the formation of galaxies and the large scale structure of the observable Universe as well as, in combination with the gravitational waves produced during inflation, for the anisotropies in the cosmic microwave background.

The standard calculation [7, 8, 9, 10, 11] of the spectrum of density perturbations produced during inflation assumes that there is only one real dynamical degree of freedom during inflation. Although this is the case in most of the models of inflation constructed up to now, it is by assumption rather than prediction. When one tries to construct models of inflation [3, 5, 12] that might arise naturally in realistic models of particle physics, such as the low energy effective supergravity theories derived from superstrings, one often gets more than one dynamical degree of freedom during inflation. The standard calculation is then generally not applicable.

In this paper we derive general analytic formulae for the spectrum and spectral index of the density perturbations produced during inflation. This work is based on earlier work by Starobinsky [13]. See also [14]. While this work was in slow preparation two other related papers [15] appeared in the gr-qc archive. After this work was completed another related paper [16] appeared in the astro-ph archive.

2 Gravity

We assume the gravitational part of the action to be

\[
S = -\frac{1}{2} \int R \sqrt{-g} \, d^4 x .
\]

2.1 The background

The background metric is

\[
d s^2 = d t^2 - a(t)^2 \delta_{ij} d x^i d x^j .
\]

Two important quantities are the Hubble parameter

\[
H = \frac{\dot{a}}{a}
\]

\(^1\)All scalar fields generically acquire effective masses at least of the order of the Hubble parameter in the early Universe [2, 3, 4]. However, this can be naturally avoided for some scalar fields during inflation in certain classes of supergravity theories [5, 6].
and the number of $e$-folds of expansion

$$N = \int H dt.$$  

(4)

2.2 $\mathcal{R}$

Scalar linear perturbations to the metric can be expressed most generally as [17, 18, 9]

$$ds^2 = (1 + 2A)dt^2 - 2\partial_i B dx^i dt - a(t)^2 [(1 + 2\mathcal{R})\delta_{ij} + 2\partial_i \partial_j \mathcal{E}] dx^i dx^j.$$  

(5)

We follow [18] most closely. $\mathcal{R}$ is the intrinsic curvature perturbation of the constant time hypersurfaces. On comoving hypersurfaces, $\mathcal{R}_c = H A_c$. On flat hypersurfaces, $\mathcal{R}_f \equiv 0$.

2.3 $N$

Let $\{\Sigma(t)\}$ be a foliation of spacetime with hypersurfaces $\Sigma(t)$ labeled by a certain coordinate time $t$ and let $v^\mu$ be the unit vector field normal to $\Sigma(t)$. Then $\theta = v^\mu \,_{;\mu}$ is the volume expansion rate of the hypersurfaces along the integral curve $\gamma(\tau)$ of $v^\mu$. For each integral curve, define

$$N = \int_{\gamma(\tau)} \frac{1}{3} \theta d\tau,$$  

(6)

where $\tau$ is the proper time along the curve.

2.4 $\mathcal{R}$ and $N$

From [18]

$$\frac{1}{3} \theta = H \left(1 - A + \frac{1}{H} \dot{\mathcal{R}} + \frac{1}{3H a^2} \partial^i \partial_i S_g\right),$$  

(7)

where $S_g = a^2 \dot{E} - B$. (If one Fourier expands $S_g$, $S_g = \sigma_g/q$, where $q = k/a$, in the notation of [18].) Assuming that the anisotropic stress perturbation is negligible, which is the case for scalar field, radiation or dust perturbations, then the spatial trace-free part of the Einstein equations gives

$$\frac{1}{H} \dot{S}_g + S_g = \frac{1}{H} (A + \mathcal{R}).$$  

(8)

From this equation we see that $S_g$ is at most of order $A/H$ or $\mathcal{R}/H$ so that it is clear that the last term in Eq. (7) is negligible compared with the other terms on superhorizon scales, that is for $q^2 \ll H^2$ when the perturbations are Fourier expanded. So we get

$$\frac{1}{3} \theta \simeq H \left(1 - A + \frac{1}{H} \dot{\mathcal{R}}\right).$$  

(9)
Also, from Eq. (5),
\[ d\tau = (1 + A) \, dt. \]  
(10)

Therefore
\[ N = \int_{\gamma(\tau)} H \left(1 - A + \frac{1}{H} \dot{H}\right) (1 + A) \, dt = \int_{\gamma(\tau)} \left(H + \dot{H}\right) \, dt, \]  
(11)

and so
\[ \delta N = \Delta R. \]  
(12)

In particular, if we choose a foliation such that the initial hypersurface is flat and the final one is comoving, we get
\[ \delta N(\Sigma_f(t_1), \Sigma_c(t_2); \gamma(\tau)) = R_c(t_2) \]  
(13)

for a given curve \( \gamma(\tau) \).

Now take \( t_1 \) to be some time during inflation soon after the relevant scale has passed outside the horizon and \( t_2 \) to be some time after complete reheating when \( R_c \) has become constant. The relevant scale is assumed to be still well outside the horizon at \( t = t_2 \). Then one may regard \( N \) as a function of the field configuration \( \phi^a(t_1, x^i) \) on \( \Sigma(t_1) \) and the time \( t_2 \),
\[ N = N(\phi^a(t_1, x^i), t_2). \]  
(14)

Note that in general \( N \) depends on both \( \phi^a(t_1) \) and \( \dot{\phi}^a(t_1) \), but as \( t_1 \) is during inflation we use the slow roll approximation to eliminate the dependence on \( \dot{\phi}^a(t_1) \). Therefore
\[ R_c(t_2, x^i_2) = \delta N = \frac{\partial N}{\partial \delta \phi^a} \delta \phi^a_f(t_1, x^i_1), \]  
(15)

where \( x^i_1 \) and \( x^i_2 \) are the spatial coordinates of \( \gamma(\tau) \) on \( \Sigma_f(t_1) \) and \( \Sigma_c(t_2) \), respectively. One can of course choose the spatial coordinates on the hypersurfaces by the condition \( B = 0 \) to make \( x^i_1 = x^i_2 \). Since the perturbations in both \( \theta \) and the density are negligibly small on comoving hypersurfaces on superhorizon scales, \( \Sigma_c(t_2) \) may be regarded as a hypersurface of constant Hubble parameter or constant energy density.

### 3 Scalar fields

We assume the scalar field part of the action to be
\[ S = \int \left[ \frac{1}{2} h_{ab} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b - V(\phi) \right] \sqrt{-g} \, d^4x. \]  
(16)
3.1 The background

The background scalar fields are spatially homogeneous

\[ \phi^a = \phi^a(t). \]  

(17)

The following formula will be useful

\[ \frac{\partial N}{\partial \phi^a} \dot{\phi}^a = -H. \]  

(18)

The background equation of motion for the scalar fields is

\[ \frac{D\dot{\phi}^a}{dt} + 3H\dot{\phi}^a + h^{ab}V_b = 0, \]  

(19)

where \( DX^a/dt = dX^a/dt + \Gamma^a_{bc} \dot{\phi}^b X^c \) and \( \Gamma^a_{bc} = \frac{1}{2} h^{ad} (h_{db,c} + h_{dc,b} - h_{bc,d}) \). The covariant slow roll approximation then gives

\[ 3H\dot{\phi}^a \simeq -h^{ab}V_b, \]  

(20)

or

\[ \frac{\dot{\phi}^a}{H} \simeq -h^{ab}V_b \frac{V}{V}. \]  

(21)

3.2 \( \delta \phi^a \)

The equation of motion for scalar field perturbations on flat hypersurfaces is

\[ \frac{D^2 \delta \phi^a}{dt^2} + 3H\frac{D\delta \phi^a}{dt} - R^a_{\ bcd} \dot{\phi}^b \dot{\phi}^d \delta \phi^c + q^2 \delta \phi^a + h^{ab}V_{;bc}\delta \phi^c = \frac{1}{a^3} \frac{D}{dt} \left( \frac{a^3}{H} \dot{\phi}^a \dot{\phi}^b \right) h_{bc}\delta \phi^c \]  

(22)

where \( X_{a,b} = X_{a,b} - \Gamma^c_{ab} X^c \) and \( R^a_{\ bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^c_{eb} \). The covariant slow roll approximation then gives

\[ 3H \frac{D\delta \phi^a}{dt} - R^a_{\ bcd} \dot{\phi}^b \dot{\phi}^d \delta \phi^c + h^{ab}V_{;bc}\delta \phi^c = 3\dot{\phi}^a \dot{\phi}^b h_{bc}\delta \phi^c. \]  

(23)

4 Calculation of the spectral index

Fourier expansion of Eq. (15) gives

\[ \mathcal{R}_k(t_2) = \frac{\partial N}{\partial \phi^a} \delta \phi_k^a(t_1). \]  

(24)

For scalar field perturbations generated from vacuum fluctuations during inflation

\[ \delta \phi_k^a = \sum_\alpha \phi_k^{\alpha a} a_k^\alpha, \]  

(25)
where $a^\alpha_k$ is a classical random variable satisfying

$$\langle a^\alpha_k a^\beta_1 \rangle = \delta^{\alpha\beta} \delta^3(k - l), \quad (26)$$

$\alpha$ runs over the number of scalar field components, and $\phi_k^{\alpha\alpha}$ is real and satisfies

$$\sum_\alpha \phi_k^{\alpha\alpha} \phi_k^{\beta\beta} = \frac{H^2}{2k^3} \left(h^{ab} + \epsilon^{ab}\right) \quad (27)$$

where $\epsilon^{ab}$ is small and slowly varying with respect to $a$ at fixed $k/a$. Therefore the spectrum of curvature perturbations is given by

$$\frac{2\pi^2}{k^3} P_\delta \delta^3(k - 1) = \langle R_k(t_2) R_l^\dagger(t_2) \rangle \quad (28)$$

$$= \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b} \langle \delta \phi_k^a(t_1) \delta \phi_l^b(t_1) \rangle \quad (29)$$

$$= \sum_\alpha \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b} \phi_k^{\alpha\alpha} \phi_k^{\beta\beta} \delta^3(k - l), \quad (30)$$

or

$$P_\delta = \frac{k^3}{2\pi^2} \sum_\alpha \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b} \phi_k^{\alpha\alpha} \phi_k^{\beta\beta}. \quad (31)$$

The spectral index is then given by

$$n_\delta - 1 = \frac{d \ln P_\delta}{d \ln k} = 3 + \frac{k^3}{2\pi^2 P_\delta} \sum_\alpha \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b} D \phi_k^{\alpha\alpha} \phi_k^{\beta\beta} \quad (32)$$

Now

$$\frac{D}{d \ln a} = \left. \frac{D}{d \ln a} \right|_{k/a=\text{constant}} = \frac{D}{\partial \ln \frac{k}{a}}, \quad (33)$$

Eq. (27) gives

$$\left. \frac{D}{d \ln a} \left( \sum_\alpha \phi_k^{\alpha\alpha} \phi_k^{\beta\beta} \right) \right|_{k/a=\text{constant}} = - \left( 3 - 2 \frac{\dot{H}}{H^2} \right) \sum_\alpha \phi_k^{\alpha\alpha} \phi_k^{\beta\beta}, \quad (34)$$

and the slow roll equation of motion Eq. (23) gives

$$\frac{D \phi_k^{\alpha\alpha}}{d \ln a} = \left( \frac{\phi_k^{\alpha\alpha}}{H^2} h_{bd} + \frac{1}{3} R_{bcd} \phi_k^{\beta\beta} \phi_k^{\gamma\gamma} - h^{ab} V_{bd} \right) \phi_k^{\alpha\alpha}. \quad (35)$$

Substituting into Eq. (32) then gives

$$n_\delta - 1 = 2 \frac{\dot{H}}{H^2} - \frac{k^3}{2\pi^2 P_\delta} \sum_\alpha \frac{\partial N}{\partial \phi^a} \left( \frac{\phi_k^{\alpha\alpha}}{H^2} h_{bd} + \frac{1}{3} R_{abcd} \phi_k^{\beta\beta} \phi_k^{\gamma\gamma} - h^{ab} V_{bd} \right) \phi_k^{\alpha\alpha} \phi_k^{\beta\beta} \frac{\partial N}{\partial \phi^e}. \quad (36)$$

\text{We are outside the horizon so everything is classical.}
Now
\[ \sum_{\alpha} \phi_\alpha^{\alpha} \phi_\alpha^{\beta} \sim \frac{H^2}{2k^3} h^{ab}, \] (37)
and so from Eq. (31)
\[ P_R = \left( \frac{H}{2\pi} \right)^2 h^{ab} \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b}, \] (38)
therefore from Eq. (36)
\[ n_R - 1 = 2 \frac{\dot{H}}{H^2} - 2 \frac{\partial N}{\partial \phi^a} \left( \frac{\dot{\phi}^a \dot{\phi}^b}{H^2} + \frac{1}{3} R_{bc} \dot{\phi}^b \dot{\phi}^c \frac{1}{H^2} - h^{ab} \frac{\partial V}{\partial \phi^c} \frac{h^{cd}}{V} \right) \frac{\partial N}{\partial \phi^d}. \] (39)
Therefore, from Eqs. (18) and (21),
\[ n_R - 1 = 2 \frac{\dot{H}}{H^2} - 2 \frac{\partial N}{\partial \phi^a} \left( \frac{1}{3} R_{abcd} \frac{V_{a} V_{c}}{V^2} \frac{h^{ab} V_{bc} h^{cd}}{V} \right) \frac{\partial N}{\partial \phi^d}. \] (40)

5 Summary

The spectrum of gravitational waves produced during inflation is [19]
\[ P_g = \left( \frac{H}{2\pi} \right)^2. \] (41)
The spectrum of curvature perturbations produced during inflation is
\[ P_R = \left( \frac{H}{2\pi} \right)^2 h^{ab} \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b}. \] (42)
The spectral index of the gravitational waves is [8]
\[ n_g = 2 \frac{\dot{H}}{H^2}. \] (43)
The spectral index of the curvature perturbations is
\[ n_R - 1 = 2 \frac{\dot{H}}{H^2} - 2 \frac{\partial N}{\partial \phi^a} \left( \frac{1}{3} R_{abcd} \frac{V_{a} V_{c}}{V^2} \frac{h^{ab} V_{bc} h^{cd}}{V} \right) \frac{\partial N}{\partial \phi^d}. \] (44)
Using Eq. (18) we see that inflation predicts
\[ \frac{P_g}{P_R} \leq |n_g|, \] (45)
so that the ratio of gravitational wave to curvature perturbation contributions to the cosmic microwave background anisotropy satisfies \( R \leq 6.2|n_g| \). Note that the
spectral index depends on the curvature of the space of scalar fields as well as the potential, though as realistic models of inflation tend to give \((V'/V)^2 \ll |V''/V|\) this may be difficult to observe. An interesting point is that in models with more than one dynamical degree of freedom there is generally not a unique inflationary trajectory and so the initial conditions might play a role in determining the spectrum and hence be observable.

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**References**


