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On a Certain Construction of Graded Lie
Algebras with Derivation

by

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Abstract

Using a unital associative $\ast$ -algebra $\mathfrak{A}$ over $\mathbb{C}$ and a certain class of Hermitian finite projective modules together with a graded involutive differential algebra, both associated with $\mathfrak{A}$, we develop a procedure for constructing graded Lie algebras with derivation. Taking, in particular, the canonical differential algebra of Connes' theory, related to the simplest two-point K-cycle, we obtain a class of graded Lie algebras with derivation, which as one special case contains the graded Lie algebra used in the Mainz-Marseille approach to model building.

1 Introduction

During the last decade there has been an increasing interest in methods related to non-commutative differential geometric structures. One of the main streams in this field was initiated and mainly developed by A. Connes ([7], [6]). Starting from the observation that the “classical” Dirac K-cycle of a Riemannian manifold X contains all information about this manifold, he invented the abstract notion of a K-cycle over an – in general – non-commutative algebra. This gives the possibility to discuss geometric structures, which – in general – do not possess an underlying “classical” manifold. Connes realized that already slight modifications of the “classical” K-cycle, namely such that the algebra remains commutative, give rise to interesting physical applications. The simplest relevant example of this type [7] is the K-cycle over the algebra $C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$ leading to a unification of gauge and Higgs bosons. If one takes the tensor product of this algebra with the vector space of fermions, one can derive a version of the classical Lagrangian of the Salam–Weinberg model of electroweak interactions with the bosonic sector described in terms of a unified non-commutative gauge field, see [7], [6], [8]. The above algebra is the simplest example of the class of algebras $C^\infty(X) \otimes (M_k \mathbb{C} \oplus M_l \mathbb{C})$, which we call two-point algebras. For the derivation of the full (classical) standard model, Connes and Lott [8] proposed to use a K-cycle over the algebra $C_\mathbb{R}^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{H})$, where $\mathbb{H}$ denotes the field of quaternions and $C_\mathbb{R}^\infty(X)$ the algebra of real smooth functions on X . A detailed presentation of this construction can be found in a series of papers by Kastler ([18], [19], [20], [21]). For an overview over the mathematical background we refer to [27] and for a physicist’s review to [10].

There is another approach to model building, proposed by Coquereaux and Sheck and further developed by their groups in Mainz and Marseille, see [13], [12], [9], [11], [14], which at first sight seems to be completely different from that of Connes and Lott. These authors postulate ad hoc a certain graded matrix Lie algebra and consider a generalized connection with values in this algebra. The connection is built both from differential one forms and zero forms, representing the classical gauge fields of the electroweak interaction and the scalar Higgs fields respectively. Adding by hand the gauge bosons of the strong interaction and choosing appropriate fermionic representations, one can derive the classical Lagrangian of the standard model in this way.

The fact that the bosonic sector in this type of models is unified, has non-trivial phenomenological consequences. In particular, in most versions one obtains a prediction of the Higgs mass at tree level. However, there are – from the phenomenological point of view – certain subtle differences between the two above-mentioned approaches. This is mainly related to the fact that within the construction of Connes and Lott one gets additional relations between boson and fermion masses. For a detailed discussion of this aspect we refer to [23].

In this paper we present a rigorous mathematical link between these two approaches. Using results from our previous paper [22] we will prove that given

the simplest two-point K-cycle together with the differential algebra Ω_D^* , which is obtained from the universal differential algebra (associated with the algebra of the K-cycle) by factorizing with respect to a canonically given ideal, and taking a finite projective module over the algebra, we are able to construct in a canonical way a graded Lie algebra. Since every finite projective module carries a canonical connection, this graded Lie algebra is naturally endowed with a derivation. If one chooses the module appropriately, then one arrives at the graded Lie algebra used by the Mainz-Marseille group for the derivation of the standard model. This way all structures, ad hoc postulated within this approach, find their natural explanation within the context of Connes' theory.

As a matter of fact, the construction of graded Lie algebras with derivation proposed in this paper is not limited to the case, when a K-cycle together with the canonically associated differential algebra Ω_D^* is given. All we need – in the most general context – is a unital associative algebra $\mathfrak{A}$ over $\mathbb{C}$ (fulfilling a certain technical condition) and a certain graded differential algebra $\Lambda_{\mathfrak{A}}^*$, associated with $\mathfrak{A}$ in a sense defined below. Then taking an arbitrary finite projective module over $\mathfrak{A}$, we can construct a graded Lie algebra with derivation – a fact, which at least from a purely mathematical point of view seems to be of some interest in itself. For physical applications as discussed above one is rather interested in the case, when $\mathfrak{A}$ and $\Lambda_{\mathfrak{A}}^*$ are endowed additionally with an involution and the module carries a Hermitian structure. It will be interesting to apply our general construction to situations more complicated than that of the simplest two-point K-cycle. In particular, a similar analysis for the N -point case would be interesting, because this case seems to be relevant for the construction of grand unified theories, see [3], [4] and [5].

The paper is organized as follows: In section 2 we present the construction of graded Lie algebras in the general context – as indicated above. In section 3 we discuss the notion of connections on finite projective modules and show how the canonical connection gives rise to a graded derivation in the graded Lie algebra constructed before. Next, in section 4 we give a matrix formulation of these structures. In section 5 we review results [22] on the differential algebra $\Lambda_{\mathfrak{A}}^*$ associated canonically with the simplest two-point K-cycle and in section 6 we distinguish a certain graded Lie subalgebra of the graded Lie algebra constructed earlier. In section 7 we pass to a standard matrix representation of the structures discussed before and in section 8 we show that the mathematical structures used in the Mainz-Marseille approach are naturally obtained from the framework developed in this paper. Finally, in section 9 we make some comments on model building.

2 Finite Projective Modules with Hermitian Structure and Graded Lie Algebras

Let $\mathfrak{A}$ be a unital associative $*$ -algebra over $\mathbb{C}$, such that $a^*a = 0$ iff $a = 0$. Moreover, let $(\Lambda_{\mathfrak{A}}^*, \bullet, *, d)$ be a graded involutive differential algebra associated with

$\mathfrak{A}$. That means $\Lambda_{\mathfrak{A}}^* = \bigoplus_{k=0}^{\infty} \Lambda_{\mathfrak{A}}^k$, $\Lambda_{\mathfrak{A}}^0 \equiv \mathfrak{A}$. The dot $\bullet$ denotes the multiplication $\Lambda_{\mathfrak{A}}^k \bullet \Lambda_{\mathfrak{A}}^l \subset \Lambda_{\mathfrak{A}}^{k+l}$, d the graded differential

$$d(\lambda^k \bullet \tilde{\lambda}^l) = (d\lambda^k) \bullet \tilde{\lambda}^l + (-1)^k \lambda^k \bullet d\tilde{\lambda}^l \in \Lambda_{\mathfrak{A}}^{k+l}, \quad d^2 \lambda^k \equiv 0, \quad (1)$$

for $\lambda^k \in \Lambda_{\mathfrak{A}}^k$, $\tilde{\lambda}^l \in \Lambda_{\mathfrak{A}}^l$, and $*$ is an involution compatible with d

$$d((\lambda^k)^*) = (-1)^k (d\lambda^k)^*, \quad \lambda^k \in \Lambda_{\mathfrak{A}}^k. \quad (2)$$

Since $\mathfrak{A} \equiv \Lambda_{\mathfrak{A}}^0$, we have a natural $\mathfrak{A}$ -bimodule structure on $\Lambda_{\mathfrak{A}}^*$. When multiplying elements of $\mathfrak{A}$ with elements of $\Lambda_{\mathfrak{A}}^*$, we omit the dot for simplicity.

We recall [27] that every finite projective right module $\mathcal{E}$ over $\mathfrak{A}$ has the structure $\mathcal{E} = e\mathfrak{A}^p$, where $p \in \mathbb{N}$ and $e \in \text{End}_{\mathfrak{A}}(\mathfrak{A}^p)$, with $e^2 = e$. Here $\mathfrak{A}^p$ is treated as $\mathbb{C}^p \otimes \mathfrak{A}$, because there is a canonical isomorphism $\mathfrak{A}^p = \underbrace{\mathfrak{A} \oplus \dots \oplus \mathfrak{A}}_p \cong$

$\mathbb{C}^p \otimes \mathfrak{A}$. Elements $\zeta \in \mathfrak{A}^p$ are of the form $\zeta = \sum_{\alpha} c_{\alpha} \otimes a_{\alpha}$, finite sum, where $c_{\alpha} \in \mathbb{C}^p$ and $a_{\alpha} \in \mathfrak{A}$. The right $\mathfrak{A}$ -module structure is given by

$$\mathfrak{A}^p \times \mathfrak{A} \ni (\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}, \tilde{a}) \mapsto (\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}) \tilde{a} := \sum_{\alpha} c_{\alpha} \otimes (a_{\alpha} \tilde{a}) \in \mathfrak{A}^p. \quad (3)$$

Every nontrivial projective module can be naturally embedded into $\mathfrak{A}^p$, for a certain p . The right $\mathfrak{A}$ -module structure of $\mathcal{E}$ is given by (3) if we restrict $\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}$ to $\mathcal{E} \subset \mathfrak{A}^p$.

Definition 1 A Hermitian finite projective right $\mathfrak{A}$ -module is a pair $(\mathcal{E}, (\cdot, \cdot)_{\mathcal{E}})$, where $(\cdot, \cdot)_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \rightarrow \mathfrak{A}$ is a sesquilinear, Hermitian, non-degenerate, positive map. This means:

- (i) $(\xi a, \tilde{\xi} \tilde{a})_{\mathcal{E}} = a^* (\xi, \tilde{\xi})_{\mathcal{E}} \tilde{a}$,
 - (ii) $(\xi, \tilde{\xi})_{\mathcal{E}}^* = (\tilde{\xi}, \xi)_{\mathcal{E}}$,
 - (iii) $(\xi, \xi)_{\mathcal{E}} \geq 0$. if $(\xi, \xi)_{\mathcal{E}} = 0$ then $\xi = 0$,
- with $\xi, \tilde{\xi} \in \mathcal{E}$, $a, \tilde{a} \in \mathfrak{A}$.

We define a Hermitian structure on $\mathfrak{A}^p$ by

$$(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}, \sum_{\beta} \tilde{c}_{\beta} \otimes \tilde{a}_{\beta})_{\mathfrak{A}^p} := \sum_{\alpha, \beta} (c_{\alpha}, \tilde{c}_{\beta})_{\mathbb{C}^p} a_{\alpha}^* \tilde{a}_{\beta}, \quad (4)$$

where $(\cdot, \cdot)_{\mathbb{C}^p}$ denotes a scalar product on $\mathbb{C}^p$. The involution of endomorphisms of $\mathfrak{A}^p$ is defined by $(\chi^* \zeta, \tilde{\zeta})_{\mathfrak{A}^p} = (\zeta, \chi \tilde{\zeta})_{\mathfrak{A}^p}$, for $\chi \in \text{End}_{\mathfrak{A}}(\mathfrak{A}^p)$. Since every idempotent e can be transformed into an orthogonal (Hermitian) projector, we can assume that $e = e^*$. The similarity transformation leading from an idempotent e to an orthogonal projector $\tilde{e}$ is given by [28]

$$\tilde{e} = z e z^{-1}, \quad z := ((2e^* - 1)(2e - 1) + 1)^{\frac{1}{2}}. \quad (5)$$

To be precise, taking the square root in this formula is in general not well defined for every $*$ -algebra. In such situations we always assume that e is orthogonal

from the very beginning. One can easily see that the right $\mathfrak{A}$ -modules $e\mathfrak{A}^p$ and $\tilde{e}\mathfrak{A}^p$ are isomorphic. One isomorphism is given by the map

$$e\mathfrak{A}^p \ni \xi \mapsto z\xi \in \tilde{e}\mathfrak{A}^p. \quad (6)$$

Restricting the Hermitian structure given by (4) to $\mathcal{E} = e\mathfrak{A}^p$, we get a Hermitian structure on $\mathcal{E}$. From now on we assume that Hermitian structures of projective modules under consideration are obtained that way.

Let us denote the tensor product of the right module $\mathcal{E}$ with the bimodule $\Lambda_{\mathfrak{A}}^k$ over the algebra $\mathfrak{A}$ by $\mathcal{E}^k = \mathcal{E} \otimes_{\mathfrak{A}} \Lambda_{\mathfrak{A}}^k$, $\mathcal{E}^0 := \mathcal{E}$ and $\mathcal{E}^* := \bigoplus_{k \in \mathbf{N}_0} \mathcal{E}^k$. This tensor product carries a natural right $\mathfrak{A}$ -module structure:

$$\left(\sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k, a \right) \mapsto \left(\sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \right) a := \sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} (\lambda_{\alpha}^k a), \quad (7)$$

with $\xi_{\alpha} \in \mathcal{E}$, $\lambda_{\alpha}^k \in \Lambda_{\mathfrak{A}}^k$, $a \in \mathfrak{A}$. On $\mathcal{E}^*$ we have the natural structure of a right $\Lambda_{\mathfrak{A}}^*$ -module inherited from the multiplication in $\Lambda_{\mathfrak{A}}^*$:

$$\mathcal{E}^k \times \Lambda_{\mathfrak{A}}^l \ni \left(\sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k, \tilde{\lambda}^l \right) \mapsto \left(\sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \right) \bullet \tilde{\lambda}^l := \sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} (\lambda_{\alpha}^k \bullet \tilde{\lambda}^l) \in \mathcal{E}^{k+l}, \quad (8)$$

for $\xi_{\alpha} \in \mathcal{E}$, $\lambda_{\alpha}^k \in \Lambda_{\mathfrak{A}}^k$, $\tilde{\lambda}^l \in \Lambda_{\mathfrak{A}}^l$. We extend the Hermitian structure on $\mathcal{E}$ to mappings $(\cdot, \cdot)_{\mathcal{E}}^{k,l} : \mathcal{E}^k \times \mathcal{E}^l \rightarrow \Lambda_{\mathfrak{A}}^{k+l}$ by

$$\left(\sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k, \sum_{\beta} \tilde{\xi}_{\beta} \otimes_{\mathfrak{A}} \tilde{\lambda}_{\beta}^l \right)_{\mathcal{E}}^{k,l} := \sum_{\alpha} \sum_{\beta} (\lambda_{\alpha}^k)^* \bullet (\xi_{\alpha}, \tilde{\xi}_{\beta})_{\mathcal{E}} \bullet \tilde{\lambda}_{\beta}^l. \quad (9)$$

Lemma 2 *We have*

- (i) $(\xi^k a, \tilde{\xi}^l \tilde{a})_{\mathcal{E}}^{k,l} = a^* (\xi^k, \tilde{\xi}^l)_{\mathcal{E}}^{k,l} \tilde{a}$,
- (ii) $((\xi^k, \tilde{\xi}^l)_{\mathcal{E}}^{k,l})^* = (\tilde{\xi}^l, \xi^k)_{\mathcal{E}}^{l,k}$,
- (iii) $(\xi^k, \tilde{\xi})_{\mathcal{E}}^{k,0} = 0 \quad \forall \xi \in \mathcal{E} \quad \text{iff} \quad \xi^k = 0$,
- (iv) $(\xi, \tilde{\xi}^l)_{\mathcal{E}}^{0,l} = 0 \quad \forall \xi \in \mathcal{E} \quad \text{iff} \quad \tilde{\xi}^l = 0$,

where $\xi^k \in \mathcal{E}^k$, $\tilde{\xi}^l \in \mathcal{E}^l$, $a, \tilde{a} \in \mathfrak{A}$.

Proof: (i) is obvious from (9) and (7).

(ii) If $\xi^k \equiv \sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k$, $\tilde{\xi}^l \equiv \sum_{\beta} \tilde{\xi}_{\beta} \otimes_{\mathfrak{A}} \tilde{\lambda}_{\beta}^l$, we have

$$((\xi^k, \tilde{\xi}^l)_{\mathcal{E}}^{k,l})^* = \sum_{\alpha, \beta} ((\lambda_{\alpha}^k)^* \bullet (\xi_{\alpha}, \tilde{\xi}_{\beta})_{\mathcal{E}} \bullet \tilde{\lambda}_{\beta}^l)^* = \sum_{\alpha, \beta} (\tilde{\lambda}_{\beta}^l)^* \bullet (\tilde{\xi}_{\beta}, \xi_{\alpha})_{\mathcal{E}} \bullet \lambda_{\alpha}^k = (\tilde{\xi}^l, \xi^k)_{\mathcal{E}}^{l,k}.$$

(iii) Let us choose an orthogonal basis $\{c_i\}_{i=1, \dots, p}$ in $\mathbb{C}^p$, $(c_i, c_j)_{\mathbb{C}^p} = \delta_{ij}$. Since we treat $\mathcal{E}$ as a submodule of the free module $\mathfrak{A}^p$, we can represent $\xi_{\alpha} \equiv \sum_{i=1}^p c_i \otimes a_{i\alpha} \in \mathcal{E}$ and $\tilde{\xi}_j \equiv e(\tilde{c}_j \otimes I_{\mathfrak{A}}) \in \mathcal{E}$, where $I_{\mathfrak{A}}$ denotes the unit element of $\mathfrak{A}$. Moreover, we denote $\xi^k \equiv \sum_{i=1}^p \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \in \mathcal{E}^k$. Now we have

$$\begin{aligned} 0 &= (\xi^k, \tilde{\xi}_j)_{\mathcal{E}}^{k,0} = \sum_{\alpha} (\lambda_{\alpha}^k)^* \left(\sum_{i=1}^p c_i \otimes a_{i\alpha}, e(c_j \otimes I_{\mathfrak{A}}) \right)_{\mathcal{E}} \\ &= \sum_{\alpha} (\lambda_{\alpha}^k)^* \left(e^* \left(\sum_{i=1}^p c_i \otimes a_{i\alpha} \right), c_j \otimes I_{\mathfrak{A}} \right)_{\mathfrak{A}^p} \\ &= \sum_{i=1}^p (c_i, c_j)_{\mathbb{C}^p} \sum_{\alpha} (\lambda_{\alpha}^k)^* a_{i\alpha}^* = \sum_{\alpha} (a_{j\alpha} \lambda_{\alpha}^k)^*. \end{aligned}$$

This means $\sum_{\alpha} a_{i\alpha} \lambda_{\alpha}^k = 0$ for all $i = 1, \dots, p$. Then we have

$$\xi^k = \sum_{i=1}^p (c_i \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} \sum_{\alpha} (a_{i\alpha} \lambda_{\alpha}^k) \equiv 0 ,$$

which means that $(\ , \)_{\mathcal{E}}^{k,0}$ is non-degenerate.

(iv) is a direct consequence of (ii) and (iii). $\square$

We denote by $\mathcal{E}'$ the dual of the module $\mathcal{E}$ (the set of $\mathfrak{A}$ -linear mappings from $\mathcal{E}$ to $\mathfrak{A}$). One has ([2], §4, n° 2)

$$\mathcal{E}' = j^t \circ e^t(\mathfrak{A} \otimes C^{p'}) , \quad (10)$$

where $j : \mathcal{E} \rightarrow \mathfrak{A}^p$ is the canonical embedding and j^t respectively e^t means taking the algebraic dual mapping. The space $\mathcal{E}'$ is a left $\mathfrak{A}$ -module:

$$(a\xi')(\xi) = a(\xi'(\xi)) , \quad a \in \mathfrak{A} , \quad \xi' \in \mathcal{E}' , \quad \xi \in \mathcal{E} . \quad (11)$$

This allows to define the tensor products $\mathcal{E}'^k := \Lambda_{\mathfrak{A}}^k \otimes_{\mathfrak{A}} \mathcal{E}'$ and $\mathcal{E}'^* := \bigoplus_{k=0}^{\infty} \mathcal{E}'^k$, $\mathcal{E}'^0 := \mathcal{E}'$. There is a natural left $\Lambda_{\mathfrak{A}}^*$ -module structure on $\mathcal{E}'^*$:

$$\Lambda_{\mathfrak{A}}^k \times \mathcal{E}'^l \ni (\lambda^k, \sum_{\alpha} \tilde{\lambda}_{\alpha}^l \otimes_{\mathfrak{A}} \xi'_{\alpha}) \mapsto \lambda^k \bullet (\sum_{\alpha} \tilde{\lambda}_{\alpha}^l \otimes_{\mathfrak{A}} \xi'_{\alpha}) := \sum_{\alpha} (\lambda^k \bullet \tilde{\lambda}_{\alpha}^l) \otimes_{\mathfrak{A}} \xi'_{\alpha} \in \mathcal{E}'^{k+l} . \quad (12)$$

We also have natural $\mathfrak{A}$ -bilinear mappings, which extend the pairing of the dual modules $\mathcal{E}$ and $\mathcal{E}'$,

$$\begin{aligned} \mathcal{E}'^k \times \mathcal{E}'^l &\ni (\sum_{\alpha} \lambda_{\alpha}^k \otimes_{\mathfrak{A}} \xi'_{\alpha}, \sum_{\beta} \xi_{\beta} \otimes_{\mathfrak{A}} \tilde{\lambda}_{\beta}^l) \\ &\mapsto (\sum_{\alpha} \lambda_{\alpha}^k \otimes_{\mathfrak{A}} \xi'_{\alpha})(\sum_{\beta} \xi_{\beta} \otimes_{\mathfrak{A}} \tilde{\lambda}_{\beta}^l) := \sum_{\alpha, \beta} \lambda_{\alpha}^k \bullet (\xi'_{\alpha}(\xi_{\beta})) \bullet \tilde{\lambda}_{\beta}^l \in \Lambda_{\mathfrak{A}}^{k+l} . \end{aligned} \quad (13)$$

Let $\mathcal{H}^k \equiv \text{Hom}_{\mathfrak{A}}(\mathcal{E}, \mathcal{E}^k)$ be the set of homomorphisms of the right $\mathfrak{A}$ -module $\mathcal{E}$ to the right $\mathfrak{A}$ -module $\mathcal{E}^k$ and $\mathcal{H} := \bigoplus_{k \in \mathbb{N}_0} \mathcal{H}^k$. Using the right $\Lambda_{\mathfrak{A}}^*$ -module structure on $\mathcal{E}^*$, see (8), we get a natural associative multiplication $\bullet$ on $\mathcal{H}$. We define $\bullet : \mathcal{H}^k \times \mathcal{H}^l \rightarrow \mathcal{H}^{k+l}$ by

$$(\varrho^k \bullet \tilde{\varrho}^l)(\xi) := (id_{\mathcal{E}} \otimes_{\mathfrak{A}} \bullet) \circ (\varrho^k \otimes_{\mathfrak{A}} id_{\Lambda_{\mathfrak{A}}^l}) \circ \tilde{\varrho}^l(\xi) , \quad (14)$$

for $\varrho^k \in \mathcal{H}^k$, $\tilde{\varrho}^l \in \mathcal{H}^l$, $\xi \in \mathcal{E}$. One has ([2], §4, n° 2) $\text{Hom}_{\mathfrak{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathfrak{A}} \Lambda_{\mathfrak{A}}^k) \equiv \mathcal{E} \otimes_{\mathfrak{A}} \Lambda_{\mathfrak{A}}^k \otimes_{\mathfrak{A}} \mathcal{E}'$. Thus, we represent elements $\varrho^k \in \mathcal{H}^k$ in the form

$$\varrho^k = \sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \otimes_{\mathfrak{A}} \xi'_{\alpha} , \quad (15)$$

where $\xi_{\alpha} \in \mathcal{E}$, $\lambda_{\alpha}^k \in \Lambda_{\mathfrak{A}}^k$, $\xi'_{\alpha} \in \mathcal{E}'$. In this notation the multiplication $\bullet$ in $\mathcal{H}$ reads

$$(\sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \otimes_{\mathfrak{A}} \xi'_{\alpha}) \bullet (\sum_{\beta} \tilde{\xi}_{\beta} \otimes_{\mathfrak{A}} \tilde{\lambda}_{\beta}^l \otimes_{\mathfrak{A}} \tilde{\xi}'_{\beta}) = \sum_{\alpha, \beta} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \bullet \xi'_{\alpha}(\tilde{\xi}_{\beta}) \bullet \tilde{\lambda}_{\beta}^l \otimes_{\mathfrak{A}} \tilde{\xi}'_{\beta} . \quad (16)$$

The Hermitian mappings $(\cdot, \cdot)_{\mathcal{E}}^{k,0}$ and $(\cdot, \cdot)_{\mathcal{E}}^{0,k}$ induce an involution on $\mathcal{H}^k$:

$$(\xi, (\varrho^k)^* \tilde{\xi})_{\mathcal{E}}^{0,k} := (\varrho^k \xi, \tilde{\xi})_{\mathcal{E}}^{k,0}, \quad \forall \xi, \tilde{\xi} \in \mathcal{E}, \quad \varrho^k \in \mathcal{H}^k. \quad (17)$$

Using (ii), (iii) and (iv) of Lemma 2, one shows that this mapping is a well defined involution.

Lemma 3 $(\varrho^k \bullet \tilde{\varrho}^l)^* = (\tilde{\varrho}^l)^* \bullet (\varrho^k)^*$, $\varrho^k \in \mathcal{H}^k$, $\tilde{\varrho}^l \in \mathcal{H}^l$.

Proof: For given $\xi, \tilde{\xi} \in \mathcal{E}$ we denote $\tilde{\varrho}^l(\xi) \equiv \sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \tilde{\lambda}_{\alpha}^l$, $\varrho^k(\xi_{\alpha}) \equiv \sum_{\beta} \xi_{\alpha\beta} \otimes_{\mathfrak{A}} \lambda_{\beta}^k$, $(\varrho^k)^*(\tilde{\xi}) \equiv \sum_{\tilde{\alpha}} \tilde{\xi}_{\tilde{\alpha}} \otimes_{\mathfrak{A}} \tau_{\tilde{\alpha}}^k$, $(\tilde{\varrho}^l)^*(\tilde{\xi}_{\tilde{\alpha}}) \equiv \sum_{\tilde{\beta}} \tilde{\xi}_{\tilde{\alpha}\tilde{\beta}} \otimes_{\mathfrak{A}} \tilde{\tau}_{\tilde{\beta}}^l$. Then we have

$$\begin{aligned} (\xi, (\varrho^k \bullet \tilde{\varrho}^l)^*(\tilde{\xi}))_{\mathcal{E}}^{0,k+l} &= ((\varrho^k \bullet \tilde{\varrho}^l)(\xi), \tilde{\xi})_{\mathcal{E}}^{k+l,0} = \sum_{\alpha, \beta} (\xi_{\alpha\beta} \otimes_{\mathfrak{A}} (\lambda_{\beta}^k \bullet \tilde{\lambda}_{\alpha}^l), \tilde{\xi})_{\mathcal{E}}^{k+l,0} \\ &= \sum_{\alpha, \beta} (\lambda_{\beta}^k \bullet \tilde{\lambda}_{\alpha}^l)^*(\xi_{\alpha\beta}, \tilde{\xi})_{\mathcal{E}} = \sum_{\alpha, \beta} (\tilde{\lambda}_{\alpha}^l)^* \bullet (\lambda_{\beta}^k)^*(\xi_{\alpha\beta}, \tilde{\xi})_{\mathcal{E}} \\ &= \sum_{\alpha} (\tilde{\lambda}_{\alpha}^l)^* \bullet (\varrho^k(\xi_{\alpha}), \tilde{\xi})_{\mathcal{E}}^{k,0} = \sum_{\alpha} (\tilde{\lambda}_{\alpha}^l)^* \bullet (\xi_{\alpha}, (\varrho^k)^*(\tilde{\xi}))_{\mathcal{E}}^{0,k} = \sum_{\tilde{\alpha}} (\tilde{\varrho}^l(\xi), \tilde{\xi}_{\tilde{\alpha}})_{\mathcal{E}}^{l,0} \bullet \tau_{\tilde{\alpha}}^k \\ &= \sum_{\tilde{\alpha}} \sum_{\tilde{\beta}} (\xi, \tilde{\xi}_{\tilde{\alpha}\tilde{\beta}})_{\mathcal{E}} \tilde{\tau}_{\tilde{\beta}}^l \bullet \tau_{\tilde{\alpha}}^k = \sum_{\tilde{\alpha}, \tilde{\beta}} (\xi, \tilde{\xi}_{\tilde{\alpha}\tilde{\beta}} \otimes_{\mathfrak{A}} (\tilde{\tau}_{\tilde{\beta}}^l \bullet \tau_{\tilde{\alpha}}^k))_{\mathcal{E}}^{0,k+l} \\ &= (\xi, ((\tilde{\varrho}^l)^* \bullet (\varrho^k)^*)(\tilde{\xi}))_{\mathcal{E}}^{0,k+l}, \end{aligned}$$

where we used (9), (14) and (17). $\square$

Thus we see that $\mathcal{H}$ is an associative, $\mathbb{N}$ -graded, unital, involutive algebra over $\mathbb{C}$.

We define

$$[\varrho^k, \tilde{\varrho}^l]_g := \varrho^k \bullet \tilde{\varrho}^l - (-1)^{kl} \tilde{\varrho}^l \bullet \varrho^k, \quad \varrho^k \in \mathcal{H}^k, \quad \tilde{\varrho}^l \in \mathcal{H}^l. \quad (18)$$

Lemma 4 With respect to the above bracket, $\mathcal{H}$ is a graded Lie algebra, i.e. we have:

$$\begin{aligned} (i) \quad & [\varrho^k, \tilde{\varrho}^l]_g = -(-1)^{kl} [\tilde{\varrho}^l, \varrho^k]_g, \\ (ii) \quad & [\varrho^k, c_1 \tilde{\varrho}^l + c_2 \tilde{\varrho}^l]_g = c_1 [\varrho^k, \tilde{\varrho}^l]_g + c_2 [\varrho^k, \tilde{\varrho}^l]_g, \quad c_1, c_2 \in \mathbb{C}, \\ (iii) \quad & (-1)^{km} [\varrho^k, [\tilde{\varrho}^l, \tilde{\varrho}^m]_g]_g + (-1)^{lk} [\tilde{\varrho}^l, [\tilde{\varrho}^m, \varrho^k]_g]_g \\ & \quad + (-1)^{ml} [\tilde{\varrho}^m, [\varrho^k, \tilde{\varrho}^l]_g]_g = 0. \end{aligned} \quad (19)$$

Proof: Properties (i) – (iii) can be easily checked using (18). $\square$

Let us define

$$\mathfrak{h} := \bigoplus_{k \in \mathbb{N}_0} \mathfrak{h}^k, \quad \mathfrak{h}^k := \{ \varrho^k \in \mathcal{H}^k : \varrho^k = (-1)^{f(k)} (\varrho^k)^*, \quad f(k) \in \mathbb{Z} \}. \quad (20)$$

We want to find $f(k)$, for which $\mathfrak{h}$ is a real graded Lie subalgebra of $\mathcal{H}$. From Lemma 3 and (18) we find for $\varrho^k \in \mathfrak{h}^k$ and $\varrho^l \in \mathfrak{h}^l$:

$$([\varrho^k, \varrho^l]_g)^* = -(-1)^{kl} [(\varrho^k)^*, (\varrho^l)^*]_g \equiv (-1)^{kl+1+f(k)+f(l)} [\varrho^k, \varrho^l]_g = (-1)^{f(k+l)} [\varrho^k, \varrho^l]_g.$$

This gives $f(k+l) = f(k) + f(l) + kl + 1 \pmod{2}$, in particular $f(0) = 1 \pmod{2}$. By recursion we get $f(k+nl) = f(k) + nf(l) + nkl + \frac{n(n-1)}{2}l^2 + n \pmod{2}$, for $n \in \mathbb{N}_0$. Putting $k=0, l=1$, we obtain

$$f(n) = nf(1) + \frac{(n+2)(n-1)}{2} \pmod{2}, \quad f(1) \in \mathbb{Z}. \quad (21)$$

Finally, we endow $\mathcal{E}^*$ naturally with the structure of a left graded $\mathcal{H}$ -module, putting

$$\varrho^k \bullet \xi^l = (id_{\mathcal{E}} \otimes_{\mathfrak{A}} \bullet) \circ (\varrho^k \otimes_{\mathfrak{A}} id_{\Lambda_{\mathfrak{A}}^l})(\xi^l), \quad (22)$$

for $\varrho^k \in \mathcal{H}^k$ and $\xi^l \in \mathcal{E}^l$. By construction, we have

$$(\varrho^k \bullet \tilde{\varrho}^l) \bullet \xi^n = \varrho^k \bullet (\tilde{\varrho}^l \bullet \xi^n), \quad \varrho^k \in \mathcal{H}^k, \quad \tilde{\varrho}^l \in \mathcal{H}^l, \quad \xi^n \in \mathcal{E}^n. \quad (23)$$

Observe that $\mathcal{E}^*$ is a natural representation space of the above constructed graded Lie algebra.

3 Connections and Graded Derivations

Now we discuss the notion of a connection on $\mathcal{E}$ associated with the differential calculus $(\Lambda_{\mathfrak{A}}^*, \bullet, *, d)$, see [8].

Definition 5 *i) A connection on $\mathcal{E}$ is given by a $\mathbb{C}$ -linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E}^1$, such that $\nabla(\xi a) = (\nabla \xi)a + \xi \odot_{\mathfrak{A}} da$, for $\xi \in \mathcal{E}$, $a \in \mathfrak{A}$.*

ii) A connection is compatible (with the Hermitian structure) iff $(\xi, \nabla \tilde{\xi})_{\mathcal{E}}^{0,1} + (\nabla \xi, \tilde{\xi})_{\mathcal{E}}^{1,0} = d(\xi, \tilde{\xi})_{\mathcal{E}}$, for $\xi, \tilde{\xi} \in \mathcal{E}$.

Definition 6 (cf. [8]) *The gauge group $\mathcal{U}(\mathcal{E})$ is the group of unitary automorphisms of $\mathcal{E}$, $\mathcal{U}(\mathcal{E}) := \{ u \in \text{End}_{\mathfrak{A}}(\mathcal{E}) : uu^* = u^*u = id_{\mathcal{E}} \}$, and gauge transformations of the connection ∇ are given by $u\nabla u^*$.*

We extend ∇ uniquely to a linear map $\nabla : \mathcal{E}^n \rightarrow \mathcal{E}^{n+1}$ by

$$\nabla(\xi \otimes_{\mathfrak{A}} \lambda^n) := (\nabla \xi) \bullet \lambda^n + \xi \odot_{\mathfrak{A}} d\lambda^n, \quad \xi \in \mathcal{E}, \quad \lambda^n \in \Lambda_{\mathfrak{A}}^n, \quad (24)$$

which satisfies $\nabla(\xi^n \bullet \lambda) = (\nabla \xi^n) \bullet \lambda + (-1)^n \xi^n \bullet d\lambda$, $\xi^n \in \mathcal{E}^n$, $\lambda \in \Lambda_{\mathfrak{A}}^*$. Therefore, $\nabla^2(\xi^n \bullet \lambda) = (\nabla^2 \xi^n) \bullet \lambda$, which shows that ∇^2 is an endomorphism of the right $\Lambda_{\mathfrak{A}}^*$ -module $\mathcal{E}^*$. It is determined by its restriction to $\mathcal{E}$, which is the curvature θ of the connection ∇ :

$$\theta = \nabla^2|_{\mathcal{E}}. \quad (25)$$

Lemma 7 *The curvature of a compatible connection is selfadjoint, $(\xi, \nabla^2 \tilde{\xi})_{\mathcal{E}}^{0,2} = (\nabla^2 \xi, \tilde{\xi})_{\mathcal{E}}^{2,0}$, for all $\xi, \tilde{\xi} \in \mathcal{E}$.*

Proof: Let $\nabla \tilde{\xi} \equiv \sum_{\alpha} \tilde{\xi}_{\alpha} \otimes_{\mathfrak{A}} \tilde{\lambda}_{\alpha}^1$ and $\nabla \xi \equiv \sum_{\beta} \xi_{\beta} \otimes_{\mathfrak{A}} \lambda_{\beta}^1$. Then we find

$$\begin{aligned}
(\xi, \nabla^2 \tilde{\xi})_{\mathcal{E}}^{0,2} &= \sum_{\alpha} (\xi, \nabla \tilde{\xi}_{\alpha} \bullet \tilde{\lambda}_{\alpha}^1 + \tilde{\xi}_{\alpha} \otimes_{\mathfrak{A}} d\tilde{\lambda}_{\alpha}^1)_{\mathcal{E}}^{0,2} \\
&= \sum_{\alpha} \{ (\xi, \nabla \tilde{\xi}_{\alpha})_{\mathcal{E}}^{0,1} \bullet \tilde{\lambda}_{\alpha}^1 + (\xi, \tilde{\xi}_{\alpha})_{\mathcal{E}} d\tilde{\lambda}_{\alpha}^1 \} \\
&= \sum_{\alpha} \{ d(\xi, \tilde{\xi}_{\alpha})_{\mathcal{E}} \bullet \tilde{\lambda}_{\alpha}^1 + (\xi, \tilde{\xi}_{\alpha})_{\mathcal{E}} d\tilde{\lambda}_{\alpha}^1 - (\nabla \xi, \tilde{\xi}_{\alpha})_{\mathcal{E}}^{1,0} \bullet \tilde{\lambda}_{\alpha}^1 \} \\
&= d(\xi, \nabla \tilde{\xi})_{\mathcal{E}}^{0,1} - \sum_{\beta} (\lambda_{\beta}^1)^* (\xi_{\beta}, \nabla \tilde{\xi})_{\mathcal{E}}^{0,1} \\
&= -d(\nabla \xi, \tilde{\xi})_{\mathcal{E}}^{1,0} + \sum_{\beta} \{ -(\lambda_{\beta}^1)^* d(\xi_{\beta}, \tilde{\xi})_{\mathcal{E}} + (\lambda_{\beta}^1)^* (\nabla \xi_{\beta}, \tilde{\xi})_{\mathcal{E}}^{1,0} \} \\
&= \sum_{\beta} \{ -d((\lambda_{\beta}^1)^*) (\xi_{\beta}, \tilde{\xi})_{\mathcal{E}} + (\lambda_{\beta}^1)^* (\nabla \xi_{\beta}, \tilde{\xi})_{\mathcal{E}}^{1,0} \} \\
&= \sum_{\beta} (\nabla \xi_{\beta} \bullet \lambda_{\beta}^1 + \xi_{\beta} \otimes_{\mathfrak{A}} d\lambda_{\beta}^1, \tilde{\xi})_{\mathcal{E}}^{2,0} = (\nabla^2 \xi, \tilde{\xi})_{\mathcal{E}}^{2,0},
\end{aligned}$$

where we have used (24), (9), (1), (2), and *ii*) of Definition 5. $\square$

Lemma 8 *There exists a canonical compatible connection ∇_0 on $\mathcal{E}$ given by*

$$\nabla_0(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}) := \sum_{\alpha} \epsilon(c_{\alpha} \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} da_{\alpha}, \quad (26)$$

for $\sum_{\alpha} c_{\alpha} \otimes a_{\alpha} \in \mathcal{E} \subset \mathfrak{A}^p$, where $I_{\mathfrak{A}}$ denotes the identity of $\mathfrak{A}$.

Proof: By definition, ∇_0 is a $\mathbb{C}$ -linear map from $\mathcal{E}$ to $\mathcal{E}^1$. Using (3) and (7) one gets

$$\begin{aligned}
\nabla_0((\sum_{\alpha} c_{\alpha} \otimes a_{\alpha})\tilde{a}) &= \sum_{\alpha} \epsilon(c_{\alpha} \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} d(a_{\alpha}\tilde{a}) \\
&= (\sum_{\alpha} \epsilon(c_{\alpha} \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} da_{\alpha})\tilde{a} + \sum_{\alpha} \epsilon(c_{\alpha} \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} a_{\alpha}d\tilde{a} \\
&= (\nabla_0(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}))\tilde{a} + (\sum_{\alpha} \epsilon(c_{\alpha} \otimes I_{\mathfrak{A}})a_{\alpha}) \otimes_{\mathfrak{A}} d\tilde{a} \\
&= (\nabla_0(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}))\tilde{a} + (\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}) \otimes_{\mathfrak{A}} d\tilde{a},
\end{aligned}$$

using $\epsilon(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}) \equiv \sum_{\alpha} c_{\alpha} \otimes a_{\alpha}$, for $\sum_{\alpha} c_{\alpha} \otimes a_{\alpha} \in \mathcal{E}$. This proves that ∇_0 is a connection. It is compatible with the Hermitian structure, because with (9) and (4) we have for $\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}$, $\sum_{\beta} \tilde{c}_{\beta} \otimes \tilde{a}_{\beta} \in \mathcal{E} \subset \mathfrak{A}^p$

$$\begin{aligned}
&(\nabla_0(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}), \sum_{\beta} \tilde{c}_{\beta} \otimes \tilde{a}_{\beta})_{\mathcal{E}}^{1,0} + (\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}, \nabla_0(\sum_{\beta} \tilde{c}_{\beta} \otimes \tilde{a}_{\beta}))_{\mathcal{E}}^{0,1} \\
&= \sum_{\alpha, \beta} \{ (da_{\alpha})^* (\epsilon(c_{\alpha} \otimes I_{\mathfrak{A}}), \tilde{c}_{\beta} \otimes \tilde{a}_{\beta})_{\mathcal{E}} + (c_{\alpha} \otimes a_{\alpha}, \epsilon(\tilde{c}_{\beta} \otimes I_{\mathfrak{A}}))_{\mathcal{E}} d\tilde{a}_{\beta} \} \\
&= \sum_{\alpha} (da_{\alpha})^* (c_{\alpha} \otimes I_{\mathfrak{A}}, e^*(\sum_{\beta} \tilde{c}_{\beta} \otimes \tilde{a}_{\beta}))_{\mathfrak{A}^p} + \sum_{\beta} (e^*(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}), \tilde{c}_{\beta} \otimes I_{\mathfrak{A}})_{\mathfrak{A}^p} d\tilde{a}_{\beta} \\
&= \sum_{\alpha, \beta} (c_{\alpha}, \tilde{c}_{\beta})_{\mathbb{C}^p} ((da_{\alpha})^* I_{\mathfrak{A}}^* \tilde{a}_{\beta} + a_{\alpha}^* I_{\mathfrak{A}} d\tilde{a}_{\beta}) = \sum_{\alpha, \beta} d\{ (c_{\alpha}, \tilde{c}_{\beta})_{\mathbb{C}^p} a_{\alpha}^* \tilde{a}_{\beta} \} \\
&= d(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}, \sum_{\beta} \tilde{c}_{\beta} \otimes \tilde{a}_{\beta})_{\mathfrak{A}^p} \equiv d(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}, \sum_{\beta} \tilde{c}_{\beta} \otimes \tilde{a}_{\beta})_{\mathcal{E}},
\end{aligned}$$

where we have used (1), (2), and $\epsilon = e^*$. $\square$

Lemma 9 Any compatible connection ∇ on $\mathcal{E}$ has the form

$$\nabla = \nabla_0 + \rho, \quad \text{with} \quad \rho = -\rho^* \in \mathcal{H}^1. \quad (27)$$

Proof: (see [27]) Let ∇_1 and ∇_2 be two connections. Then from *i*) of Definition 5 we obtain $(\nabla_1 - \nabla_2)(\xi a) = ((\nabla_1 - \nabla_2)\xi)a$, for all $\xi \in \mathcal{E}$ and $a \in \mathfrak{A}$. This means $\nabla_1 - \nabla_2 \in \mathcal{H}^1$, i.e. any two connections differ by an element of $\mathcal{H}^1$. Therefore, any connection ∇ can be written as $\nabla = \nabla_0 + \rho$, with $\rho \in \mathcal{H}^1$. Inserting this into *ii*) of Definition 5, using (17) and the fact that ∇_0 is a compatible connection, we obtain

$$d(\xi, \tilde{\xi})_{\mathcal{E}} = ((\nabla_0 + \rho)\xi, \tilde{\xi})_{\mathcal{E}}^{1,0} + (\xi, (\nabla_0 + \rho)\tilde{\xi})_{\mathcal{E}}^{0,1} = d(\xi, \tilde{\xi})_{\mathcal{E}} + (\xi, (\rho + \rho^*)\tilde{\xi})_{\mathcal{E}}^{0,1},$$

for all $\xi, \tilde{\xi} \in \mathcal{E}$. Now the Lemma follows, because $(\cdot, \cdot)_{\mathcal{E}}^{0,1}$ is non-degenerate. $\square$

The existence of the canonical connection ∇_0 on $\mathcal{E}$ ensures that we have a canonical graded derivation $D_{\mathcal{H}} : \mathcal{H}^k \rightarrow \mathcal{H}^{k+1}$, which is defined by

$$(D_{\mathcal{H}}\varrho^k)(\xi) := \nabla_0(\varrho^k \xi) - (-1)^k \varrho^k \bullet (\nabla_0 \xi), \quad (28)$$

where $\xi \in \mathcal{E}$, $\varrho^k \in \mathcal{H}^k$. Using (22) one shows

$$(D_{\mathcal{H}}\varrho^k) \bullet \xi^l = \nabla_0(\varrho^k \bullet \xi^l) - (-1)^k \varrho^k \bullet (\nabla_0 \xi^l). \quad (29)$$

for $\xi^l \in \mathcal{E}^l$. Moreover, one easily shows that $(D_{\mathcal{H}}\varrho^k)(\xi a) = ((D_{\mathcal{H}}\varrho^k)(\xi))a$, for $a \in \mathfrak{A}$. Then by a simple calculation we obtain the graded Leibniz rule

$$D_{\mathcal{H}}(\varrho^k \bullet \tilde{\varrho}^l) = (D_{\mathcal{H}}\varrho^k) \bullet \tilde{\varrho}^l + (-1)^k \varrho^k \bullet D_{\mathcal{H}}\tilde{\varrho}^l. \quad (30)$$

In particular, we have

$$D_{\mathcal{H}}[\varrho^k, \tilde{\varrho}^l]_g = [D_{\mathcal{H}}\varrho^k, \tilde{\varrho}^l]_g + (-1)^k [\varrho^k, D_{\mathcal{H}}\tilde{\varrho}^l]_g, \quad \varrho^k \in \mathcal{H}^k, \quad \tilde{\varrho}^l \in \mathcal{H}^l. \quad (31)$$

This means that $D_{\mathcal{H}}$ is a graded derivation of the graded Lie algebra $\mathcal{H}$. Note, however, that $D_{\mathcal{H}}$ – in general – is *not* a differential of $\mathcal{H}$, because – denoting by $\theta_0 := \nabla_0^2$ the curvature of the canonical connection ∇_0 – we get from (28)

$$D_{\mathcal{H}}^2 \varrho \equiv \theta_0 \bullet \varrho - \varrho \bullet \theta_0, \quad \varrho \in \mathcal{H}. \quad (32)$$

We introduce the dual connection $\nabla'_0 : \mathcal{E}' \rightarrow \mathcal{E}'$, putting

$$d(\xi'(\xi)) = (\nabla'_0 \xi')(\xi) + \xi'(\nabla_0 \xi). \quad (33)$$

In the notations of (15) we have

$$D_{\mathcal{H}}\varrho^k = \sum_{\alpha} (\nabla_0 \xi_{\alpha} \bullet \lambda_{\alpha}^k \otimes_{\mathfrak{A}} \xi'_{\alpha} + \xi_{\alpha} \otimes_{\mathfrak{A}} d\lambda_{\alpha}^k \otimes_{\mathfrak{A}} \xi'_{\alpha} + (-1)^k \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \bullet \nabla'_0 \xi'_{\alpha}). \quad (34)$$

see also (11). Indeed, this equation is consistent with (28):

$$\begin{aligned}
(D_{\mathcal{H}}\varrho^k)(\xi) &= \sum_{\alpha} (\nabla_0 \xi_{\alpha} \bullet \lambda_{\alpha}^k \xi'_{\alpha}(\xi) + \xi_{\alpha} \otimes_{\mathfrak{A}} d(\lambda_{\alpha}^k) \xi'_{\alpha}(\xi) \\
&\quad + (-1)^k \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \bullet (\nabla'_0 \xi'_{\alpha})(\xi)) \\
&= \sum_{\alpha} (\nabla_0 \xi_{\alpha} \bullet \lambda_{\alpha}^k \xi'_{\alpha}(\xi) + \xi_{\alpha} \otimes_{\mathfrak{A}} d(\lambda_{\alpha}^k) \xi'_{\alpha}(\xi) + (-1)^k \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \bullet (d(\xi'_{\alpha}(\xi)) - \xi'_{\alpha}(\nabla_0 \xi))) \\
&= \sum_{\alpha} (\nabla_0 \xi_{\alpha} \bullet \lambda_{\alpha}^k \xi'_{\alpha}(\xi) + \xi_{\alpha} \otimes_{\mathfrak{A}} d(\lambda_{\alpha}^k \xi'_{\alpha}(\xi)) - (-1)^k \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \bullet \xi'_{\alpha}(\nabla_0 \xi)) \\
&= \nabla_0(\varrho^k \xi) - (-1)^k \varrho^k \bullet (\nabla_0 \xi) .
\end{aligned}$$

Lemma 10 *We have $(D_{\mathcal{H}}\varrho^k)^* = (-1)^k D_{\mathcal{H}}((\varrho^k)^*)$, for all $\varrho^k \in \mathcal{H}^k$.*

Proof: We denote $\varrho^k(\xi) \equiv \sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k$, $\nabla_0 \xi_{\alpha} \equiv \sum_{\beta} \xi_{\alpha\beta} \otimes_{\mathfrak{A}} \lambda_{\beta}^1$. Then we have

$$\begin{aligned}
(\nabla_0(\varrho^k \xi), \tilde{\xi})_{\mathcal{E}}^{k+1,0} &= \sum_{\alpha,\beta} (\xi_{\alpha\beta} \otimes_{\mathfrak{A}} (\lambda_{\beta}^1 \bullet \lambda_{\alpha}^k), \tilde{\xi})_{\mathcal{E}}^{k+1,0} + \sum_{\alpha} (\xi_{\alpha} \otimes_{\mathfrak{A}} d\lambda_{\alpha}^k, \tilde{\xi})_{\mathcal{E}}^{k+1,0} \\
&= \sum_{\alpha,\beta} (\lambda_{\alpha}^k)^* \bullet (\lambda_{\beta}^1)^* (\xi_{\alpha\beta}, \tilde{\xi})_{\mathcal{E}} + \sum_{\alpha} (d\lambda_{\alpha}^k)^* (\xi_{\alpha}, \tilde{\xi})_{\mathcal{E}} \\
&= \sum_{\alpha} \{ (\lambda_{\alpha}^k)^* \bullet (\nabla_0 \xi_{\alpha}, \tilde{\xi})_{\mathcal{E}}^{1,0} + (-1)^k d((\lambda_{\alpha}^k)^*) (\xi_{\alpha}, \tilde{\xi})_{\mathcal{E}} \} \\
&= \sum_{\alpha} \{ (\lambda_{\alpha}^k)^* \bullet (d(\xi_{\alpha}, \tilde{\xi})_{\mathcal{E}} - (\xi_{\alpha}, \nabla_0 \tilde{\xi})_{\mathcal{E}}^{0,1}) + (-1)^k d((\lambda_{\alpha}^k)^*) (\xi_{\alpha}, \tilde{\xi})_{\mathcal{E}} \} \\
&= \sum_{\alpha} \{ (-1)^k d((\lambda_{\alpha}^k)^* (\xi_{\alpha}, \tilde{\xi})_{\mathcal{E}}) - (\lambda_{\alpha}^k)^* \bullet (\xi_{\alpha}, \nabla_0 \tilde{\xi})_{\mathcal{E}}^{0,1} \} ,
\end{aligned}$$

where we have used (24), (9), (1), (2), and Lemma 8. Denoting $\nabla_0 \tilde{\xi} \equiv \sum_{\hat{\alpha}} \tilde{\xi}_{\hat{\alpha}} \otimes_{\mathfrak{A}} \lambda_{\hat{\alpha}}^1$, $(\varrho^k)^*(\tilde{\xi}_{\hat{\alpha}}) \equiv \sum_{\hat{\beta}} \tilde{\xi}_{\hat{\alpha}\hat{\beta}} \otimes_{\mathfrak{A}} \lambda_{\hat{\beta}}^k$, we obtain for the second term

$$\begin{aligned}
\sum_{\alpha} (\lambda_{\alpha}^k)^* \bullet (\xi_{\alpha}, \nabla_0 \tilde{\xi})_{\mathcal{E}}^{0,1} &= \sum_{\hat{\alpha}} (\xi, (\varrho^k)^* \tilde{\xi}_{\hat{\alpha}})_{\mathcal{E}}^{0,k} \bullet \lambda_{\hat{\alpha}}^1 = \sum_{\hat{\alpha}, \hat{\beta}} (\xi, \tilde{\xi}_{\hat{\alpha}\hat{\beta}} \otimes_{\mathfrak{A}} (\lambda_{\hat{\beta}}^k \bullet \lambda_{\hat{\alpha}}^1))_{\mathcal{E}}^{0,k+1} \\
&= (\xi, (\varrho^k)^* \bullet \nabla_0 \tilde{\xi})_{\mathcal{E}}^{0,k+1} .
\end{aligned}$$

We remark that these arguments cannot be simplified, because – in general – $(\ , \)_{\mathcal{E}}^{k,l}$ is non-degenerate only if $k = 0$ or $l = 0$. This gives

$$(\nabla_0(\varrho^k \xi), \tilde{\xi})_{\mathcal{E}}^{k+1,0} = (-1)^k d(\varrho^k \xi, \tilde{\xi})_{\mathcal{E}}^{k,0} - (\xi, (\varrho^k)^* \bullet \nabla_0 \tilde{\xi})_{\mathcal{E}}^{0,k+1} . \quad (35)$$

Replacing in (35) $\xi \leftrightarrow \tilde{\xi}$, $\varrho^k \leftrightarrow (\varrho^k)^*$, applying the involution, and using (2) and (ii) of Lemma 2, we get

$$(\xi, \nabla_0((\varrho^k)^* \tilde{\xi}))_{\mathcal{E}}^{0,k+1} = d(\varrho^k \xi, \tilde{\xi})_{\mathcal{E}}^{k,0} - (\varrho^k \bullet \nabla_0 \xi, \tilde{\xi})_{\mathcal{E}}^{k+1,0} . \quad (36)$$

Multiplying (36) by $(-1)^{k+1}$ and adding (35) we obtain, using (28),

$$((D_{\mathcal{H}}\varrho^k)(\xi), \tilde{\xi})_{\mathcal{E}}^{k+1,0} = (-1)^k (\xi, (D_{\mathcal{H}}(\varrho^k)^*)(\tilde{\xi}))_{\mathcal{E}}^{0,k+1} ,$$

which is equivalent to the assertion. $\square$

Demanding that the graded derivation $D_{\mathcal{H}}$ has to preserve the real graded Lie subalgebra $\mathfrak{h}$, we obtain from (20) and Lemma 10

$$(D_{\mathcal{H}}\varrho^k)^* = (-1)^k D_{\mathcal{H}}((\varrho^k)^*) = (-1)^{k+f(k)} D_{\mathcal{H}}\varrho^k \equiv (-1)^{f(k+1)} D_{\mathcal{H}}\varrho^k .$$

This gives $f(k+1) = f(k) + k \bmod 2$ and – using (21) – $f(1) = 1 \bmod 2$, which proves

Lemma 11 *The real graded Lie algebra $\mathfrak{h}$ defined by (20) is a graded Lie subalgebra of $\mathcal{H}$ with graded derivation $D_{\mathcal{H}}$ iff $f(k) = \frac{(k+2)(k+1)}{2} \bmod 2$. $\square$*

Using (28) and $\theta_0 = \nabla_0^2$ one finds for the curvature θ of the connection (27)

$$\theta = \theta_0 + D_{\mathcal{H}}\rho + \rho \bullet \rho \in \mathfrak{h}^2 . \quad (37)$$

Observe that $End(\mathcal{E}) \equiv \mathcal{H}^0$, therefore, $D_{\mathcal{H}}u$ is well-defined as an element of $\mathcal{H}^1$, where $u \in \mathcal{U}(\mathcal{E})$. Now we have

$$u\nabla(u^*\xi) = u(D_{\mathcal{H}}u^*)(\xi) + \nabla_0\xi + u\rho u^*\xi ,$$

or, equivalently,

$$u\nabla u^* = \nabla_0 + uD_{\mathcal{H}}u^* + u\rho u^* . \quad (38)$$

Then, the gauge transformation of the connection form ρ is given by

$$\gamma_u(\rho) = uD_{\mathcal{H}}u^* + u\rho u^* . \quad (39)$$

For the gauge transformation of the curvature one finds $\gamma_u(\theta) = u\theta u^*$.

4 Matrix Representation

Now we choose the canonical basis $\{\varepsilon_i\}_{i=1,\dots,p}$ in C^p , together with the canonical scalar product. This enables us to embed all structures discussed in the previous two sections into the tensor product $\Lambda_{\mathfrak{A}}^* \otimes M_p\mathbb{C}$. Observe that $\{\varepsilon_i \otimes I_{\mathfrak{A}}\}_{i=1,\dots,p}$ is the canonical basis of the free right $\mathfrak{A}$ -module $\mathfrak{A}^p \cong C^p \otimes \mathfrak{A}$ and

$$\epsilon(\varepsilon_i \otimes I_{\mathfrak{A}}) = \sum_{j=1}^p \varepsilon_j \otimes e_{ji} . \quad (40)$$

Thus, the projector ϵ is represented by the Hermitian $p \times p$ -matrix (e_{ji}) , $e_{ji} \in \mathfrak{A}$. Therefore, elements

$$\xi \equiv \epsilon\xi = \sum_{\alpha} c_{\alpha} \otimes a_{\alpha} = \sum_{i=1}^p \varepsilon_i \otimes \sum_{\alpha} c_{\alpha i} a_{\alpha} \in \mathcal{E} , \quad c_{\alpha} = \sum_{i=1}^p \varepsilon_i c_{\alpha i} , \quad (41)$$

are naturally identified with columns

$$\xi = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_p \end{pmatrix} , \quad b_i = \sum_{\alpha} c_{\alpha i} a_{\alpha} \in \mathfrak{A} . \quad (42)$$

Observe that $e\xi = \xi$ means $\sum_{j=1}^p e_{ij}b_j = b_i$. In this notation, the right action of $\mathfrak{A}$ on $\mathcal{E}$ is given by

$$\mathcal{E} \times \mathfrak{A} \ni (\xi, a) \mapsto \xi a := \begin{pmatrix} b_1 a \\ \vdots \\ b_p a \end{pmatrix} \in \mathcal{E}. \quad (43)$$

The Hermitian structure on $\mathcal{E}$ takes the form

$$(\xi, \tilde{\xi})_{\mathcal{E}} := \sum_{i=1}^p b_i^* \tilde{b}_i, \quad \xi, \tilde{\xi} \in \mathcal{E}. \quad (44)$$

Observe that for $\xi^k = e\xi^k \in \mathcal{E}^k$ we get

$$\xi^k = \sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k = \sum_{i=1}^p (\varepsilon_i \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} \sum_{\alpha, \beta} c_{\alpha\beta i} a_{\beta} \lambda_{\alpha}^k, \quad (45)$$

with $\xi_{\alpha} = \sum_{\beta} c_{\alpha\beta} \otimes a_{\beta}$, $c_{\alpha\beta} = \sum_{i=1}^p c_{\alpha\beta i} \varepsilon_i$. Therefore, elements $\xi^k \in \mathcal{E}^k$ are naturally identified with columns

$$\xi^k = \begin{pmatrix} \xi_1^k \\ \vdots \\ \xi_p^k \end{pmatrix}, \quad \xi_i^k = \sum_{\alpha, \beta} c_{\alpha\beta i} a_{\beta} \lambda_{\alpha}^k \in \Lambda_{\mathfrak{A}}^k. \quad (46)$$

Again, $e\xi^k = \xi^k$ means $\sum_{j=1}^p e_{ij}\xi_j^k = \xi_i^k$. The right $\Lambda_{\mathfrak{A}}^*$ -module structure of $\mathcal{E}^*$ is obtained as

$$\mathcal{E}^* \times \Lambda_{\mathfrak{A}}^l \ni (\xi^k, \lambda^l) \mapsto \xi^k \bullet \lambda^l = \begin{pmatrix} \xi_1^k \bullet \lambda^l \\ \vdots \\ \xi_p^k \bullet \lambda^l \end{pmatrix} \in \mathcal{E}^{k+l}. \quad (47)$$

The canonical compatible connection ∇_0 on $\mathcal{E}^*$, see (26) and (24), takes the form

$$\begin{aligned} \nabla_0 \xi^k &= \sum_{\alpha} \nabla_0 \xi_{\alpha} \bullet \lambda_{\alpha}^k + \xi_{\alpha} \otimes_{\mathfrak{A}} d\lambda_{\alpha}^k = \sum_{i,j=1}^p \sum_{\alpha} (\varepsilon_j \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} e_{ji} d(a_{i\alpha} \lambda_{\alpha}^k) \\ &\equiv \sum_{i,j=1}^p (\varepsilon_j \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} e_{ji} d(\xi_i^k), \end{aligned} \quad (48)$$

where we have denoted $\xi^k = \sum_{\alpha} \xi_{\alpha} \otimes_{\mathfrak{A}} \lambda_{\alpha}^k \in \mathcal{E}^k$, $\xi_{\alpha} = \sum_{i=1}^p \varepsilon_i \otimes a_{i\alpha} \in \mathcal{E}$ and $\xi_i^k = \sum_{\alpha} a_{i\alpha} \lambda_{\alpha}^k \in \Lambda_{\mathfrak{A}}^k$. Thus, $\nabla_0 \xi^k \in \mathcal{E}^{k+1}$ can be represented by

$$\nabla_0 \xi^k = e \begin{pmatrix} d\xi_1^k \\ \vdots \\ d\xi_p^k \end{pmatrix}. \quad (49)$$

It follows from (10) that $\{I_{\mathfrak{A}} \otimes \varepsilon'_i\}_{i=1,\dots,p}$ is the basis in $\mathfrak{A}^{p'}$ dual to the basis $\{\varepsilon_i \otimes I_{\mathfrak{A}}\}_{i=1,\dots,p}$ in $\mathfrak{A}^p$, where, of course, $\{\varepsilon'_i\}_{i=1,\dots,p}$ denotes the basis dual to $\{\varepsilon_i\}$. Thus, elements of $\mathcal{E}'$ are given as $\xi' = \sum_{i=1}^p b'_i \otimes \varepsilon'_i$, with $b'_i = \sum_{j=1}^p b_j e_{ji}$. Then, using (15) and (41) we get

$$\varrho^k = \sum_{i,j=1}^p (\varepsilon_i \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} \left(\sum_{\alpha} a_{i\alpha} \lambda_{\alpha}^k a'_{\alpha j} \right) \otimes_{\mathfrak{A}} (I_{\mathfrak{A}} \otimes \varepsilon'_j), \quad (50)$$

where $\xi_{\alpha} = \sum_{i=1}^p \varepsilon_i \otimes a_{i\alpha}$ and $\xi'_{\alpha} = \sum_{j=1}^p a'_{\alpha j} \otimes \varepsilon'_j$. Thus, an element $\varrho^k \in \mathcal{H}^k$ can be represented by a matrix

$$\varrho^k = \begin{pmatrix} \varrho_{11}^k & \cdots & \varrho_{1j}^k & \cdots & \varrho_{1p}^k \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \varrho_{i1}^k & \cdots & \varrho_{ij}^k & \cdots & \varrho_{ip}^k \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \varrho_{p1}^k & \cdots & \varrho_{pj}^k & \cdots & \varrho_{pp}^k \end{pmatrix}, \quad \varrho_{ij}^k = \sum_{\alpha} a_{i\alpha} \lambda_{\alpha}^k a'_{\alpha j} \in \Lambda_{\mathfrak{A}}^k. \quad (51)$$

We have $\varepsilon \varrho^k e = \varrho^k$ or in matrix representation $\sum_{i,j,m,n=1}^p \varepsilon_{im} \varrho_{mn}^k \varepsilon_{nj} = \varrho_{ij}^k$. For $\varrho^k \in \mathcal{H}^k$ given by (50) and $\tilde{\xi}^l = \sum_{\beta} \tilde{\xi}_{\beta} \otimes_{\mathfrak{A}} \tilde{\lambda}_{\beta}^l = \sum_{n=1}^p (\varepsilon_n \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} \tilde{\xi}_n^l \in \mathcal{E}^l$, $\tilde{\xi}_{\beta} = \sum_{n=1}^p \varepsilon_n \otimes \tilde{a}_{n\beta}$, $\tilde{\xi}_n^l = \sum_{\beta} \tilde{a}_{n\beta} \tilde{\lambda}_{\beta}^l$, we calculate - using (22) and (15)

$$\begin{aligned} \varrho^k \bullet \tilde{\xi}^l &= \left(\sum_{i,j=1}^p (\varepsilon_i \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} \varrho_{ij}^k \otimes_{\mathfrak{A}} (I_{\mathfrak{A}} \otimes \varepsilon'_j) \right) \bullet \left(\sum_{n=1}^p (\varepsilon_n \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} \tilde{\xi}_n^l \right) \\ &= \sum_{i,j=1}^p (\varepsilon_i \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} (\varrho_{ij}^k \bullet \tilde{\xi}_j^l). \end{aligned} \quad (52)$$

This means that ϱ^k acts on $\tilde{\xi}^l \in \mathcal{E}^l$ by left matrix multiplication:

$$(\varrho^k \bullet \tilde{\xi}^l)_i = \sum_{j=1}^p \varrho_{ij}^k \bullet \tilde{\xi}_j^l. \quad (53)$$

The product $\bullet$, see (16) and (50), is represented by matrix multiplication:

$$(\varrho^k \bullet \tilde{\varrho}^l)_{ij} = \sum_{n=1}^p \varrho_{in}^k \bullet \tilde{\varrho}_{nj}^l, \quad (54)$$

with $\varrho_{ij}^k \in \Lambda_{\mathfrak{A}}^k$, $\tilde{\varrho}_{ij}^l \in \Lambda_{\mathfrak{A}}^l$, $i, j = 1, \dots, p$, and the involution (17) is given by

$$((\varrho^k)^*)_{ij} = (\varrho_{ji}^k)^*. \quad (55)$$

We observe that $\mathcal{H}$ can be treated as an involutive subalgebra of $\Lambda_{\mathfrak{A}}^* \otimes M_p \mathbb{C}$.

Using (48) and the calculus just developed one gets the curvature of the connection ∇_0 :

$$(\theta_0)_{ij} = \sum_{k,l,m=1}^p \varepsilon_{ik} d(\varepsilon_{kl}) \bullet d(\varepsilon_{lm}) \varepsilon_{mj}, \quad (56)$$

where in particular one has to use $\sum_{m,n=1}^p e_{im} d(\epsilon_{mn}) e_{nj} = 0$. Using (28), (48), (50), (52) and (53) one calculates

$$\begin{aligned} (D_{\mathcal{H}} \varrho^k)(\xi) &= \sum_{i,j,n=1}^p (\epsilon_i \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} e_{ij} d(\varrho_{jn}^k b_n) - (-1)^k \sum_{i,j,n=1}^p (\epsilon_i \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} \varrho_{ij}^k e_{jn} d(b_n) \\ &= \sum_{i,j,n,m=1}^p (\epsilon_i \otimes I_{\mathfrak{A}}) \otimes_{\mathfrak{A}} \{e_{ij} d(\varrho_{jn}^k) e_{nm} b_m\} . \end{aligned} \quad (57)$$

Thus, $(D_{\mathcal{H}} \varrho^k)$ can be represented by the following matrix

$$D_{\mathcal{H}} \varrho^k = \epsilon d(\varrho^k) \epsilon \equiv \epsilon \begin{pmatrix} d(\varrho_{11}^k) & \dots & d(\varrho_{1p}^k) \\ \vdots & \ddots & \vdots \\ d(\varrho_{p1}^k) & \dots & d(\varrho_{pp}^k) \end{pmatrix} \epsilon . \quad (58)$$

For later purposes it is convenient to represent also $\mathcal{E}^*$ and $\Lambda_{\mathfrak{A}}^*$ as subspaces of $\Lambda_{\mathfrak{A}}^* \otimes M_p \mathbb{C}$. This goes as follows: First, $\mathcal{E}^*$ is embedded as a vector subspace, putting

$$j_{\mathcal{E}}(\xi^*) := \left(\underbrace{\xi^* \mid \xi^* \mid \dots \mid \xi^*}_p \right) , \quad (59)$$

which means building the $p \times p$ -block matrix $j_{\mathcal{E}}(\xi^*)$ from the $p \times 1$ -column $\xi^* \in \mathcal{E}^*$. To preserve the right $\Lambda_{\mathfrak{A}}^*$ -module structure of $\mathcal{E}^*$, we embed $\Lambda_{\mathfrak{A}}^*$ as a subalgebra, putting

$$j_{\Lambda}(\lambda) := \begin{pmatrix} \lambda & & O \\ & \ddots & \\ O & & \lambda \end{pmatrix}_{p \times p} , \quad \lambda \in \Lambda_{\mathfrak{A}}^* . \quad (60)$$

Under this embedding the right module structure is transported as follows:

$$j_{\mathcal{E}}(\xi^* \bullet \lambda) = j_{\mathcal{E}}(\xi^*) \bullet j_{\Lambda}(\lambda) , \quad \xi^* \in \mathcal{E}^* , \quad \lambda \in \Lambda_{\mathfrak{A}}^* . \quad (61)$$

Moreover, the transport of the left action of $\mathcal{H}$ on $\mathcal{E}^*$ is given by

$$j_{\mathcal{E}}(\varrho \bullet \xi^*) = \varrho \bullet j_{\mathcal{E}}(\xi^*) , \quad \xi^* \in \mathcal{E}^* , \quad \varrho \in \mathcal{H} . \quad (62)$$

5 The Differential Algebra $\Lambda_{\mathcal{A}}^*$

The construction presented above can be, in particular, applied to the special case of a K-cycle and its canonically associated differential algebra Ω_D^* , see [7], [8]. For the rest of the paper we restrict ourselves to this situation. We consider the simplest two-point K-cycle, whose differential algebra Ω_D^* was analysed in [22]. To keep this paper selfcontained, we review some results obtained there.

Let X be a compact even dimensional Riemannian spin manifold, $\dim X =: N$. We denote by $L^2(X, S)$ the Hilbert space of square integrable sections of the

spinor bundle over X , by C the Clifford bundle over X , and by C^k the set of those sections of C , whose values at each point $x \in X$ belong to the subspace spanned by products of less than or equal k elements of T_x^*X of the same parity. We consider the even K-cycle $(\mathcal{A}, h, D, \Gamma)$, see [7], [6], [8]. The Hilbert space h is

$$h := L^2(X, S) \otimes \tilde{F}, \quad (63)$$

where $\tilde{F}$ is a finite dimensional Hilbert space, which in physical applications carries fermionic degrees of freedom. We assume that there exists a selfadjoint grading operator Γ acting on h , $\Gamma^2 = id_h$,

$$\Gamma = \gamma^{N+1} \otimes \tilde{\Gamma}, \quad \tilde{\Gamma} \in End(\tilde{F}), \quad (64)$$

with $\gamma^{N+1} := i^{\frac{N}{2}} \gamma^1 \gamma^2 \dots \gamma^{N-1} \gamma^N$ and $\tilde{\Gamma}$ denoting the grading operators on $L^2(X, S)$ and $\tilde{F}$ respectively. The $\{\gamma^\mu\}_{\mu=1, \dots, N}$ are chosen as local orthonormal selfadjoint sections of C^1 . We have the decomposition

$$\tilde{F} \equiv \frac{1}{2}(id_{\tilde{F}} + \tilde{\Gamma})\tilde{F} \oplus \frac{1}{2}(id_{\tilde{F}} - \tilde{\Gamma})\tilde{F} \equiv F_+ \oplus F_- . \quad (65)$$

This gives the decomposition $h \equiv h_+ \oplus h_-$, with $h_\pm := L^2(X, S) \otimes F_\pm$. Thus, elements $\psi \in h$ naturally decompose as $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, where $\psi_+ \in h_+$ and $\psi_- \in h_-$. Then, Γ can be represented by $\Gamma = \begin{pmatrix} \gamma^{N+1} \otimes id_{F_+} & 0 \\ 0 & -\gamma^{N+1} \otimes id_{F_-} \end{pmatrix}$.

The algebra $\mathcal{A}$ of the K-cycle is

$$\mathcal{A} := C^\infty(X) \otimes (C \oplus C) \cong C^\infty(X) \oplus C^\infty(X). \quad (66)$$

We consider the following involutive representation π of $\mathcal{A}$ on h :

$$\pi((c, \tilde{c}))(\psi, \tilde{\psi}) := ((c \otimes id_{F_+})(\psi), (\tilde{c} \otimes id_{F_-})(\tilde{\psi})), \quad (67)$$

for $c, \tilde{c} \in C^\infty(X)$ and $\psi \in h_+$, $\tilde{\psi} \in h_-$. This implies that Γ commutes with $\pi(\mathcal{A})$. In the above representation we get

$$\pi(\mathcal{A}) = \left\{ a = \begin{pmatrix} c \otimes id_{F_+} & 0 \\ 0 & \tilde{c} \otimes id_{F_-} \end{pmatrix}, \quad c, \tilde{c} \in C^0 \cong C^\infty(X) \right\}. \quad (68)$$

The selfadjoint generalized Dirac operator D of the K-cycle is

$$D := D^\alpha \otimes id_{\tilde{F}} + \gamma^{N+1} \otimes \mathcal{M}, \quad (69)$$

where D^α is the classical Dirac operator on $L^2(X, S)$ and $\mathcal{M}$ is an endomorphism of $\tilde{F}$. One demands $D\Gamma + \Gamma D = 0$, which implies $\tilde{\Gamma}\mathcal{M} = -\mathcal{M}\tilde{\Gamma}$. From

the selfadjointness of D there follows $\mathcal{M}_+ = (\mathcal{M}_-)^*$. Thus, we have a natural decomposition $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$,

$$\mathcal{M}_+ := \mathcal{M} \frac{1}{2}(id_{\hat{F}} + \hat{\Gamma}) = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_- := \mathcal{M} \frac{1}{2}(id_{\hat{F}} - \hat{\Gamma}) = \begin{pmatrix} 0 & 0 \\ M^* & 0 \end{pmatrix}, \quad (70)$$

where $M \in Hom(F_-, F_+)$. We define

$$\begin{aligned} \mathcal{M}_+^0 &= \begin{pmatrix} id_{F_+} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_-^0 = \begin{pmatrix} 0 & 0 \\ 0 & id_{F_-} \end{pmatrix}, \\ \mathcal{M}_+^{2t} &= (\mathcal{M}_- \mathcal{M}_+)^t = \begin{pmatrix} M_1^t & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_-^{2t} = (\mathcal{M}_+ \mathcal{M}_-)^t = \begin{pmatrix} 0 & 0 \\ 0 & M_4^t \end{pmatrix}, \\ \mathcal{M}_+^{2t+1} &= \mathcal{M}_+ (\mathcal{M}_- \mathcal{M}_+)^t = \begin{pmatrix} 0 & 0 \\ M_3^t & 0 \end{pmatrix}, \\ \mathcal{M}_-^{2t+1} &= \mathcal{M}_- (\mathcal{M}_+ \mathcal{M}_-)^t = \begin{pmatrix} 0 & M_2^t \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (71)$$

where

$$M_1^r := (MM^*)^r, \quad M_2^r := M(M^*M)^r, \quad M_3^r := M^*(MM^*)^r, \quad M_4^r := (M^*M)^r. \quad (72)$$

There exists an involutive representation π of the universal differential algebra Ω^* over $\mathcal{A}$ on h , giving the algebra [8]

$$\begin{aligned} \pi(\Omega^*) &= \bigoplus_{k=0}^{\infty} \pi(\Omega^k), \quad \pi(\Omega^0) = \pi(\mathcal{A}), \\ \pi(\Omega^k) &= \{ (-i)^k \sum_{\alpha} \pi(a_{\alpha}^0) [D, \pi(a_{\alpha}^1)] \cdots [D, \pi(a_{\alpha}^k)] \mid a_{\alpha}^i \in \mathcal{A} \}, \quad k \geq 1. \end{aligned} \quad (73)$$

We restrict ourselves to the case $F_+ \cong F_- \equiv F$ and demand additionally that $\mathcal{M}^2 \notin C id_{F \in F}$. In this case one can show, see [22], that

$$\pi(\Omega^k) = \left[\begin{array}{cc} \bigoplus_{t=0}^m C^{k-2t} \otimes C M_1^t & ; \quad \bigoplus_{t=0}^m C^{k-2t-1} \gamma^{N+1} \otimes C M_2^t \\ \bigoplus_{t=0}^m C^{k-2t-1} \gamma^{N+1} \otimes C M_3^t & ; \quad \bigoplus_{t=0}^m C^{k-2t} \otimes C M_4^t \end{array} \right], \quad (74)$$

where $m+1$ is the number of linear independent elements $\mathcal{M}_+^{2t}$. We denote $L^n \equiv C^n / C^{n-2}$, for $n \geq 2$, and put $L^0 \equiv C^0$, $L^1 \equiv C^1$ and $L^n = \{0\}$ for $n < 0$. We have $L^n = \{0\}$ for $n > N$. There is a graded algebra $\Lambda_{\mathcal{A}}^*$ associated with $\pi(\Omega^*)$ defined as follows:

$$\begin{aligned} \Lambda_{\mathcal{A}}^* &= \bigoplus_{k=0}^{\infty} \Lambda_{\mathcal{A}}^k, \\ \Lambda_{\mathcal{A}}^k &\equiv \sigma_k \circ \pi(\Omega^k) := \begin{cases} \pi(\Omega^k) / \pi(\Omega^{k-2}) & \text{for } k \geq 2, \\ \pi(\Omega^k) & \text{for } k = 0, 1, \end{cases} \end{aligned} \quad (75)$$

with multiplication

$$\Lambda_{\mathcal{A}}^k \times \Lambda_{\mathcal{A}}^l \ni (\lambda^k, \tilde{\lambda}^l) \mapsto \lambda^k \bullet \tilde{\lambda}^l := \sigma_{k+l}(\tau^k \tilde{\tau}^l) \in \Lambda_{\mathcal{A}}^{k+l}, \quad (76)$$

where $\tau^k \in \pi(\Omega^k)$, $\tilde{\tau}^l \in \pi(\Omega^l)$, such that $\sigma_k(\tau^k) = \lambda^k$, $\sigma_l(\tilde{\tau}^l) = \tilde{\lambda}^l$. One can show [22] that

$$\Lambda_{\mathcal{A}}^k \cong \left[\begin{array}{cc} \bigoplus_{t=0}^m L^{k-2t} \otimes \mathbb{C} M_1^t & ; \quad \bigoplus_{t=0}^m L^{k-2t-1} \gamma^{N+1} \otimes \mathbb{C} M_2^t \\ \bigoplus_{t=0}^m L^{k-2t-1} \gamma^{N+1} \otimes \mathbb{C} M_3^t & ; \quad \bigoplus_{t=0}^m L^{k-2t} \otimes \mathbb{C} M_4^t \end{array} \right]. \quad (77)$$

Elements $\lambda^k \in \Lambda_{\mathcal{A}}^k$ are of the form

$$\lambda^k = \left(\begin{array}{cc} \sum_{t=0}^m \alpha_1^{k-2t} \otimes M_1^t & ; \quad \sum_{t=0}^m \alpha_2^{k-2t-1} \gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^m \alpha_3^{k-2t-1} \gamma^{N+1} \otimes M_3^t & ; \quad \sum_{t=0}^m \alpha_4^{k-2t} \otimes M_4^t \end{array} \right), \quad \alpha_q^n \in L^n. \quad (78)$$

Thus, we see that λ^k is completely characterized by the sequence of elements α_1^{k-2t} , α_2^{k-2t-1} , α_3^{k-2t-1} , α_4^{k-2t} , where $t = 0, \dots, m$. Denoting by ι_k the classical vector space isomorphism $\iota_k : L^k \equiv C^k / C^{k-2} \rightarrow \Lambda^k(X)$, where $\Lambda^k(X)$ is the set of complex-valued k -forms on X , and denoting the transport by the isomorphism ι of the exterior product $\wedge$ in $\Lambda^*(X) = \bigoplus_{k=0}^N \Lambda^k(X)$ by the same symbol, we get: If α_q^n , $\tilde{\alpha}_q^n \in L^n$ are the characterizing elements of $\lambda^k \in \Lambda_{\mathcal{A}}^k$, $\tilde{\lambda}^l \in \Lambda_{\mathcal{A}}^l$, then the characterizing elements β_q^n of $\lambda^k \bullet \tilde{\lambda}^l \in \Lambda_{\mathcal{A}}^{k+l}$ are

$$\begin{aligned} \beta_1^{k+l-2t} &= \sum_{r=0}^t (\alpha_1^{k-2r} \wedge \tilde{\alpha}_1^{l-2(t-r)} + (-1)^{l-1} \alpha_2^{k-2r-1} \wedge \tilde{\alpha}_3^{l-2(t-r)+1}), \\ \beta_2^{k+l-2t-1} &= \sum_{r=0}^t (\alpha_1^{k-2r} \wedge \tilde{\alpha}_2^{l-2(t-r)-1} + (-1)^l \alpha_2^{k-2(t-r)-1} \wedge \tilde{\alpha}_4^{l-2r}), \\ \beta_3^{k+l-2t-1} &= \sum_{r=0}^t (\alpha_4^{k-2r} \wedge \tilde{\alpha}_3^{l-2(t-r)-1} + (-1)^l \alpha_3^{k-2(t-r)-1} \wedge \tilde{\alpha}_1^{l-2r}), \\ \beta_4^{k+l-2t} &= \sum_{r=0}^t (\alpha_4^{k-2r} \wedge \tilde{\alpha}_4^{l-2(t-r)} + (-1)^{l-1} \alpha_3^{k-2r-1} \wedge \tilde{\alpha}_2^{l-2(t-r)+1}), \end{aligned} \quad (79)$$

where $t = 0, \dots, m$.

We have an involution on $\Lambda_{\mathcal{A}}^*$ given by $(\lambda^k)^* := \sigma_k((\tau^k)^*)$, with $\sigma_k(\tau^k) = \lambda^k$. Explicitly, for elements $\lambda^k \in \Lambda_{\mathcal{A}}^k$ represented as in (78) we find

$$(\lambda^k)^* = \left(\begin{array}{cc} \sum_{t=0}^m (\alpha_1^{k-2t})^* \otimes M_1^t ; & \sum_{t=0}^m (-1)^{k-1} (\alpha_2^{k-2t-1})^* \gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^m (-1)^{k-1} (\alpha_3^{k-2t-1})^* \gamma^{N+1} \otimes M_3^t ; & \sum_{t=0}^m (\alpha_4^{k-2t})^* \otimes M_4^t \end{array} \right). \quad (80)$$

We define the following operations

$$\begin{aligned}
[\gamma^{N+1} \otimes \mathcal{M}, \lambda^k]_g &:= (\gamma^{N+1} \otimes \mathcal{M}) \bullet \lambda^k - (-1)^k \lambda^k \bullet (\gamma^{N+1} \otimes \mathcal{M}), \\
\mathbf{d}\alpha^k &:= \iota_{k+1}^{-1} \circ d \circ \iota_k(\alpha^k), \\
\mathbf{d}^* &:= \gamma^{N+1} \mathbf{d} \gamma^{N+1}, \\
\mathbf{D}\lambda^k &:= pr_{k+1} \circ ((\mathbf{d} - \mathbf{d}^*) \otimes id_{\hat{F}})(\lambda^k),
\end{aligned} \tag{81}$$

for $\alpha^k \in L^k$, $\lambda^k \in \Lambda_{\mathcal{A}}^k$, and pr_{k+1} denoting the projection from $\Lambda_{\mathcal{A}}^{k+1} \oplus \Lambda_{\mathcal{A}}^{k-1}$ onto $\Lambda_{\mathcal{A}}^{k+1}$. One easily proves that $\mathbf{D}$ is a graded differential on $\Lambda_{\mathcal{A}}^*$. Moreover, one shows that

$$\hat{d} := \mathbf{D} - i[\gamma^{N+1} \otimes \mathcal{M}, \cdot]_g \tag{82}$$

is a graded differential on $\Lambda_{\mathcal{A}}^*$, too, which can be characterized as follows: If α_q^n are the characterizing elements of $\lambda^k \in \Lambda_{\mathcal{A}}^k$, then the characterizing elements β_q^n of $\hat{d}\lambda^k \in \Lambda_{\mathcal{A}}^{k+1}$ are:

$$\begin{aligned}
\beta_1^{k-2t+1} &= \mathbf{d}\alpha_1^{k-2t} + (-1)^k i(\alpha_2^{k-2t+1} + \alpha_3^{k-2t+1}), \\
\beta_2^{k-2t} &= \mathbf{d}\alpha_2^{k-2t-1} + (-1)^k i(\alpha_1^{k-2t} - \alpha_4^{k-2t}), \\
\beta_3^{k-2t} &= \mathbf{d}\alpha_3^{k-2t-1} + (-1)^k i(\alpha_4^{k-2t} - \alpha_1^{k-2t}), \\
\beta_4^{k-2t+1} &= \mathbf{d}\alpha_4^{k-2t} + (-1)^k i(\alpha_3^{k-2t+1} + \alpha_2^{k-2t+1}),
\end{aligned} \tag{83}$$

where $t = 0, \dots, m$. The consistency relation (2) is fulfilled for the differential algebra $(\Lambda_{\mathcal{A}}^*, \bullet, *, \hat{d})$.

In [22] we have shown that $\Lambda_{\mathcal{A}}^*$ coincides with the differential algebra Ω_D^* of Connes and Lott associated with the even K-cycle $(\mathcal{A}, h, D, \Gamma)$. The result (77) for Ω_D^* can also be obtained from a different procedure presented in [17].

For later purposes we define on $\Lambda_{\mathcal{A}}^*$ a linear map $T_{\Lambda} : \Lambda_{\mathcal{A}}^* \rightarrow L^*$ by

$$T_{\Lambda} \left(\begin{pmatrix} \sum_{t=0}^m \alpha_1^{k-2t} \otimes M_1^t & \sum_{t=0}^m \alpha_2^{k-2t-1} \gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^m \alpha_3^{k-2t-1} \gamma^{N+1} \otimes M_3^t & \sum_{t=0}^m \alpha_4^{k-2t} \otimes M_4^t \end{pmatrix} \right) := \sum_{t=0}^m (\alpha_1^{k-2t} + \alpha_4^{k-2t}), \tag{84}$$

see (78). We define a special element $\Gamma_{\Lambda} \in \mathcal{A} \equiv \Lambda_{\mathcal{A}}^0$, putting

$$\Gamma_{\Lambda} = \begin{pmatrix} 1 \otimes id_F & 0 \\ 0 & -1 \otimes id_F \end{pmatrix}, \tag{85}$$

where 1 denotes the unit element of $C^\infty(X)$. We recall (Proposition 6 in [22]) that, putting

$$e^\mu := \hat{d} \begin{pmatrix} x^\mu \otimes id_F & 0 \\ 0 & x^\mu \otimes id_F \end{pmatrix}, \quad e^0 := \hat{d} \begin{pmatrix} \frac{1}{2} \otimes id_F & 0 \\ 0 & -\frac{1}{2} \otimes id_F \end{pmatrix},$$

and $\epsilon : \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \mapsto \begin{pmatrix} f_2 & 0 \\ 0 & f_1 \end{pmatrix}$, we have

$$a \bullet e^\mu = e^\mu \bullet a, \quad a \bullet e^0 = e^0 \bullet \epsilon(a), \quad a \in \mathcal{A}, \quad (86)$$

$$e^\mu \bullet e^\nu = -e^\nu \bullet e^\mu, \quad e^\mu \bullet e^0 = -e^0 \bullet e^\mu, \quad (87)$$

for a local coordinate system $(U, \{x^\mu\}_{\mu=1, \dots, N})$ of X . Moreover, every $\lambda^k \in \Lambda_{\mathcal{A}}^k$ has a representation

$$\lambda^k = \sum_{l=0, \dots, \min(k, 2m+1)} \sum_{1 \leq \mu_1 < \dots < \mu_{k-l} \leq N} a_{l\mu_1 \dots \mu_{k-l}} \bullet e^{\mu_1} \bullet \dots \bullet e^{\mu_{k-l}} \bullet (\epsilon^0)^l, \quad (88)$$

with unique $a_{l\mu_1 \dots \mu_{k-l}} \in \mathcal{A}$.

Lemma 12 *We have $T_\Lambda(\Gamma_\Lambda \bullet (\lambda^k \bullet \tilde{\lambda}^l - (-1)^{kl} \tilde{\lambda}^l \bullet \lambda^k)) = 0$, for any $\lambda^k \in \Lambda_{\mathcal{A}}^k$, $\tilde{\lambda}^l \in \Lambda_{\mathcal{A}}^l$.*

Proof: It is sufficient to prove the Lemma locally. Thus, we can use the representation (88). Because of the linearity of T_Λ it is enough to prove the assertion for $\lambda^k, \tilde{\lambda}^l$ given by $\lambda^k = a \bullet e^{\mu_1} \bullet \dots \bullet e^{\mu_{k-p}} \bullet (\epsilon^0)^p$ and $\tilde{\lambda}^l = \tilde{a} \bullet e^{\nu_1} \bullet \dots \bullet e^{\nu_{l-q}} \bullet (\epsilon^0)^q$, with $a, \tilde{a} \in \mathcal{A}$.

We can assume that all $e^{\mu_\alpha}, e^{\nu_\beta}$ are different, because otherwise the Lemma is trivially fulfilled, see (87). Then with (86) and (87) we obtain

$$\begin{aligned} \lambda^k \bullet \tilde{\lambda}^l &= (-1)^{p(l-q)} a \epsilon^p(\tilde{a}) \bullet e^{\mu_1} \bullet \dots \bullet e^{\mu_{k-p}} \bullet e^{\nu_1} \bullet \dots \bullet e^{\nu_{l-q}} \bullet (\epsilon^0)^{p+q}, \\ \tilde{\lambda}^l \bullet \lambda^k &= (-1)^{q(k-p)} \tilde{a} \epsilon^q(a) \bullet e^{\nu_1} \bullet \dots \bullet e^{\nu_{l-q}} \bullet e^{\mu_1} \bullet \dots \bullet e^{\mu_{k-p}} \bullet (\epsilon^0)^{p+q} \\ &= (-1)^{l(k-p)} \epsilon^q(a) \tilde{a} \bullet e^{\mu_1} \bullet \dots \bullet e^{\mu_{k-p}} \bullet e^{\nu_1} \bullet \dots \bullet e^{\nu_{l-q}} \bullet (\epsilon^0)^{p+q}, \end{aligned}$$

where $\epsilon^q(a) = \epsilon(a)$ for q being odd and $\epsilon^q(a) = a$ for q being even. Thus we have

$$T_\Lambda(\Gamma_\Lambda \bullet (\lambda^k \bullet \tilde{\lambda}^l - (-1)^{kl} \tilde{\lambda}^l \bullet \lambda^k)) = (-1)^{p(l-q)} \times$$

$$T_\Lambda\{(\Gamma_\Lambda a \epsilon^p(\tilde{a}) - (-1)^{pq} \Gamma_\Lambda \epsilon^q(a) \tilde{a}) \bullet e^{\mu_1} \bullet \dots \bullet e^{\mu_{k-p}} \bullet e^{\nu_1} \bullet \dots \bullet e^{\nu_{l-q}} \bullet (\epsilon^0)^{p+q}\}.$$

The only nonzero contribution is obtained for $p+q$ even, because otherwise the term in the braces is off-diagonal and vanishes under T_Λ , see (84). First let p and q be even. Then $\Gamma_\Lambda a \epsilon^p(\tilde{a}) - (-1)^{pq} \Gamma_\Lambda \epsilon^q(a) \tilde{a} \equiv 0$. Next, let p and q be odd and $a = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix} \otimes id_F$, $\tilde{a} = \begin{pmatrix} \tilde{a}_1 & 0 \\ 0 & \tilde{a}_4 \end{pmatrix} \otimes id_F$, where $a_q, \tilde{a}_q \in C^0$. Then we have

$$\Gamma_\Lambda a \epsilon^p(\tilde{a}) - (-1)^{pq} \Gamma_\Lambda \epsilon^q(a) \tilde{a} = \begin{pmatrix} a_1 \tilde{a}_4 + a_4 \tilde{a}_1 & 0 \\ 0 & -a_4 \tilde{a}_1 - a_1 \tilde{a}_4 \end{pmatrix} \otimes id_F \quad \text{and}$$

$$e^{\mu_1} \bullet \dots \bullet e^{\mu_{k-p}} \bullet e^{\nu_1} \bullet \dots \bullet e^{\nu_{l-q}} \bullet (\epsilon^0)^{p+q} = \begin{pmatrix} \alpha \otimes M_1^{\frac{p+q}{2}} & 0 \\ 0 & \alpha \otimes M_4^{\frac{p+q}{2}} \end{pmatrix},$$

where $\alpha = dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{k-p}} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{l-q}}$. Now the assertion follows from the definition of T_Λ , see (84). $\square$

6 A certain Lie Subalgebra of $\mathcal{H}$

We denote by ∇_0 and $\tilde{\nabla}_0$ the canonical compatible connections on $\mathcal{E}$, which are defined according to (26) using the differential $\hat{d}$ respectively $\mathbf{D}$ on $\Lambda_{\mathcal{A}}^*$:

$$\begin{aligned}\nabla_0(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}) &:= \sum_{\alpha} e(c_{\alpha} \otimes \mathbf{I}_{\mathcal{A}}) \otimes_{\mathcal{A}} \hat{d}a_{\alpha} , \\ \tilde{\nabla}_0(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}) &:= \sum_{\alpha} e(c_{\alpha} \otimes \mathbf{I}_{\mathcal{A}}) \otimes_{\mathcal{A}} \mathbf{D}a_{\alpha} .\end{aligned}\quad (89)$$

Moreover, we denote by $D_{\mathcal{H}}$ and $\tilde{D}_{\mathcal{H}}$ the derivations on $\mathcal{H}$ associated to ∇_0 and $\tilde{\nabla}_0$ respectively, see (28):

$$\begin{aligned}(D_{\mathcal{H}}\varrho^k)(\xi) &= \nabla_0(\varrho^k\xi) - (-1)^k \varrho^k \bullet (\nabla_0\xi) , \\ (\tilde{D}_{\mathcal{H}}\varrho^k)(\xi) &= \tilde{\nabla}_0(\varrho^k\xi) - (-1)^k \varrho^k \bullet (\tilde{\nabla}_0\xi) .\end{aligned}\quad (90)$$

We introduce a special element $\mu \in \mathcal{H}^1$ by

$$\mu(\sum_{\alpha} c_{\alpha} \otimes a_{\alpha}) := \sum_{\alpha} e(c_{\alpha} \otimes \mathbf{I}_{\mathcal{A}}) \otimes_{\mathcal{A}} (-i\gamma^{N+1} \otimes \mathcal{M})a_{\alpha} , \quad (91)$$

where $c_{\alpha} \in \mathbb{C}^p$, $a_{\alpha} \in \mathcal{A}$. From (52) we get the matrix form

$$\mu = e(\mathbf{I}_{p \times p} \otimes (-i\gamma^{N+1} \otimes \mathcal{M}))e , \quad (92)$$

or more explicitly $\mu_{ij} = \sum_{k,l=1}^p e_{ik}(-i\gamma^{N+1} \otimes \mathcal{M})\delta_{kl}e_{lj}$.

Lemma 13 *For the graded Lie algebra $\mathcal{H}$ associated to the differential algebra $\Lambda_{\mathcal{A}}^*$ we have*

$$D_{\mathcal{H}}\varrho^k = \tilde{D}_{\mathcal{H}}\varrho^k + [\mu, \varrho^k]_g , \quad \varrho^k \in \mathcal{H}^k . \quad (93)$$

Proof: Let $\varrho^k = \sum_{\alpha} \xi_{\alpha} \otimes_{\mathcal{A}} \lambda_{\alpha}^k \otimes_{\mathcal{A}} \xi'_{\alpha} \in \mathcal{H}^k$, with $\xi_{\alpha} = \sum_{\beta} c_{\beta} \otimes a_{\beta\alpha} \in \mathcal{E}$. Moreover, let $\xi = \sum_{\gamma} c_{\gamma} \otimes \tilde{a}_{\gamma} \in \mathcal{E}$. Using (82), (24), (91), (16) and (26) we find

$$\begin{aligned}(D_{\mathcal{H}}\varrho^k)(\xi) &= \nabla_0(\varrho^k(\xi)) - (-1)^k \varrho^k \bullet (\nabla_0\xi) \\ &= \sum_{\alpha} \{ \nabla_0\xi_{\alpha} \bullet \lambda_{\alpha}^k \xi'_{\alpha}(\xi) + \xi_{\alpha} \otimes_{\mathcal{A}} \hat{d}(\lambda_{\alpha}^k \xi'_{\alpha}(\xi)) \\ &\quad - (-1)^k \sum_{\gamma} \xi_{\alpha} \otimes_{\mathcal{A}} \lambda_{\alpha}^k \xi'_{\alpha}(e(c_{\gamma} \otimes \mathbf{I}_{\mathcal{A}})) \bullet \hat{d}\tilde{a}_{\gamma} \} \\ &= \sum_{\alpha, \beta, \gamma} e(c_{\beta} \otimes \mathbf{I}_{\mathcal{A}}) \otimes_{\mathcal{A}} \hat{d}\{a_{\beta\alpha} \lambda_{\alpha}^k \xi'_{\alpha}(e(c_{\gamma} \otimes \mathbf{I}_{\mathcal{A}}))a_{\gamma}\} \\ &\quad - (-1)^k \sum_{\alpha, \beta, \gamma} e(c_{\beta} \otimes \mathbf{I}_{\mathcal{A}}) \otimes_{\mathcal{A}} \{a_{\beta\alpha} \lambda_{\alpha}^k \xi'_{\alpha}(e(c_{\gamma} \otimes \mathbf{I}_{\mathcal{A}})) \bullet \hat{d}(\tilde{a}_{\gamma})\} \\ &= \sum_{\alpha, \beta, \gamma} e(c_{\beta} \otimes \mathbf{I}_{\mathcal{A}}) \otimes_{\mathcal{A}} \hat{d}\{a_{\beta\alpha} \lambda_{\alpha}^k \xi'_{\alpha}(e(c_{\gamma} \otimes \mathbf{I}_{\mathcal{A}}))\} \tilde{a}_{\gamma} \\ &= (\tilde{D}_{\mathcal{H}}\varrho^k)(\xi) + \sum_{\alpha, \beta, \gamma} e(c_{\beta} \otimes \mathbf{I}_{\mathcal{A}}) \otimes_{\mathcal{A}} [-i\gamma^{N+1} \otimes \mathcal{M}, \{a_{\beta\alpha} \lambda_{\alpha}^k \xi'_{\alpha}(e(c_{\gamma} \otimes \mathbf{I}_{\mathcal{A}}))\}]_g \tilde{a}_{\gamma} \\ &= (\tilde{D}_{\mathcal{H}}\varrho^k)(\xi) + \mu \bullet (\varrho^k(\xi)) - (-1)^k \varrho^k \bullet (\mu(\xi)) .\end{aligned}$$

Now formula (93) follows from (18). □

Next we transport the structures discussed at the end of section 5 to the graded Lie algebra $\mathcal{H}$. For elements $\varrho^k \in \mathcal{H}^k$ represented by the matrix (51) we define a linear map $T_{\mathcal{H}} : \mathcal{H} \rightarrow L^*$ by

$$T_{\mathcal{H}}(\varrho) := \sum_{i=1}^p T_{\Lambda}(\Gamma_{\Lambda} \varrho_{ii}) . \quad (94)$$

Lemma 14 *For all $\varrho^k \in \mathcal{H}^k$ and $\tilde{\varrho}^l \in \mathcal{H}^l$ we have $T_{\mathcal{H}}([\varrho^k, \tilde{\varrho}^l]_g) = 0$.*

Proof: Using formulae (54), (18) and (94), we obtain

$$T_{\mathcal{H}}([\varrho^k, \tilde{\varrho}^l]_g) = \sum_{i,j=1}^p T_{\Lambda} \{ \Gamma_{\Lambda} (\varrho_{ij}^k \bullet \tilde{\varrho}_{ji}^l - (-1)^{kl} \tilde{\varrho}_{ji}^l \bullet \varrho_{ij}^k) \} .$$

Applying Lemma 12, we get $T_{\mathcal{H}}([\varrho^k, \tilde{\varrho}^l]_g) = 0$. $\square$

Putting $l = 0$, $\varrho^k = u\varrho$ and $\tilde{\varrho}^0 = u^*$ in Lemma 14, for $u \in \mathcal{U}(\mathcal{E})$, we get

$$T_{\mathcal{H}}(u\varrho u^*) = T_{\mathcal{H}}(\varrho) . \quad (95)$$

This means that $T_{\mathcal{H}}(\cdot)$ is invariant under unitary automorphisms of the module. We define

$$\mathcal{H}_0 := \bigoplus_{k=0}^{\infty} \mathcal{H}_0^k, \quad \mathcal{H}_0^k := \{ \varrho^k \in \mathcal{H}^k : T_{\mathcal{H}}(\varrho^k) = 0 \} . \quad (96)$$

Lemma 15 *$\mathcal{H}_0$ is a graded Lie subalgebra of $\mathcal{H}$, and $D_{\mathcal{H}}$ is a graded derivation of $\mathcal{H}_0$.*

Proof: Because of Lemma 14, $\mathcal{H}_0$ is a graded Lie subalgebra of $\mathcal{H}$. For any $\varrho^k \in \mathcal{H}_0^k$ we have with (93), Lemma 14, (94) and (58)

$$T_{\mathcal{H}}(D_{\mathcal{H}}\varrho^k) = T_{\mathcal{H}}(\tilde{D}_{\mathcal{H}}\varrho^k + [\mu, \varrho^k]_g) = T_{\mathcal{H}}(\tilde{D}_{\mathcal{H}}\varrho^k) = \sum_{i=1}^p T_{\Lambda}(\Gamma_{\Lambda} e_i \mathbf{D}(\varrho_{ii}^k) e_i) .$$

Now observe that $\mathbf{D}$ commutes with Γ_{Λ} and e_i and that due to (81) we have

$$T_{\Lambda}(\mathbf{D}\lambda^k) = \mathbf{d}T_{\Lambda}(\lambda^k), \quad \lambda^k \in \Lambda_{\mathcal{A}}^k . \quad (97)$$

This gives $T_{\mathcal{H}}(D_{\mathcal{H}}\varrho^k) = \mathbf{d}\{\sum_{i=1}^p T_{\Lambda}(\Gamma_{\Lambda} \varrho_{ii}^k)\} = 0$, for $\varrho^k \in \mathcal{H}_0$. $\square$

Finally, we put, see (20),

$$\mathfrak{h}_0 := \bigoplus_{k=0}^{\infty} \mathfrak{h}_0^k, \quad \mathfrak{h}_0^k := \mathcal{H}_0 \cap \mathfrak{h}^k, \quad (98)$$

which is a real $\mathbb{N}$ -graded Lie algebra as an intersection of two $\mathbb{N}$ -graded Lie algebras.

7 Standard Matrix Representation

In this section we analyse the matrix structures discussed in section 4 for the case of the differential algebra presented in sections 5 and 6. For this purpose we use the fact that $\Lambda_{\mathcal{A}}^*$ can be treated as a subspace of $L^* \otimes \text{End}(F) \otimes M_2\mathbb{C}$. Of course, elements of the tensor product $\Lambda_{\mathcal{A}}^* \otimes M_p\mathbb{C}$, which in section 6 were treated as $p \times p$ -matrices with $L^* \otimes \text{End}(F) \otimes M_2\mathbb{C}$ -valued entries, can be treated as 2×2 -matrices with $L^* \otimes \text{End}(F) \otimes M_p\mathbb{C}$ -valued entries. This natural mapping can be realized as an inner automorphism of $L^* \otimes \text{End}(F) \otimes M_{2p}\mathbb{C}$. It turns out that after applying this automorphism combined with another natural mapping, we find that the matrix structure of $\mathcal{H}_0$ coincides with a graded Lie subalgebra of the special graded linear Lie algebra $\Lambda^*(X) \otimes \text{spl}(p, p)$. This is the appropriate formulation for deriving the mathematical structure of the Mainz–Marseille approach, as will be shown in the next section.

Let $W = (W_{ij})_{i,j=1,\dots,p} \in M_p\mathbb{C}$ and $w = (w_{AB})_{A,B=1,2} \in M_2\mathbb{C}$. We denote

$$w \otimes W = \begin{pmatrix} wW_{11} & wW_{12} & \dots & wW_{1p} \\ wW_{21} & wW_{22} & \dots & wW_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ wW_{p1} & wW_{p2} & \dots & wW_{pp} \end{pmatrix} \quad (99)$$

and define

$$i_1(w \otimes W) = \begin{pmatrix} Ww_{11} & Ww_{12} \\ Ww_{21} & Ww_{22} \end{pmatrix}. \quad (100)$$

We extend this mapping naturally to the algebra $L^* \otimes \text{End}(F) \otimes M_2\mathbb{C} \otimes M_p\mathbb{C}$ and denote it by the same letter, the restriction to the subspace $\Lambda_{\mathcal{A}}^* \otimes M_p\mathbb{C}$ will also be denoted by i_1 . It is easy to convince oneself that the mapping (100) can be also realized as an inner automorphism of the algebra $M_{2p}\mathbb{C}$. This goes as follows:

$$i_1(\mathbf{W}) := \mathbf{J}\mathbf{W}\mathbf{J}^{-1}, \quad \mathbf{W}, \mathbf{J} \in M_{2p}\mathbb{C}, \quad \mathbf{J}_{ij} = \delta_{j,2i-1} + \delta_{j+2p,2i}, \quad (101)$$

for $i, j = 1, \dots, 2p$. Moreover, it is easy to convince oneself that this operation consists in applying the permutation $(1, 2, 3, 4, \dots, 2p-1, 2p) \mapsto (1, 3, \dots, 2p-1, 2, 4, \dots, 2p)$ to both rows and columns.

Note that due to (78) after applying the operation (101) to elements of $\Lambda_{\mathcal{A}}^* \otimes M_p\mathbb{C}$ the grading operator γ^{N+1} occurs exactly in every component of the two off-diagonal blocks. The next step consists in removing γ^{N+1} from these blocks and applying the classical isomorphism $\iota_k : L^k \rightarrow \Lambda^k(X)$. For this purpose we define the following vector space isomorphism i_2 from $i_1(\Lambda_{\mathcal{A}}^* \otimes M_p\mathbb{C})$ onto its image:

$$\begin{aligned}
i_2 \left(\begin{array}{cc} \sum_{t=0}^m \alpha_1^{k-2t} \otimes M_1^t & ; \quad \sum_{t=0}^m \alpha_2^{k-2t-1} \gamma^{N+1} \otimes M_2^t \\ \sum_{t=0}^m \alpha_3^{k-2t-1} \gamma^{N+1} \otimes M_3^t & ; \quad \sum_{t=0}^m \alpha_4^{k-2t} \otimes M_4^t \end{array} \right) \\
:= \left(\begin{array}{cc} \sum_{t=0}^m A_1^{k-2t} \otimes M_1^t & ; \quad \sum_{t=0}^m A_2^{k-2t-1} \otimes M_2^t \\ \sum_{t=0}^m A_3^{k-2t-1} \otimes M_3^t & ; \quad \sum_{t=0}^m A_4^{k-2t} \otimes M_4^t \end{array} \right), \tag{102}
\end{aligned}$$

where $\alpha_q^n \in L^n \otimes M_p \mathbb{C}$ and $A_q^n := \iota_n(\alpha_q^n) \in \Lambda^n(X) \otimes M_p \mathbb{C}$. The composition of these two mappings gives the embedding $i : \Lambda_{\mathcal{A}}^* \otimes M_p \mathbb{C} \rightarrow \Lambda^*(X) \otimes \text{End}(F) \otimes M_{2p} \mathbb{C}$,

$$i = i_2 \circ i_1, \tag{103}$$

often we will treat i as an isomorphism onto its image.

Now it is easy to characterize elements

$$\left(\begin{array}{cc} \sum_{t=0}^m A_1^{k-2t} \otimes M_1^t & ; \quad \sum_{t=0}^m A_2^{k-2t-1} \otimes M_2^t \\ \sum_{t=0}^m A_3^{k-2t-1} \otimes M_3^t & ; \quad \sum_{t=0}^m A_4^{k-2t} \otimes M_4^t \end{array} \right), \quad A_q^n \in \Lambda^n(X) \otimes M_p \mathbb{C}, \tag{104}$$

of $\mathcal{H}$, $\mathcal{E}^*$ and $\Lambda_{\mathcal{A}}^*$, see (51), (59) and (60), transported by i . First, observe that $\epsilon \in \mathcal{H}^0$ and, therefore, we have

$$i(\epsilon) \equiv \mathbf{e} = \begin{pmatrix} \mathbf{e}_1 \otimes id_F & 0 \\ 0 & \mathbf{e}_4 \otimes id_F \end{pmatrix}, \tag{105}$$

with

$$\mathbf{e}_q = \mathbf{e}_q^2 = \mathbf{e}_q^* \in M_p \mathbb{C}, \quad q = 1, 4. \tag{106}$$

Since for elements $\varrho \in \mathcal{H}$ we have $\epsilon \varrho \epsilon = \varrho$, we get for elements $i(\varrho) \in i(\mathcal{H})$, given in the representation (104),

$$A_1^n = \mathbf{e}_1 A_1^n \mathbf{e}_1, \quad A_2^n = \mathbf{e}_1 A_2^n \mathbf{e}_4, \quad A_3^n = \mathbf{e}_4 A_3^n \mathbf{e}_1, \quad A_4^n = \mathbf{e}_4 A_4^n \mathbf{e}_4. \tag{107}$$

Defining $i_{\mathcal{E}} : \mathcal{E}^k \rightarrow i(\Lambda_{\mathcal{A}}^k \otimes M_p \mathbb{C})$ and $i_{\Lambda} : \Lambda_{\mathcal{A}}^k \rightarrow i(\Lambda_{\mathcal{A}}^k \otimes M_p \mathbb{C})$ by putting

$$i_{\mathcal{E}} := i \circ j_{\mathcal{E}}, \quad i_{\Lambda} := i \circ j_{\Lambda}, \tag{108}$$

we can represent elements of $\mathcal{E}^*$ and $\Lambda_{\mathcal{A}}^*$ as elements of $\Lambda^*(X) \otimes \text{End}(F) \otimes M_{2p} \mathbb{C}$. For elements of $i_{\mathcal{E}}(\mathcal{E}^k)$ we get from the representation (104):

$$A_q^n \in E_{(i)} \otimes \Lambda^n(X), \quad (i) = 1 \text{ for } q = 1, 2, \quad (i) = 2 \text{ for } q = 3, 4, \quad \text{where} \tag{109}$$

$$E_{(i)} = \mathbf{e}_{(i)} \left(C \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} + C \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} + \dots + C \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \right).$$

Analogously, for elements of $i_\Lambda(\Lambda_{\mathcal{A}}^k)$ we have

$$A_q^n \in I_{p \times p} \otimes \Lambda^n(X), \quad q = 1, \dots, 4. \quad (110)$$

We define the multiplication in $i(\Lambda_{\mathcal{A}}^* \otimes M_p \mathbb{C})$ as the transport of the multiplication $\bullet$ in $\Lambda_{\mathcal{A}}^* \otimes M_p \mathbb{C}$ and denote it by the same symbol $\bullet$:

$$i(\chi^k) \bullet i(\tilde{\chi}^l) := i(\chi^k \bullet \tilde{\chi}^l), \quad (111)$$

for $\chi^k \in \Lambda_{\mathcal{A}}^k \otimes M_p \mathbb{C}$, $\tilde{\chi}^l \in \Lambda_{\mathcal{A}}^l \otimes M_p \mathbb{C}$. Denoting $i(\chi^k) = n^k$ and $i(\tilde{\chi}^l) = \tilde{n}^l$, which we represent as in (104), and using (79) we get:

$$n^k \bullet \tilde{n}^l = \left(\begin{array}{c|c} \sum_{t=0}^m \sum_{s=0}^t (A_1^{k-2s} \wedge \tilde{A}_1^{l-2(t-s)+1} + (-1)^{l-1} A_2^{k-2s-1} \wedge \tilde{A}_3^{l-2(t-s)+1}) \otimes M_1^t & \sum_{t=0}^m \sum_{s=0}^t (A_1^{k-2s} \wedge \tilde{A}_2^{l-2(t-s)-1} + (-1)^l A_2^{k-2(t-s)-1} \wedge \tilde{A}_4^{l-2s}) \otimes M_2^t \\ \hline \sum_{t=0}^m \sum_{s=0}^t (A_4^{k-2s} \wedge \tilde{A}_3^{l-2(t-s)-1} + (-1)^l A_3^{k-2(t-s)-1} \wedge \tilde{A}_1^{l-2s}) \otimes M_3^t & \sum_{t=0}^m \sum_{s=0}^t (A_4^{k-2s} \wedge \tilde{A}_4^{l-2(t-s)+1} + (-1)^{l-1} A_3^{k-2s-1} \wedge \tilde{A}_2^{l-2(t-s)+1}) \otimes M_4^t \end{array} \right). \quad (112)$$

In particular, we have

$$i_{\mathcal{E}}(\varrho \bullet \xi^*) = i(\varrho) \bullet i_{\mathcal{E}}(\xi^*), \quad \varrho \in \mathcal{H}, \quad \xi^* \in \mathcal{E}^*, \quad (113)$$

Next we transport the remaining structures via i :

$$[i(\varrho^k), i(\tilde{\varrho}^l)]_g := i([\varrho^k, \tilde{\varrho}^l]_g), \quad (114)$$

$$(i(\varrho^k))^* := i((\varrho^k)^*), \quad (115)$$

$$\nabla^{(i)}(i_{\mathcal{E}}(\xi^k)) := i_{\mathcal{E}}(\nabla \xi^k), \quad (116)$$

$$D_{i(\mathcal{H})}i(\varrho^k) := i(D_{\mathcal{H}}\varrho^k), \quad (117)$$

where $\varrho^k \in \mathcal{H}^k$, $\tilde{\varrho}^l \in \mathcal{H}^l$ and $\xi^k \in \mathcal{E}^k$. Using (93) we find for $D_{i(\mathcal{H})}(r^k)$, $r^k \in i(\mathcal{H}^k)$,

$$\begin{aligned} D_{i(\mathcal{H})}r^k &= dr^k + [\mathbf{m}, r^k]_g, \\ \mathbf{m} &:= i(\mu) \equiv \mathbf{ene}, \\ \mathbf{n} &:= i(I_{p \times p} \otimes (-i\gamma^{N+1} \otimes \mathcal{M})) = \begin{pmatrix} 0 & -i I_{p \times p} \otimes M_2^0 \\ -i I_{p \times p} \otimes M_3^0 & 0 \end{pmatrix}, \end{aligned} \quad (118)$$

where d is the classical exterior differential acting componentwise on r^k . For the involution (115) we get in the representation (104)

$$\begin{aligned}
& \left(\begin{array}{cc} \sum_{t=0}^m A_1^{k-2t} \otimes M_1^t & ; \quad \sum_{t=0}^m A_2^{k-2t-1} \otimes M_2^t \\ \sum_{t=0}^m A_3^{k-2t-1} \otimes M_3^t & ; \quad \sum_{t=0}^m A_4^{k-2t} \otimes M_4^t \end{array} \right)^* \\
& = \left(\begin{array}{cc} \sum_{t=0}^m (A_1^{k-2t})^* \otimes M_1^t & ; \quad \sum_{t=0}^m (-1)^{k-1} (A_3^{k-2t-1})^* \otimes M_2^t \\ \sum_{t=0}^m (-1)^{k-1} (A_2^{k-2t-1})^* \otimes M_3^t & ; \quad \sum_{t=0}^m (A_4^{k-2t})^* \otimes M_4^t \end{array} \right).
\end{aligned} \tag{119}$$

Next, we observe that we can also transport the generalized trace defined in (94):

$$T_{i(\mathcal{H})} i(\varrho^k) := \iota_k(T_{\mathcal{H}}(\varrho^k)), \quad \varrho^k \in \mathcal{H}^k. \tag{120}$$

For elements $r^k \in i(\mathcal{H}^k)$ represented as in (104) we get

$$T_{i(\mathcal{H})}(r^k) = \sum_{t=0}^m (tr(A_1^{k-2t}) - tr(A_4^{k-2t})). \tag{121}$$

Thus, elements $r^k \in i(\mathcal{H}_0^k)$ are characterized by

$$tr(A_1^n) = tr(A_4^n), \quad \text{for all } n. \tag{122}$$

With the general form $\nabla = \nabla_0 + \rho$ of a connection on $\mathcal{E}$ and (113) one finds

$$\begin{aligned}
\nabla^{(i)} &= \nabla_0^{(i)} + i(\rho), \quad \text{where} \\
\nabla_0^{(i)} x^k &= i_{\mathcal{E}} \circ \nabla_0 \circ i_{\mathcal{E}}^{-1}(x^k) = dx^k + \mathbf{e}(\mathbf{n} \bullet x^k - (-1)^k x^k \bullet \mathbf{n}) \\
&\equiv dx^k + \mathbf{e}[\mathbf{n}, x^k]_g.
\end{aligned} \tag{123}$$

Here d is the exterior differential acting componentwise on $x^k \in i_{\mathcal{E}}(\mathcal{E}^k)$, $\mathbf{e}\mathbf{n} \bullet x^k$ is the application of the homomorphism $\mathbf{e}\mathbf{n}\mathbf{e} \in i(\mathcal{H}^1)$ to x^k , and $x^k \bullet \mathbf{n}$ is the right action of $\mathbf{n} \in i_{\Lambda}(\Lambda_{\mathcal{A}}^1)$ on x^k . Next, using (56), one easily calculates

$$i(\theta_0) = \mathbf{e}[\mathbf{n}, \mathbf{e}][\mathbf{n}, \mathbf{e}]\mathbf{e} = \begin{pmatrix} \mathbf{e}_1(\mathbf{e}_1 - \mathbf{e}_4) \otimes M_1^1 & 0 \\ 0 & \mathbf{e}_4(\mathbf{e}_4 - \mathbf{e}_1) \otimes M_4^1 \end{pmatrix}. \tag{124}$$

In order to fully expose the structure of the graded linear Lie algebra $spl(p, p)$, we have to apply one further operation. However, before doing that we recall – for the sake of completeness – the definition of this algebra together with their generalizations to graded Lie algebras with values in differential forms used in the Mainz–Marseille approach to model building.

Defining the grading operator $\Gamma_0 := \begin{pmatrix} I_{p \times p} & 0 \\ 0 & -I_{p \times p} \end{pmatrix} \in M_{2p}\mathbb{C}$, we introduce a $\mathbb{Z}_2$ -grading structure in $M_{2p}\mathbb{C}$ and denote for $\mathbf{M} \in M_{2p}\mathbb{C}$

$$\mathbf{M}_0 := \frac{1}{2}(\mathbf{M} + \Gamma_0 \mathbf{M} \Gamma_0), \quad \mathbf{M}_1 := \frac{1}{2}(\mathbf{M} - \Gamma_0 \mathbf{M} \Gamma_0). \tag{125}$$

We have $\mathbf{M} = \mathbf{M}_0 \oplus \mathbf{M}_1$, the submatrix $\mathbf{M}_0$ is called the even part and $\mathbf{M}_1$ the odd part of $\mathbf{M}$. We denote the degree of a matrix $\mathbf{M}$ by $\partial\mathbf{M}$ and define $\partial\mathbf{M}_0 = 0$ and $\partial\mathbf{M}_1 = 1$. Defining the graded commutator

$$[\mathbf{M}, \mathbf{N}]_g := \sum_{i,j=0}^1 (\mathbf{M}_i \mathbf{N}_j - (-1)^{\partial\mathbf{M}_i \partial\mathbf{N}_j} \mathbf{N}_j \mathbf{M}_i), \quad \mathbf{M}, \mathbf{N} \in \mathbf{M}_{2p}\mathbb{C}, \quad (126)$$

we get the structure of a graded Lie algebra on $\mathbf{M}_{2p}\mathbb{C}$, called $pl(p, p)$. There is a non-simple graded Lie subalgebra $spl(p, p) \subset \mathbf{M}_{2p}\mathbb{C}$ of graded-tracefree matrices [25] defined by

$$spl(p, p) := \{ \mathbf{M} \in \mathbf{M}_{2p}\mathbb{C} : \text{tr}(\Gamma_0 \mathbf{M}) = 0 \}. \quad (127)$$

In $spl(p, p)$ there exists a differential d_M given by

$$\begin{aligned} d_M \mathbf{M} &:= [\mathbf{m}, \mathbf{M}]_g, \quad \mathbf{M} \in spl(p, p), \\ \mathbf{m} &= z \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} \in spl(p, p), \end{aligned} \quad (128)$$

where u is an arbitrary element of $U(p)$ and $z \in \mathbb{C}$. We choose, however, from the very beginning $u = I_{p \times p}$ and $z = -i$. The reason for this choice will become clear below.

Now one defines the $\mathbb{Z}_2$ -graded algebra $\Lambda^*(X) \otimes \mathbf{M}_{2p}\mathbb{C}$ as the $\mathbb{Z}_2$ -graded tensor product of the $\mathbb{Z}_2$ -graded algebras $\Lambda^*(X)$ and $\mathbf{M}_{2p}\mathbb{C}$. This means: The total degree of $\mathbf{b} = \beta \otimes \mathbf{M} \in \Lambda^*(X) \otimes \mathbf{M}_{2p}\mathbb{C}$ is $\partial\mathbf{b} = (\partial\beta + \partial\mathbf{M}) \pmod{2}$, where $\partial\beta$ is the ordinary differential form degree modulo 2. Thus, we have a natural decomposition

$$\Lambda^*(X) \otimes \mathbf{M}_{2p}\mathbb{C} = \mathcal{F}_0 \oplus \mathcal{F}_1 \quad (129)$$

into elements of even ($\mathcal{F}_0$) and odd ($\mathcal{F}_1$) degree. Both ($\mathcal{F}_0$) and ($\mathcal{F}_1$) are naturally filtrated, $\mathcal{F}_0 = \bigcup_k \mathcal{F}_0^{2k}$ and $\mathcal{F}_1 = \bigcup_k \mathcal{F}_1^{2k+1}$. Elements of $\mathcal{F}_0^{2k}$ have differential $(2k - 2i)$ -forms in the diagonal blocks and $(2k - 2i - 1)$ -forms in the off-diagonal blocks. Analogously, elements of $\mathcal{F}_1^{2k+1}$ have differential $(2k - 2i + 1)$ -forms in the diagonal blocks and $(2k - 2i)$ -forms in the off-diagonal blocks. The product in $\Lambda^*(X) \otimes \mathbf{M}_{2p}\mathbb{C}$, here denoted by $\odot$, is

$$(\beta \otimes \mathbf{M}) \odot (\nu \otimes \mathbf{N}) := (-1)^{\partial\nu \partial\mathbf{M}} (\beta \wedge \nu) \otimes (\mathbf{M} \mathbf{N}), \quad (130)$$

where $\mathbf{M}$ and ν are homogeneous elements. In terms of 2×2 -block matrices one has

$$\begin{aligned}
& \left(\begin{array}{cc} \sum_{r=0}^{k_1} A_1^{k_1-r} & \sum_{r=0}^{k_2} A_2^{k_2-r} \\ \sum_{r=0}^{k_3} A_3^{k_3-r} & \sum_{r=0}^{k_4} A_4^{k_4-r} \end{array} \right) \odot \left(\begin{array}{cc} \sum_{s=0}^{l_1} \tilde{A}_1^{l_1-s} & \sum_{s=0}^{l_2} \tilde{A}_2^{l_2-s} \\ \sum_{s=0}^{l_3} \tilde{A}_3^{l_3-s} & \sum_{s=0}^{l_4} \tilde{A}_4^{l_4-s} \end{array} \right) \\
&= \left(\begin{array}{cc} \sum_{t=0}^{k_1+l_1} \sum_{r=0}^t A_1^{k_1-r} \wedge \tilde{A}_1^{l_1-(t-r)} + & \sum_{t=0}^{k_1+l_2} \sum_{r=0}^t A_1^{k_1-r} \wedge \tilde{A}_2^{l_2-(t-r)} + \\ \sum_{t=0}^{k_2+l_3} \sum_{r=0}^t (-1)^{l_3-(t-r)} A_2^{k_2-r} \wedge \tilde{A}_3^{l_3-(t-r)} & \sum_{t=0}^{k_2+l_4} \sum_{r=0}^t (-1)^{l_4-(t-r)} A_2^{k_2-r} \wedge \tilde{A}_4^{l_4-(t-r)} \\ \hline \sum_{t=0}^{k_4+l_3} \sum_{r=0}^t A_4^{k_4-r} \wedge \tilde{A}_3^{l_3-(t-r)} + & \sum_{t=0}^{k_4+l_4} \sum_{r=0}^t A_4^{k_4-r} \wedge \tilde{A}_4^{l_4-(t-r)} + \\ \sum_{t=0}^{k_3+l_1} \sum_{r=0}^t (-1)^{l_1-(t-r)} A_3^{k_3-r} \wedge \tilde{A}_1^{l_1-(t-r)} & \sum_{t=0}^{k_3+l_2} \sum_{r=0}^t (-1)^{l_2-(t-r)} A_3^{k_3-r} \wedge \tilde{A}_2^{l_2-(t-r)} \end{array} \right), \tag{131}
\end{aligned}$$

where $A_q^n, \tilde{A}_q^n \in M_p \mathbb{C} \otimes \Lambda^n(X)$ and elements with negative superscripts on the r.h.s. are by definition equal to zero. Obviously, $\Lambda^*(X) \otimes M_{2p} \mathbb{C}$ has the structure of a graded Lie algebra with

$$[\mathbf{b}_1, \mathbf{b}_2]_g := \mathbf{b}_1 \odot \mathbf{b}_2 - (-1)^{\partial \mathbf{b}_1 \partial \mathbf{b}_2} \mathbf{b}_2 \odot \mathbf{b}_1. \tag{132}$$

Moreover, $\Lambda^*(X) \otimes M_{2p} \mathbb{C}$ is a graded differential algebra with differential

$$\mathfrak{d}(\beta \otimes \mathbf{M}) := (d\beta) \otimes \mathbf{M} + (-1)^{\partial \beta} \beta \otimes (d_M \mathbf{M}), \tag{133}$$

where d is the exterior differential on $\Lambda^*(X)$. Using the above definitions one easily calculates

$$\mathfrak{d} \mathbf{b} = d\mathbf{b} + [1 \otimes \mathbf{m}, \mathbf{b}]_g, \tag{134}$$

where $d\mathbf{b}$ denotes the action of the exterior differential on the first component of $\Lambda^*(X) \otimes M_{2p} \mathbb{C}$.

On $\Lambda^*(X) \otimes M_{2p} \mathbb{C}$ we introduce an involution $*$ by

$$(\beta \otimes \mathbf{M})^* := (-1)^{\partial \beta \partial \mathbf{M}} \beta^* \otimes \mathbf{M}^*, \tag{135}$$

where $(\beta \wedge \nu)^* = \nu^* \wedge \beta^*$. In terms of 2×2 -block matrices one has

$$\left(\begin{array}{cc} A_1^{k_1} & A_2^{k_2} \\ A_3^{k_3} & A_4^{k_4} \end{array} \right)^* = \left(\begin{array}{cc} (A_1^{k_1})^* & (-1)^{k_3} (A_3^{k_3})^* \\ (-1)^{k_2} (A_2^{k_2})^* & (A_4^{k_4})^* \end{array} \right). \tag{136}$$

A straightforward calculation gives $([\mathbf{b}_1, \mathbf{b}_2]_g)^* = -(-1)^{\partial \mathbf{b}_1 \partial \mathbf{b}_2} [\mathbf{b}_1^*, \mathbf{b}_2^*]_g$ and $(\mathfrak{d} \mathbf{b})^* = (-1)^{\partial \mathbf{b}} \mathfrak{d} \mathbf{b}^*$ for $\mathbf{m} = -\mathbf{m}^*$.

One easily shows that $\Lambda^*(X) \otimes spl(p, p)$ is a graded Lie subalgebra of $\Lambda^*(X) \otimes M_{2p} \mathbb{C}$. Moreover, the graded differential $\mathfrak{d}$ defined in (133) respects

the Lie subalgebra $\Lambda^*(X) \otimes spl(p, p)$. Note, however, that the involution (135) cannot be used to define a real graded Lie subalgebra in $\Lambda^*(X) \otimes spl(p, p)$, because imposing a graded selfadjointness condition by means of (135) requires a $\mathbb{Z}_4$ -grading, see (21).

Using the projection operator

$$\epsilon = \begin{pmatrix} \mathbf{e}_1 & 0 \\ 0 & \mathbf{e}_4 \end{pmatrix}, \quad (137)$$

with $\mathbf{e}_1$ and $\mathbf{e}_4$ fulfilling (106), we define a graded Lie subalgebra of $\Lambda^*(X) \otimes spl(p, p)$:

$$\mathfrak{H}_\epsilon := \{ \mathbf{b} \in \Lambda^*(X) \otimes spl(p, p) : \mathbf{b} = \epsilon \mathbf{b} \epsilon \}. \quad (138)$$

We underline that $\mathfrak{H}_\epsilon$ is – in general – not a differential algebra. What remains is a derivation $\mathfrak{D} = \epsilon \mathfrak{d}(\cdot) \epsilon$ on $\mathfrak{H}_\epsilon$. Explicitly, one has

$$\mathfrak{D} \mathbf{b} = d\mathbf{b} + [1 \otimes \epsilon \mathbf{m} \epsilon, \mathbf{b}]_g, \quad \mathbf{b} \in \mathfrak{H}_\epsilon. \quad (139)$$

Now, recalling the representation (104) for $i(\Lambda_{\mathcal{A}}^* \otimes M_p \mathbb{C})$, we can define a surjective mapping

$$\begin{aligned} \mathfrak{p} : i(\Lambda_{\mathcal{A}}^* \otimes M_p \mathbb{C}) &\rightarrow \Lambda^*(X) \otimes M_{2p} \mathbb{C}, \\ \mathfrak{p} : \begin{pmatrix} \sum_{t=0}^m A_1^{k-2t} \otimes M_1^t & \sum_{t=0}^m A_2^{k-2t-1} \otimes M_2^t \\ \sum_{t=0}^m A_3^{k-2t-1} \otimes M_3^t & \sum_{t=0}^m A_4^{k-2t} \otimes M_4^t \end{pmatrix} &\mapsto \begin{pmatrix} \sum_{t=0}^m A_1^{k-2t} & \sum_{t=0}^m A_2^{k-2t-1} \\ \sum_{t=0}^m A_3^{k-2t-1} & \sum_{t=0}^m A_4^{k-2t} \end{pmatrix}. \end{aligned} \quad (140)$$

Observe that $\epsilon = \mathfrak{p}(\mathbf{e}) = \text{diag}(\mathbf{e}_1, \mathbf{e}_4)$, see (105) and (137).

Proposition 16 i) $\mathfrak{p} \circ i(\mathcal{H}_0) = \mathfrak{H}_\epsilon$.

ii) $(\mathfrak{p} \circ i(\varrho))^* = \mathfrak{p} \circ i(\varrho^*)$, $\varrho \in \mathcal{H}_0$.

iii) For $k+l \leq 2m+1$ we have $\mathfrak{p} \circ i([\varrho^k, \tilde{\varrho}^l]_g) = [\mathfrak{p} \circ i(\varrho^k), \mathfrak{p} \circ i(\tilde{\varrho}^l)]_g$,
 $\varrho^k \in \mathcal{H}_0^k$, $\tilde{\varrho}^l \in \mathcal{H}_0^l$.

iv) For $k \leq 2m$ we have $\mathfrak{p} \circ i(D_{\mathcal{H}} \varrho^k) = \mathfrak{D}(\mathfrak{p} \circ i(\varrho^k))$, $\varrho^k \in \mathcal{H}_0^k$.

Proof: i) From the property (122) of elements of $\mathcal{H}_0$ we obtain immediately

$$\text{tr}(\Gamma_0 \circ \mathfrak{p} \circ i(\varrho)) = 0, \quad \varrho \in \mathcal{H}_0. \quad (141)$$

This together with $\epsilon \mathbf{r} \epsilon = \mathbf{r}$ for any $\mathbf{r} = \mathfrak{p} \circ i(\varrho) \in \mathfrak{p} \circ i(\mathcal{H}_0)$, see (107), means $\mathfrak{p} \circ i(\mathcal{H}_0) = \mathfrak{H}_\epsilon$.

ii) follows immediately from (136) and (119).

iii) Using (131) and (112) one can show for $k+l \leq 2m+1$

$$\mathfrak{p}(n^k \bullet \tilde{n}^l) = \mathfrak{p}(n^k) \odot \mathfrak{p}(\tilde{n}^l), \quad n^k \in i(\Lambda_{\mathcal{A}}^k \otimes M_p \mathbb{C}), \quad \tilde{n}^l \in i(\Lambda_{\mathcal{A}}^l \otimes M_p \mathbb{C}). \quad (142)$$

For $k + l > 2m + 1$ certain terms in $n^k \bullet \tilde{n}^l$ disappear, because the summation in (112) only runs from $t = 0$ to $t = m$. These terms will in general not vanish in the product $\odot$ of the projected terms. Then, since the total degree of $\mathfrak{p} \circ i(\varrho^k) \in \Lambda^*(X) \otimes M_{2p}\mathbb{C}$ equals k , for $\varrho^k \in \mathcal{H}_0^k$, we find with (18), (132), (111) and (114)

$$\mathfrak{p}oi([\varrho^k, \tilde{\varrho}^l]_g) = [\mathfrak{p}oi(\varrho^k), \mathfrak{p}oi(\tilde{\varrho}^l)]_g, \quad \varrho^k \in \mathcal{H}_0^k, \quad \tilde{\varrho}^l \in \mathcal{H}_0^l, \quad k+l \leq 2m+1. \quad (143)$$

Here, on the l.h.s., $[\ , \]_g$ is the graded commutator in $\mathcal{H}_0$, while on the r.h.s., $[\ , \]_g$ is the graded commutator in $\mathfrak{H}_\epsilon$.

iv) Since $\mathfrak{p}(\mathfrak{m}) = 1 \otimes \epsilon \mathfrak{m} \epsilon$, see (128) and (118), for the choice made for u and z , we obtain *iv)* for $k \leq 2m$ from (117), (118) and (139). The restriction to $k \leq 2m$ is due to the same reasons as in *iii)*, because in $i(D_{\mathcal{H}}\varrho^k)$ there appears a graded commutator. $\square$

The mapping $\mathfrak{p}$ is not injective, we have $\mathfrak{p} \circ i(\Lambda_{\mathcal{A}}^k \otimes M_p\mathbb{C}) \subset \mathfrak{p} \circ i(\Lambda_{\mathcal{A}}^{k+2} \otimes M_p\mathbb{C})$ for $k \leq 2m - 1$. These inclusions are compatible with the natural filtrations discussed above, see (129). We have

$$\mathfrak{p} \circ i(\Lambda_{\mathcal{A}}^k \otimes M_p\mathbb{C}) \subset \mathcal{F}_i^k, \quad (144)$$

where $i = k \bmod 2$. For $k \leq 2m + 1$ we obtain

$$\mathfrak{p} \circ i(\Lambda_{\mathcal{A}}^k \otimes M_p\mathbb{C}) = \mathcal{F}_i^k. \quad (145)$$

Next we observe that $\mathfrak{p}|_{i(\Lambda_{\mathcal{A}}^k \otimes M_p\mathbb{C})}$ is injective for each fixed k and that $\mathfrak{p}$ restricted to $i(\mathcal{H}^0)$ is an isomorphism of algebras. Since $MM^* \notin \text{Cid}_F$, we have $m \geq 1$. Thus, the product of elements of $i(\mathcal{H}^1)$ by elements of $i(\mathcal{H}^0)$ or $i(\mathcal{H}^1)$ is transported via $\mathfrak{p}$ isomorphically. The same is true for the transport of the derivation (139) of elements of $i(\mathcal{H}^0)$ and $i(\mathcal{H}^1)$. We stress that applying $\mathfrak{p}$, one loses (besides the physical information about the fermion masses encoded in the matrix M) the $\mathbb{N}$ -grading structure of $i(\mathcal{H})$. This is inevitable, because on $\mathfrak{H}_\epsilon$ there is only a $\mathbb{Z}_2$ -grading structure.

Next, we discuss the transport of the gauge group of the module $\mathcal{E}$, see Definition 6, and the structure of the transported connection form. We have $\text{End}(\mathcal{E}) = \mathcal{H}^0$ and, therefore, from (104), (107), (119) and Definition 6 we find

$$\mathfrak{U} := \mathfrak{p} \circ i(\mathcal{U}(\mathcal{E})) \quad (146)$$

$$= \left\{ u = \begin{pmatrix} u_1 & 0 \\ 0 & u_4 \end{pmatrix}, \quad u_1 = \mathbf{e}_1 u_1 \mathbf{e}_1, \quad u_4 = \mathbf{e}_4 u_4 \mathbf{e}_4, \quad u_1 u_1^* = \mathbf{e}_1, \quad u_4 u_4^* = \mathbf{e}_4 \right\},$$

where $u_1, u_4 \in \Lambda^0(X) \otimes M_p\mathbb{C}$.

The transported connection form belongs to $\mathfrak{p} \circ i(\mathfrak{h}^1)$ and has according to Lemma 9, (98), (104), (107), (119) and (140) the structure

$$\mathfrak{a} := \mathfrak{p} \circ i(\rho) = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}, \quad R_1 = -R_1^*, \quad R_2 = -R_3^*, \quad R_4 = -R_4^*, \quad (147)$$

$$R_1 = \mathbf{e}_1 R_1 \mathbf{e}_1 \in \Lambda^1(X) \otimes M_p\mathbb{C}, \quad R_2 = \mathbf{e}_1 R_2 \mathbf{e}_4 \in \Lambda^0(X) \otimes M_p\mathbb{C},$$

$$R_3 = \mathbf{e}_4 R_3 \mathbf{e}_1 \in \Lambda^0(X) \otimes M_p\mathbb{C}, \quad R_4 = \mathbf{e}_4 R_4 \mathbf{e}_4 \in \Lambda^1(X) \otimes M_p\mathbb{C}.$$

For physical reasons it is interesting to restrict the connection form $\mathfrak{a}$ to $\mathfrak{p} \circ \mathfrak{i}(\mathfrak{h}_0^1)$. This means, see (122),

$$\text{tr}(R_1) = \text{tr}(R_4) . \quad (148)$$

Thus, $\mathfrak{a}$ is a skew-adjoint element of $\mathfrak{H}_\epsilon$. Using (37), (124), and *iv*) of Proposition 16 one gets for the transported curvature

$$\mathfrak{f} := \mathfrak{p} \circ \mathfrak{i}(\theta) = \epsilon(\mathfrak{D}\epsilon)(\mathfrak{D}\epsilon)\epsilon + \mathfrak{D}\mathfrak{a} + (1/2)[\mathfrak{a}, \mathfrak{a}]_g . \quad (149)$$

Observe that the curvature – in general – does not take values in $\mathfrak{p} \circ \mathfrak{i}(\mathfrak{h}_0)$ because of (124). The transport of the gauge transformed connection form, see (39), is due to Proposition 16 given by

$$\gamma_u(\mathfrak{a}) = u\mathfrak{D}u^* + u\mathfrak{a}u^* , \quad (150)$$

and in the representation (147) it takes the form

$$\gamma_u(\mathfrak{a}) = \left(\begin{array}{c|c} u_1 d(u_1^*) + u_1 R_1 u_1^* & u_1 (R_2 - i\mathbf{e}_1 \mathbf{e}_4) u_4^* + i\mathbf{e}_1 \mathbf{e}_4 \\ \hline u_4 (R_3 - i\mathbf{e}_4 \mathbf{e}_1) u_1^* + i\mathbf{e}_4 \mathbf{e}_1 & u_4 d(u_4^*) + u_4 R_4 u_4^* \end{array} \right) . \quad (151)$$

Since $\gamma_u(\mathfrak{a})$ must also be an element of $\mathfrak{p} \circ \mathfrak{i}(\mathfrak{h}_0^1)$, the group of gauge transformations has to be restricted to

$$\mathfrak{U}_0 := \{ u \in \mathfrak{U} : \text{tr}(u_1 d(u_1^*)) = \text{tr}(u_4 d(u_4^*)) \} . \quad (152)$$

Putting $u = \epsilon + \mathfrak{t} + \dots \in \mathfrak{U}$, with $\mathfrak{t} \in \mathfrak{p} \circ \mathfrak{i}(\mathfrak{h}_0^0)$, we obtain the infinitesimal version of gauge transformations:

$$\gamma_{\mathfrak{t}}(\mathfrak{a}) = \mathfrak{a} + \mathfrak{D}\mathfrak{t} + [\mathfrak{t}, \mathfrak{a}]_g , \quad (153)$$

where we have used $\mathfrak{D}\epsilon \equiv 0$ and $\mathfrak{t} = \epsilon \mathfrak{t} \epsilon$. The condition $\gamma_u(\mathfrak{a}) \in \mathfrak{p} \circ \mathfrak{i}(\mathfrak{h}_0^1)$ gives $d\mathfrak{t} \in \mathfrak{p} \circ \mathfrak{i}(\mathfrak{h}_0^1)$. Neglecting global (additional) gauge transformations, we integrate $d\mathfrak{t} \in \mathfrak{p} \circ \mathfrak{i}(\mathfrak{h}_0^1)$ and obtain

$$\mathfrak{t} \in \mathfrak{p} \circ \mathfrak{i}(\mathfrak{h}_0^0) \quad (154)$$

for the generator of infinitesimal gauge transformations.

Now we give a local description of the gauge groups $\mathfrak{U}$ and $\mathfrak{U}_0$. Since the algebra under consideration is commutative, there corresponds a classical (in general nontrivial) vector bundle E over two copies of X to the Hermitian module $\mathcal{E}$. We choose a covering $\{\mathcal{O}_i\}$ of X , such that E is trivializable over this covering. Then, we can locally – on every $\mathcal{O}_i$ – diagonalize $\mathbf{e}_1$ and $\mathbf{e}_4$, using pointwise unitary matrices. Since $\mathbf{e}_1$ and $\mathbf{e}_4$ are idempotent, they can locally be written as

$$\mathbf{e}_1 = \text{diag}(\underbrace{1, \dots, 1}_{p_1}, \underbrace{1, \dots, 1}_{p_2}, \underbrace{0, \dots, 0}_{p_3}), \quad \mathbf{e}_4 = \text{diag}(\underbrace{1, \dots, 1}_{p_1}, \underbrace{0, \dots, 0}_{p_2}, \underbrace{1, \dots, 1}_{p_3}), \quad (155)$$

with $p = p_1 + p_2 + p_3$.

Inserting (155) into (146) we see that the matrices u_1 and u_4 can be locally characterized as follows:

$$u_1 \in C_{\mathbf{R}}^\infty(\mathcal{O}_i) \otimes U(p_1 + p_2), \quad u_4 \in C_{\mathbf{R}}^\infty(\mathcal{O}_i) \otimes U(p_1 + p_3), \quad (156)$$

where $C_{\mathbf{R}}^\infty(\mathcal{O}_i)$ denotes the algebra of real smooth functions on $\mathcal{O}_i$ and a representation of $U(p_1 + p_2)$ in $p \times p$ -matrices containing p_3 zero-rows and -columns is used (analogously for $U(p_1 + p_3)$). This means that the gauge group $\mathfrak{U}$ is locally isomorphic to

$$\mathfrak{U}^i = C_{\mathbf{R}}^\infty(\mathcal{O}_i) \otimes (U(p_1 + p_2) \times U(p_1 + p_3)). \quad (157)$$

There is a natural homeomorphism of $U(n)$ onto $SU(n) \times U(1)$:

$$u = u_0 \begin{pmatrix} \det u & \mathbf{0} \\ \mathbf{0}^T & I_{(n-1) \times (n-1)} \end{pmatrix}, \quad (158)$$

where $u \in U(n)$, $u_0 \in SU(n)$, $\det u \in U(1)$ and $\mathbf{0} = \underbrace{(0, \dots, 0)}_{n-1}$. Using $\text{tr}(u_0 d(u_0^*)) \equiv 0$, for $u_0 \in SU(n)$, we obtain from the condition $\text{tr}(u_1 d(u_1^*)) = \text{tr}(u_4 d(u_4^*))$ characterizing elements of $\mathfrak{U}_0$, see (152),

$$\det u_1 d(\det u_1)^{-1} = \det u_4 d(\det u_4)^{-1}. \quad (159)$$

Integrating this result, we obtain $\det u_1 = \text{const.}$ $\det u_4$. We put the integration constant equal to one. This shows that we have locally

$$\mathfrak{U}_0^i = C_{\mathbf{R}}^\infty(\mathcal{O}_i) \otimes (SU(p_1 + p_2) \times SU(p_1 + p_3) \times U(1)). \quad (160)$$

Of course, the collection $\{\mathfrak{U}_0^i\}$ can be used to reconstruct the gauge group $\mathfrak{U}_0$ – or in the bundle terminology – the group of vertical automorphisms of the principal bundle associated with E (the group of local gauge transformations).

In particular, for $p_1 + p_3 = 1$ the group of local gauge transformations is locally given by $\mathfrak{U}_0^i = C_{\mathbf{R}}^\infty(\mathcal{O}_i) \otimes (SU(p_1 + p_2) \times U(1))$ and for $p_1 + p_2 = 1$ by $\mathfrak{U}_0^i = C_{\mathbf{R}}^\infty(\mathcal{O}_i) \otimes (SU(p_1 + p_3) \times U(1))$. In the case $p_1 + p_3 = 0$ the group of local gauge transformations is

$$\mathfrak{U}^i = C_{\mathbf{R}}^\infty(\mathcal{O}_i) \otimes U(p_2), \quad (161)$$

and the group of special local gauge transformations $\mathfrak{U}_0$ is reduced to

$$\mathfrak{U}_0^i = C_{\mathbf{R}}^\infty(\mathcal{O}_i) \otimes SU(p_2). \quad (162)$$

Analogous results can be obtained in the case $p_1 + p_2 = 0$.

Finally, we comment on the local representation of the connection form $\mathfrak{a}$, see (147). Using the above described local diagonalization procedure for the

projection operators $\mathbf{e}_1$ and $\mathbf{e}_4$, one finds local representatives $A_{(i)}$, $B_{(i)}$ and $\Phi_{(i)}$ of R_1 , R_4 and $R_2 = -R_3^*$ respectively, with $A_{(i)} = -A_{(i)}^*$, $B_{(i)} = -B_{(i)}^*$ and $\text{tr}(A_{(i)}) = \text{tr}(B_{(i)})$. The fields $A_{(i)}$ and $B_{(i)}$ constitute the local representative of a classical gauge connection, that means a classical differential 1-form on $\mathcal{O}_i$ with values in the Lie-algebra of $(SU(p_1 + p_2) \times SU(p_1 + p_3) \times U(1))$. The field $\Phi_{(i)}$ is a vector-space-valued function on $\mathcal{O}_i$ and can be physically interpreted as a matter field – as done in the next section. The fact that two different classical objects are unified in one non-commutative connection form is, of course, due to the fact that we started with a non-commutative differential calculus.

8 Mathematical Structure of the Mainz–Marseille Approach to Model Building

Here we will show that the mathematical structures underlying an approach to the derivation of the standard model, proposed by Coquereaux et al. ([13], [12], [9], [11]), can be obtained as a special case of the structures derived in the previous section. Partly our notations and sign conventions differ from the original ones, due to the fact that we started essentially with the conventions of Connes. We put $N = 4$ for the dimension of the manifold X and assume that X is topologically trivial, for many physical applications it has the topology of $\mathbf{R}^4$. In that case all local considerations of the previous section concerning the group of local gauge transformations and the non-commutative connection form $\mathbf{a}$ become global.

The starting point in the Mainz–Marseille approach is the differential algebra $\Lambda^*(X) \otimes M_4\mathbb{C}$, or rather [13] $M_4\mathbb{C} \otimes \Lambda^*(X)$, giving in general a different sign in (130). This means that we put $p = 2$ in formulae of the previous section. Putting for ϵ , see (137), $\epsilon = \text{diag}(1, 1, 1, 0)$, we get a graded Lie subalgebra of $\Lambda^*(X) \otimes spl(2, 2)$, see (138), which we denote by $\Lambda^*(X) \otimes spl(2, 1)$. We stress that this graded Lie algebra was denoted in [12] by $\Lambda^*(X) \otimes SU(2|1)$. We have changed the notation, because the latter one could suggest that we are dealing with a real graded Lie algebra – which is not the case as explained above.

The authors of [12] formally define a connection putting

$$\nabla = \epsilon \mathfrak{d} + \mathbf{a}, \quad (163)$$

where $\mathfrak{d}$ is the natural differential on $\Lambda^*(X) \otimes M_4\mathbb{C}$, see (134). For the gauge potential $\mathbf{a}$ one postulates the form

$$\mathbf{a} = -\mathbf{a}^* = \begin{pmatrix} A_{11} & A_{12} & -i\Phi_1 & 0 \\ A_{21} & A_{22} & -i\Phi_2 & 0 \\ -i\bar{\Phi}_1 & -i\bar{\Phi}_2 & B & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \Lambda^*(X) \otimes spl(2, 1), \quad (164)$$

$$A_{ij} = -\bar{A}_{ji} \in \Lambda^1(X), \quad B = -\bar{B} \in \Lambda^1(X), \quad A_{11} + A_{22} = B, \quad \Phi_i \in \Lambda^0(X).$$

A certain module, on which this connection can act, was defined in [15]. But a deeper explanation for the choice of the connection form $\mathbf{a}$ was not given. The curvature of this connection is [12]

$$\mathbf{f} = \nabla^2 = \mathbf{e}(\mathfrak{d}\mathbf{e})(\mathfrak{d}\mathbf{e})\mathbf{e} + \mathfrak{D}\mathbf{a} + (1/2)[\mathbf{a}, \mathbf{a}]_g, \quad (165)$$

where $\mathfrak{D}$ is given by (139), and the bosonic action is $S_b = \int_X \text{tr}(\mathbf{f} \wedge * \mathbf{f})$, with $*$ denoting the Hodge star.

It is unclear in this approach what the group of gauge transformations is. Instead of this, only infinitesimal gauge transformations are defined – completely ad hoc – see [12],

$$\gamma_t(\mathbf{a}) := \mathbf{a} + \mathfrak{D}\mathbf{t} + [\mathbf{t}, \mathbf{a}]_g, \quad \mathbf{t} = -\mathbf{t}^* \in \Lambda^*(X) \otimes \mathfrak{spl}(2, 1). \quad (166)$$

The authors of [12] notice that for the standard model only those $\mathbf{t}$ make sense, which have the form

$$\mathbf{t} = \begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ T_{21} & T_{22} & 0 & 0 \\ 0 & 0 & T_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_{ij} = -\overline{T}_{ji} \in \Lambda^0(X), \quad \text{tr}(\Gamma_0 \mathbf{t}) = 0. \quad (167)$$

A deeper explanation why one should restrict $\mathbf{t}$ to the form (167) was not given. For an extended theory including differential forms of higher degree there were discussed more general “superbosonic” gauge transformations [12].

Finally, we notice that there exists a formulation of the Mainz–Marseille model in terms of 3×3 -matrices ([16], [24]), for a parallel treatment of both formulations see [14]. However, this formulation leads to additional mathematical difficulties. In particular, the term $\mathbf{e}(\mathfrak{d}\mathbf{e})(\mathfrak{d}\mathbf{e})\mathbf{e}$ of the curvature is missing in this formulation.

Now we show that all structures occurring here find their natural explanation within the framework developed in the previous sections. For this purpose we put $p_1 = 1$, $p_2 = 1$ and $p_3 = 0$, see (155), giving $p = 2$ and the module $\mathcal{E} = e\mathcal{A}^2$. We define the module for the Mainz–Marseille approach as $\mathfrak{p} \circ i_{\mathcal{E}}(\mathcal{E})$, which is a right module over the algebra $\mathfrak{p} \circ i_{\Lambda}(\mathcal{A})$. The explicit form of an element $\mathfrak{x} = \mathfrak{p} \circ i_{\mathcal{E}}(\xi)$, $\xi \in \mathcal{E}$, is given by

$$\mathfrak{x} = \begin{pmatrix} x_1 & x_1 & 0 & 0 \\ x_2 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad x_i \in C^\infty(X), \quad (168)$$

and elements $\mathfrak{c} = \mathfrak{p} \circ i_{\Lambda}(c)$, $c \in \mathcal{A}$, are given by

$$\mathfrak{c} = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 \\ 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & c_4 \end{pmatrix}, \quad c_i \in C^\infty(X), \quad (169)$$

see (109) and (110). Next, $\mathfrak{m}$ occurring in formula (134) takes the form

$$\mathfrak{m} = -i \begin{pmatrix} 0 & I_{2 \times 2} \\ I_{2 \times 2} & 0 \end{pmatrix}, \quad (170)$$

see also (128). Thus, from (123) and (134) one finds

$$\mathfrak{p} \circ i_{\mathcal{E}}(\nabla_0 \xi) = \epsilon \partial \mathfrak{x}, \quad \mathfrak{x} \in \mathfrak{p} \circ i_{\mathcal{E}}(\mathcal{E}), \quad (171)$$

and – using (113) and (147) – one gets

$$\mathfrak{p} \circ i_{\mathcal{E}}(\nabla \xi) = \mathfrak{p} \circ i_{\mathcal{E}}(\nabla_0 \xi + \rho \xi) = \epsilon \partial \mathfrak{x} + \mathfrak{a} \odot \mathfrak{x}, \quad (172)$$

with $\mathfrak{a} \in \Lambda^*(X) \otimes \mathfrak{spl}(2, 1)$. Moreover, $\mathfrak{a}$ given by (147) fulfils additionally (148). Changing the notations $R_1 = A$, $R_4 = \hat{B}$, $R_2 = -i\Phi$, $R_3 = -i\Phi^*$, we obtain exactly the form of the gauge potential postulated in the Mainz–Marseille approach, see (164),

$$\begin{aligned} \mathfrak{a} &= \begin{pmatrix} A & -i\Phi \\ -i\Phi^* & \hat{B} \end{pmatrix}, \quad \text{tr}(A) = \text{tr}(\hat{B}), \\ A = -A^* &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \hat{B} = -\hat{B}^* = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi_1 & 0 \\ \Phi_2 & 0 \end{pmatrix}. \end{aligned} \quad (173)$$

We note that the transported connection $\nabla \equiv (\mathfrak{p} \circ i_{\mathcal{E}}) \nabla (\mathfrak{p} \circ i_{\mathcal{E}})^{-1} : \mathfrak{p} \circ i_{\mathcal{E}}(\mathcal{E}) \rightarrow \mathfrak{p} \circ i_{\mathcal{E}}(\mathcal{E}^1)$ fulfils

$$\nabla(\mathfrak{x}c) = (\nabla \mathfrak{x})c + \mathfrak{x} \partial(c), \quad \mathfrak{x} \in \mathfrak{p} \circ i_{\mathcal{E}}(\mathcal{E}), \quad c \in \mathfrak{p} \circ i_{\Lambda}(\mathcal{A}), \quad (174)$$

which is exactly the transport of the defining equation of a connection, see Definition 5. Finally, observe that formula (149) for the curvature adapted to the case under consideration gives exactly (165).

2. We define the group of gauge transformations in the Mainz–Marseille model as $\mathcal{U}_0 := \mathfrak{p} \circ i \circ \mathcal{U}_0(\mathcal{E})$, the group of unitary automorphisms of the module $\mathfrak{p} \circ i_{\mathcal{E}}(\mathcal{E})$ with identity $\mathfrak{p} \circ i(\epsilon) = \epsilon$, see (152). From (160) we find in the case under consideration

$$\mathcal{U}_0 = C_{\mathbf{R}}^{\infty}(X) \otimes (SU(2) \times U(1)), \quad (175)$$

which is just the group of local gauge transformations of the Salam–Weinberg model. Writing down local gauge transformations, see (150), or rather infinitesimal gauge transformations, see (153) and (154), we get exactly (167) postulated in the Mainz–Marseille approach:

$$\begin{aligned} \mathfrak{t} &= \begin{pmatrix} T_1 & 0 \\ 0 & T_4 \end{pmatrix}, \quad \text{tr}(T_1) = \text{tr}(T_4), \\ T_1 = -T_1^* &= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \Lambda^0(X) \otimes M_2\mathbb{C}, \\ T_4 = -T_4^* &= \begin{pmatrix} T_{33} & 0 \\ 0 & 0 \end{pmatrix} \in \Lambda^0(X) \otimes \mathbf{e}_4(M_2\mathbb{C})\mathbf{e}_4, \end{aligned} \quad (176)$$

see (104), (107), (119) and (122). Thus, $\mathfrak{t}$ coincides with (167) of the Mainz-Marseille approach. This justifies the choice of infinitesimal gauge transformations in the model of Coquereaux and Scheck. But extended “superbosonic” gauge transformations $\mathfrak{t} \in \Lambda^*(X) \otimes \mathfrak{spl}(2, 1)$, which were suggested in [12], are within this context not allowed. We stress that – in contrary to classical differential geometry – the Lie algebra of the structure group $SU(2) \times U(1)$ does not coincide with the Lie algebra $\mathfrak{spl}(2, 1)$, where the gauge potential takes its values.

9 Comments on Model Building

Finally, we make some remarks on the derivation of the standard model. The K-cycle $(\mathcal{A}, h, D)$ reviewed in section 5, together with the finite projective module $\mathcal{E} = e\mathcal{A}^2$ from section 2, was used by Connes in [7], [6] and by Connes and Lott in [8] to obtain a unification of the Salam–Weinberg model – the theory of electroweak interactions of leptons. The K-cycle under consideration corresponds to a double sheeted structure of the universe and the non-commutative gauge potential contains both the ordinary gauge potential and the Higgs field.

Using the above K-cycle together with the module $\mathcal{E}$ and the canonical prescription for the physical Hilbert space, $H = \mathcal{E} \otimes_{\mathcal{A}} h$, of Connes and Lott, it is impossible to derive the full standard model. That is why Connes and Lott proposed a different K-cycle, namely $(\mathcal{A}_s \oplus \mathcal{B}_s, h_s, D_s)$, where

$$\begin{aligned} h_s &= L^2(X, S) \otimes (F_- \oplus F_+) , \\ F_- &= (C^2 \oplus (C^2 \otimes C^3)) \otimes C^{N_F} , \\ F_+ &= (C \oplus (C^2 \otimes C^3)) \otimes C^{N_F} , \\ \mathcal{A}_s &= C_{\mathbb{R}}^{\infty}(X) \otimes (C \oplus \mathbb{H}) , \\ \mathcal{B}_s &= C_{\mathbb{R}}^{\infty}(X) \otimes (C \oplus M_3C) . \end{aligned} \tag{177}$$

Here $N_F = 3$ is the number of generations of fermions and $\mathbb{H}$ is the real algebra of quaternions. All tensor products occurring in (177) are over $\mathbb{R}$, which means in particular that the algebras $\mathcal{A}_s$ and $\mathcal{B}_s$ are real algebras. The differential operator D_s has the same structure as the operator D in (69) for an appropriate choice of $F_{\pm}$ and $\mathcal{M}$. In this approach one uses a free module, namely $\mathcal{E}_s = \mathcal{A}_s \otimes \mathcal{B}_s$. A detailed exposition of these ideas was presented by Kastler in [19] and [20], see [18] for an earlier version.

It is worthwhile to notice that in this approach one obtains certain constraints between the masses of the fermions and the masses of the W -, Z -, and Higgs-bosons. Moreover, one gets a prediction of the Weinberg angle on tree level. In the “grand unification case” [21] Kastler and Schücker obtained

$$(g_3/g_2)^2 = 1 , \quad \sin^2 \theta_W = 3/8 , \quad m_t/m_W = 2 , \quad m_H/m_W \approx 3.14 , \tag{178}$$

where g_2 and g_3 are the coupling constants of the electroweak and strong interactions, θ_W is the Weinberg angle, m_t , m_W and m_H are the masses of the top-quark, the W -boson and the Higgs-boson.

Another way of obtaining the standard model by non-commutative geometry is the Mainz-Marseille approach ([13], [12]), which is based upon the mathematical structures reviewed in the previous section. In a first step one writes down the bosonic action of the electroweak sector using the $spl(2, 1)$ -gauge connection discussed in section 8, see [13] and [12]. The bosonic action of the chromodynamics sector is added in the same form as in classical gauge field theory. To write down the fermionic sector, one uses the theory of representations of the graded Lie algebra $spl(2, 1)$ in a finite dimensional vector space $[y, I]$, see [26] and [12], where y means hypercharge and I isospin. One builds the Hilbert space $L^2(X, S) \otimes [y, I]$, leptons live in $L^2(X, S) \otimes [1, \frac{1}{2}]$ and quarks in $L^2(X, S) \otimes [\frac{1}{3}, \frac{1}{2}] \otimes \mathbb{C}^3$.

We note that there do not exist representations of the full graded Lie algebra $\Lambda^*(X) \otimes spl(2, 1)$ in these Hilbert spaces. To define a fermionic action one has to define a covariant derivative. For this purpose the connection form $\mathfrak{a}$ has to be considered as an element of $(C^1 \oplus C^0) \otimes spl(2, 1)$, acting with the first (Clifford) part on $L^2(X, S)$ and with the $spl(2, 1)$ -part on $[y, I]$. The fermion masses are obtained from free relative normalization constants of $sl(2, \mathbb{C}) \times gl(1, \mathbb{C})$ -subrepresentations. In contrast to the model of Connes, Lott and Kastler, the fermion masses are not related to the masses of the intermediate vector and Higgs-bosons. Using reducible indecomposable representations of $spl(2, 1)$ one describes family mixing [12].

It turns out that one can use the ideas of the Mainz-Marseille approach for a construction of the standard model based upon the structures discussed in this paper. For the electroweak interaction part one has to take the module $\mathcal{E} = \epsilon \mathcal{A}^2$, see section 8, with $\epsilon = \text{diag}(1, 1, 1, 0) \otimes id_F$. For the chromodynamics part one has to use the module $\mathcal{E}_c = \epsilon_c \mathcal{A}^3$, with $\epsilon_c = \text{diag}(1, 0, 1, 0, 1, 0) \otimes id_F$. Both modules are over the simplest two-point algebra $\mathcal{A}$ defined in (66), and the vector space F , which in our construction plays an auxiliary role, is taken to be $F = \hat{F} \oplus \hat{F}$, $\hat{F} = \mathbb{C}^3$. Constructing the graded Lie algebra $\mathcal{H}_0$, see (96), one gets for the module $\mathcal{E}$ the unified bosonic sector of the Weinberg-Salam model and for $\mathcal{E}_c$ the gluons of chromodynamics. Instead of the canonical Hilbert space H defined above, we take the Hilbert space $\hat{H} = L^2(X, S) \otimes ([1, \frac{1}{2}] \oplus ([\frac{1}{3}, \frac{1}{2}] \otimes \mathbb{C}^3)) \otimes \mathbb{C}^3$. The last $\mathbb{C}^3$ -factor is a representation space of $End(\hat{F})$, labeling the fermion generations. Both the unified electroweak and chromodynamics connections contain tensor product components built from the mass matrix M , which act on $\hat{F}$. The mass matrix parameters enter both the boson and fermion masses of the theory, in contrary to the Mainz-Marseille scheme. Although there do not exist representations of $\mathcal{H}_0$ in the Hilbert space $\hat{H}$ (just as in the Mainz-Marseille model), one can easily define a canonical action of elements of $\mathcal{H}_0^k$, $k = 0, 1, 2$, on elements of $\hat{H}$ using again the representations of $spl(2, 1)$. This suffices for the construction of the bosonic and fermionic actions.

Phenomenologically, our approach leads to similar results as the construction by Connes, Lott and Kastler. The fermion masses and the masses of the W -, Z - and Higgs-bosons depend on both the parameters of the mass matrix M , as in the model of Connes, Lott and Kastler and on additional free relative normalization constants, occurring in the theory of representations of super Lie algebras [26], as in the Mainz–Marseille model. We find [29]

$$(g_3/g_2)^2 = 1, \quad \sin^2 \theta_W = \frac{3}{8}, \quad \sqrt{2} m_W < m_t \leq \sqrt{\frac{8}{3}} m_W, \quad m_H \leq 2.43 m_W. \quad (179)$$

However, we stress that the relations (178) and (179) are on classical (tree) level, they rather do not survive the renormalization procedure. But there seems to be only a weak scale dependence [1]. We remark that in our approach (in contrast to the version of Kastler) there occurs a cosmological constant in the Lagrangian due to the term θ_0 of the curvature (37). This is typical for models with nontrivial projective modules.

In our construction of the standard model one immediately obtains the correct hypercharges of the fermions, whereas in Kastler’s version this is achieved by including the algebra $\mathcal{B}_s$ and imposing a generalized Poincaré duality condition ([19], [21]). This yields a constraint between the three $U(1)$ -subgroups of the local gauge group $\mathcal{U}(\mathcal{A}_s \otimes \mathcal{B}_s) = C_{\mathbf{R}}^\infty(X) \otimes (SU(2) \times U(1) \times SU(3) \times U(1) \times U(1))$ giving the local gauge group $C_{\mathbf{R}}^\infty(X) \otimes (SU(2) \times U(1)_Y \times SU(3))$ of the standard model. The reason for that is that our $U(1)$ -subgroup of $\mathfrak{U}_0$ acts on both the right-handed and the left-handed fermions (see (146) with $\det u_1 = \det u_4$), while the $U(1)$ -subgroup of Kastler’s electroweak gauge group $SU(2) \times U(1)$ acts only on the right-handed fermions.

In conclusion, the K-cycle $(\mathcal{A}, h, D)$ of Connes and Lott can be equally well used for a derivation of the standard model as the K-cycle $(\mathcal{A}_s \otimes \mathcal{B}_s, h_s, D_s)$. A detailed description of our method will be given elsewhere [29].

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