Two-Body wave equations in curved spacetime

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Two scalar particles undergoing mutual interaction are considered in a prescribed curved spacetime. Both masses are finite (recoil is not neglected). For description of this system we propose a pair of coupled Klein-Gordon equations; they involve two-body-sector extensions of the Laplace-Beltrami operator, plus a term which takes mutual interaction into account. Besides the problem of compatibility, we discuss several requirements that seem more or less necessary from a physical point of view. Particular attention is devoted to the preservation of spacetime symmetries (isometric invariance). Composition of curvature with mutual interaction is a nonlinear problem, but it can be explicitly solved in a toy-model of static orthogonal spacetime. Moreover we check that isometric invariance and other physical requirements are satisfied in this example.

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1. Introduction

The perturbation of energy levels of an atom in curved spacetime is of considerable theoretical interest. Several years ago, L.Parker pointed out the possibility to consider an hydrogen atom as a probe of spacetime curvature [1]. We would like to add that *in principle*, also the elementary quark model of composite hadrons should be somehow affected by the presence of an extremely strong gravitational field.

Of course, curvature effects on the *internal* structure of composite systems are of very small magnitude and cannot be directly observed in accessible regions of spacetime. For an atomic system, a significant effect would require a radius of curvature of the order of 10^{-3} cm (a molecule should be more sensitive).

Nevertheless, regions of such large curvature might exist in the neighborhood of small black holes left over from the early stage of the universe. And one must bear in mind that observations of more and more remote regions of the universe provides us with information about the early universe, a laboratory of large curvature.

All these reasons have motivated an abundant flow of papers about quantum mechanics of atoms in curved spacetime [1-3]. Most authors have systematically neglected the recoil of the nucleus. It was the merit of E.Fischbach, B.S.Freeman and Wen-Kwei Cheng [3] to point out that taking recoil into account may, in weak gravitational fields, introduce a correction of the same magnitude as the gravitational perturbations (and this remark is more drastically relevant for positronium). They realized that a fully relativistic treatment of two-body dynamics in the presence of gravity is a difficult problem, and they attempted to solve it along the lines of the Bakamdjian-Thomas formalism [4].

However, the modern developments of relativistic particle dynamics [5-8] allow for a more sophisticated and more efficient framework. In our opinion it has now become natural and reasonable to look after a manifestly covariant description of mutually interacting particles moving in a prescribed Riemannian spacetime.

Insofar as pair creation can be neglected (and taking recoil into account does not change the conditions under which this approximation is valid, so one is referred to the literature quoted above in order to discuss this point) any arbitrary, but fixed, number of particles could be in principle considered. But in practice, we are in a position to undertake the study of *two-body* quantum mechanics because the progresses made up to now in flat spacetime have essentially concerned the two-body sector.

The basic equations of relativistic quantum mechanics for two interacting scalar particles are a pair of Klein-Gordon equations coupled by appropriate terms which carry the mutual interaction and must ensure compatibility [5]. This term can be either phenomenological or motivated by consideration of quantum field theory. In this latter case, the contact between coupled Klein-Gordon equations and the standard methods of quantum electrodynamics in the two-body sector has been firmly established, see Todorov for the quasipotential approach [8] and Sazdjian [9] for the relationship to Bethe-Salpeter equation (BS equation). The coupled equations of relativistic quantum mechanics present several advantages over the traditional BS equation.

Turning now to the situation where gravity is present as an external field, we observe that, in spite of promising advances, quantum theory is not yet developed in curved spacetime as far as to produce a tractable version of the BS equation taking curvature effects into account. Thus a formulation of two-body dynamics in curved spacetime using a pair of wave equations would be very convenient.

Generalization of quantum mechanics to a Lorentzian manifold M_4 is straightforward if the D'Alembert symbol is replaced by the Laplace Beltrami operator (for simplicity we consider minimal coupling). So let us write

$$2H_a\Psi = m_a^2\Psi \quad a = 1, 2 \quad (1.1)$$

where the particle masses are m_1, m_2 . The wave function Ψ depends on eight arguments, namely the coordinates of a couple q_1, q_2 taken in $M_4 \times M_4$. It is always possible to write $H_a = K_a + V_a$ where K_1, K_2 respectively refer particles 1 and 2, namely

$$K_a = -\frac{1}{2}g(a)^{\alpha\beta}\nabla_{a\alpha}\nabla_{a\beta}$$

whereas V_1, V_2 are considered as describing the mutual interaction. In view of our knowledge of the flat-spacetime theory [5], we expect to have in most cases of interest the unipotential simplification $V_1 = V_2 = V$. To avoid unnecessary complications we assume that spacetime M_4 has the trivial topology.

It is clear that equations (1.1) must be mutually compatible. By natural generalization of the flat-spacetime theory we require a strong **compatibility condition**, namely

$$[H_1, H_2] = 0 \quad (1.2)$$

At this stage it is already a practical problem to determine in closed form which interaction term can be added to the Laplace-Beltrami operators without violating condition (1.3). This problem was solved years ago [5] for an isolated system in Minkowski space. Even if one remains in flat spacetime, the presence of external fields is a difficulty, because *relativistic interactions cannot be linearly composed*. In the flat case however, some results have been obtained, for external potentials with special symmetries [10-13].

In the present case we consider a system which is submitted to no external force but gravity. Physically, this situation is analogous to the flat case in the presence of an external potential. But we have a lot of extra complications due to the geometric nature of General Relativity.

We shall first discuss axiomatical matters and propose, in Section 2, a set of necessary conditions restricting the possible choice of H_1 and H_2 . Then it will be interesting to exhibit an example satisfying all (or part of) these minimal requirements. With this goal in mind, a preliminary work is in order. That is why in Section 3 we recollect some standard facts about the motion of a test particle *alone* in a curved spacetime with possible isometries. In addition we systematically introduce a flat background metric and put special emphasis on the role of isometries.

Then we attack the two-body case in Section 4. A specific form of the metric tensor is assumed. The particular symmetry enjoyed by this metric singles out a natural flat background where standard methods of *special relativistic quantum mechanics in external fields* can be finally applied. Moreover this metric is invariant under time translation in a strong sense, which allows for solving (in closed form) the compatibility problem. We ultimately discuss whether the outcome of this procedure satisfies the conditions we have previously demanded. Section 5 is devoted to concluding remarks.

Notation

Particles labels $a, b = 1, 2$. Partial differentiation with respect to coordinates are noted as $\partial_\mu = \partial/\partial q^\mu$, $\partial_{a\mu} = \partial/\partial q_a^\mu$. The use of q_1, q_2 is not a typographic fancy. We mean to recall that in the classical limit ($\hbar \rightarrow 0$) the arguments of the wave function should in principle reduce to the canonical coordinates of a covariant hamiltonian formulation and cannot coincide with the physical coordinates (say x_1, x_2) in the large (that is unconstrained) phase space where the standard Poisson brackets are implemented [5].

Greek labels are omitted whenever no confusion is possible. When we consider a flat background metric $\bar{g}_{\mu\nu}$, its reciprocal tensor is $\bar{g}^{\alpha\beta}$. The determinants are g and $\bar{g}$ respectively. Of course, Cartesian coordinates refer to $\bar{g}_{\mu\nu}$.

Scalar products indicated by a dot always refer to the flat metric. For instance $p^2 = p \cdot p$ stands for $\bar{g}^{\mu\nu} p_\mu p_\nu$. Indices will be raised (lowered) with help of the flat metric only. $p_{a\alpha}$ is the operator with components $-i\partial/\partial q_a^\alpha$ in the cartesian coordinates.

Angular momentum in flat spacetime is $m^{\alpha\beta} = q^\alpha p^\beta - q^\beta p^\alpha$.

Dependence on the coordinates and/or momenta of particle a is indicated as in $g(a) = g(q_a)$, etc.

Relative and center-of-mass variables are defined in cartesian coordinates as follows

$$\begin{aligned} z^\alpha &= q_1^\alpha - q_2^\alpha & y &= \frac{1}{2}(p_1 - p_2) \\ P &= p_1 + p_2 & Q &= \frac{1}{2}(q_1^\alpha + q_2^\alpha) \end{aligned}$$

hence the commutators

$$[z^\alpha, y_\beta] = [Q^\alpha, P_\beta] = i\delta_\beta^\alpha$$

As in [11] we define

$$Z = z^2 P^2 - (z \cdot P)^2$$

The superscript (0), like for instance in $V^{(0)}$, refers to the isolated system (no curvature involved).

Throughout this paper, symmetric operators will be referred to as *hermitian*. Isometry always refers to the Riemannian metric, never to the metric of Hilbert space.

Definition:

A flat metric $\bar{g}$ is said *isometrically admissible* with respect to g if any isometry of g is an isometry of $\bar{g}$.

2. General framework

a) Compatibility

The wave equations have the form (1.1). We naturally require the strong condition (1.2) which ensures compatibility. Assuming that $V_1 = V_2 = V$ condition (1.2) reduces to

$$[K_1 - K_2, V] = 0 \quad (2.1)$$

b) Hermiticity

In order to have a consistent mathematical setting, the squared-mass operators in the lefthand-side of (1.1) (and eventually the generators of transformations induced by spacetime symmetries, if any) must be considered as acting in some well-defined Hilbert space; this remains true even if the wave function cannot actually belong to this Hilbert space. Therefore we shall naturally assume that H_1, H_2 are operators in

$$L^2(\mathbf{R}^8) = L^2(\mathbf{R}^8, \sqrt{|g(1)|}\sqrt{|g(2)|}d^4q_1d^4q_2)$$

Here the following observation is in order. In the absence of mutual interaction we can write (1.1) in the form of independent Klein-Gordon equations

$$2K_a\Psi = m^2\Psi$$

where $2K_a$ is the Laplace Beltrami operator in terms of the coordinates of particle a . It is well-known [14] that this operator is hermitian when considered as acting in $L_a^2 = L^2(\mathbf{R}^4, \sqrt{|g(a)|}d^4q_a)$, thus K_1 and K_2 , extended as operators acting in $L^2(\mathbf{R}^8) = L_1^2 \otimes L_2^2$ are both hermitian. Despite the fact that the fourfold integrals defining L_1^2 and L_2^2 have no direct physical interpretation, this property of hermiticity is very fortunate from a formal point of view. Already in special relativity, it has been demonstrated that off-shell Hilbert spaces of the form $L^2(\mathbf{R}^4)$ or $L^2(\mathbf{R}^8)$ provide the most convenient framework for handling Feynmann's propagators in a manifestly covariant way (for instance $\frac{1}{p^2+U-m^2\pm i\epsilon}$ where U is an external potential, naturally makes sense as the resolvent of an operator in $L^2(\mathbf{R}^4, d^4q)$ and similar observations can be made when interparticle interaction is considered [15]). We extend this philosophy to curved spacetime and require hermiticity as well in the presence of mutual interactions.

In other words we demand that H_1, H_2 are hermitian operators in the Hilbert space $L^2(\mathbf{R}^8)$ defined by (2.1).

At a deeper level of accuracy one would rise the question of self-adjointness, in connexion with the possibility of developing a manifestly covariant scattering theory in terms of scalar evolution parameters. Indeed there are two possibilities in relativistic dynamics:

(For simplicity of the present argument let us provisionally consider the case of a single particle). The most popular point of view consists in using the time coordinate as indicator of evolution (say q^0 in some coordinate system); then the Klein-Gordon equation is interpreted as the relativistic generalization of the time- depending Schroedinger equation.

Another approach, in the spirit of Schwinger's "proper time method" [16] assigns this role to a scalar parameter which coincides with proper time (or a suitable generalization of it) in the classical limit $\hbar \rightarrow 0$; now the Klein-Gordon equation is a mass-shell reduction of the Stückelberg equation [17], and the Dalember operator is trivially proportional to the generator of the motion. In this case, self-adjointness of the Dalember operator with respect to the *four-space* scalar product (involving integration over d^4q) would be a precious information in order to set up a complete quantum theory.

Although we have a preference for the latter point of view (which has been extended to many-body systems [15,18]) we shall not discuss further the evolutive aspects of the wave equations.

Returning to the problem of two-body quantum mechanics in curved spacetime we shall be contented, in this first attempt, with the property of hermiticity (symmetry) and leave the question of self-adjointness for future investigations (notice that even in the one-body case, self-adjointness in the four-space scalar product has received so far very little attention if any).

c) Coupling separability

The above conditions are not yet sufficient from a physical point of view. Indeed one expects to find the *correct limits when either the curvature or the mutual interaction is absent*.

When the curvature identically vanishes, one should recover an isolated system of particles interacting in Minkowski spacetime.

When the mutual interaction is zero, the wave equations should describe independent particles, each one being free in curved spacetime (in other words, submitted only to the curvature field associated with the metric).

These situations will be referred to as the no-field limit and the independent-particle limit respectively.

As simple as it might seem, the formulation of correct limits is problematic in curved spacetime. First of all, *the notion of an isolated system is generally ambiguous*. Strictly speaking, the absence of curvature does not correspond to a well-specified Minkowski space.

But in practice, there are cases where this difficulty can be by-passed.

i) One may consider a particular solution of the Einstein equations. This solution necessarily depends on the Newton constant, and we obtain a flat metric if we replace this constant by zero.

ii) Particular symmetries may select a flat metric in a natural way (Asymptotically flat spacetimes, plane waves etc). Another example: $ds^2 = dt^2 - d\sigma^2$, where $d\sigma^2$ enjoys spherical symmetry (but not more).

Beside this problem about the unicity of a flat background, additional care is needed because, the most natural flat background and the Riemannian metric do not define the same Hilbert space. Indeed the quotient $\bar{g}/g$ of scalar densities is a scalar, and in general it is not a constant (the case of a plane wave, in the weak field approximation and TT-gauge, is an exception). One might be tempted to consider only flat metrics satisfying $\bar{g}/g = \text{const.}$ as eligible for a flat background. But generally, this procedure does not respect the symmetries of the pseudo-Riemannian metric $g_{\mu\nu}$. Indeed it may happen that none of the background metrics selected in this manner is isometrically admissible.

Fortunately, the technical complication of having distinct Hilbert spaces will be easily handled by a change of wave function which amounts to reformulating the wave equations in terms of a new wave function $\bar{\Psi}$, an (improper) element of

$$\bar{L}^2(\mathbf{R}^8) = L^2(\mathbf{R}^8, \sqrt{\bar{g}(1)\bar{g}(2)}d^4q_1d^4q_2)$$

and in terms of modified operators which are hermitian (symmetric) in this new Hilbert space. This approach can be considered as a change of representation, performed with the help of a unitary transformation from L^2 to $\bar{L}^2$. This method allows for application of flat-space techniques : cartesian coordinates, etc.

d) Isometric Invariance

It may happen that spacetime admits some isometries. If we first consider the motion of a single test particle, each isometry of spacetime is associated with a conserved quantity: its generator.

A natural generalization of flat-space theory consists in requiring that the generators of these isometries must be constants of the two-body motion, even in the presence of a mutual interaction. Whereas an *isolated system* in Minkowski space is characterized by Poincaré symmetry (Poincaré invariance from the active point of view), *it is natural to require that an isolated system in curved spacetime be at least invariant under the infinitesimal unitary transformations associated with spacetime isometries*. This should hold true as well for two-body as for one-body systems, and also when the constituent particles undergo mutual interaction. For this aim it is not sufficient that the interaction term be Poincaré invariant in its no-field limit, say $V^{(0)}$. Indeed this term is modified by the presence of curvature (namely $V \neq V^{(0)}$) in order to keep compatibility satisfied.

The question whether this condition is sufficient goes beyond the scope of this paper (when spacetime has no isometry at all, this condition becomes empty; is it still possible to define isolated systems?).

Remark

When we perform a change of wave function, some unitary transformation of the whole picture is necessary. This transformation maps $L^2(\mathbf{R}^8)$ either onto itself or onto a different Hilbert space, and obviously keeps the compatibility conditions satisfied. When identifying the no-field and independent- particle limits, one must take this change of representation into account. For instance, after transformation (3.5) below, the independent-particle limit is characterized by the half-squared-mass operators $\tilde{K}_1, \tilde{K}_2$ which are hermitian in $\bar{L}^2(\mathbf{R}^8)$.

We first recall a few results of Riemannian geometry (see Appendix).

If ξ generates infinitesimal isometries of g then, in the standard notation of differential geometry the Lie derivative operator satisfies $\mathcal{L}_\xi \Delta = \Delta \mathcal{L}_\xi$. When f is a 0-form, that is a scalar function, we simply have $\mathcal{L}_\xi f = \xi^\alpha \partial_\alpha f$. Moreover $\xi^\alpha \partial_\alpha$ is skew-symmetric with respect to the scalar product which defines $L^2(\mathbf{R}^4)$.

Isometries are implemented as quantum operators in the following way. With any vector field ξ^α leaving $g_{\mu\nu}$ invariant we associate the operator $X = -i\xi^\mu \partial_\mu$ acting in $L^2(\mathbf{R}^4, d^4q)$. It is clear that X is hermitian.

In the two-body sector ones deal with $L^2(\mathbf{R}^8) = L_1 \otimes L_2$. The isometric vector ξ^α induces the operators

$$J_1 = -i\xi^\alpha(1)\partial_{1\alpha} \quad J_2 = -i\xi^\alpha(2)\partial_{2\alpha}$$

respectively acting in L_1, L_2 . Then J_1, J_2 are extended to $L^2(\mathbf{R}^8)$ as $J_1 \otimes 1$ and $1 \otimes J_2$ respectively. Considered as operators in $L^2(\mathbf{R}^8)$, they are the *individual isometries* (for instance, in the limit of free motion in flat spacetime, linear and angular momenta of each particle are trivial examples of individual isometries). Finally the *total isometry* induced by ξ^μ is $J = J_1 + J_2$.

3. The one-body problem

In the Klein-Gordon equation $2K\psi = m^2\psi$ the D'Alembert symbol is replaced by the Laplace-Beltrami operator. In other words

$$-2K\psi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi)$$

K transforms as a scalar and is hermitian [14] in the space $L^2(\mathbf{R}^4, \sqrt{|g|} d^4q)$.

Moreover, if the contravariant vector field $\xi^\mu(q)$ is the infinitesimal generator of a one-parameter group of transformations leaving g invariant, then a *classical* constant of geodesic motion is the function $\xi^\mu \pi_\mu$, a scalar on the cotangent bundle $T^*(M_4)$ endowed with coordinates q^α, π_β and with the standard Poisson brackets.

Its quantum counterpart is the differential operator $X = -i\xi^\mu \partial_\mu$.

But the partial differentiation operator ∂_μ present in the above expression of $K\psi$ is neither a vector nor hermitian in the sense of Hilbert space L^2 .

Nevertheless, if we now introduce any flat metric $\bar{g}_{\mu\nu}$, then the operator defined as $p_\mu = -i\partial_\mu$ in the cartesian coordinates associated with $\bar{g}$ does transform as a vector and is hermitian in the space $\bar{L}^2(\mathbf{R}^4) = L^2(\mathbf{R}^4, \sqrt{|\bar{g}|} d^4q)$ endowed with the scalar product

$$\langle \chi, \omega \rangle = \int \chi^* \omega \sqrt{|\bar{g}|} d^4q \quad (3.1)$$

Moreover we have $X = \xi \cdot p$

Using cartesian coordinates makes it trivial to check that we can write

$$2K = \frac{1}{\sqrt{|g|}} p_\mu \sqrt{|g|} g^{\mu\nu} p_\nu \quad (3.2)$$

This situation suggests a change of wave function, say

$$\tilde{\psi} = \Gamma \psi$$

where the "bimetric" scalar Γ is

$$\Gamma = |g/\bar{g}|^{\frac{1}{4}} \quad (3.3)$$

that is to say $\sqrt{|g|} = \Gamma^2 \sqrt{|\bar{g}|}$. The Klein-Gordon equation can now be written as

$$2\tilde{K}\tilde{\psi} = m^2\tilde{\psi} \quad (3.4)$$

where, for all operator A

$$\tilde{A} = \Gamma A \Gamma^{-1} \quad (3.5)$$

In this formula Γ is understood as a multiplicative operator acting on functions $f(q)$. Both $\tilde{K}$ and F are hermitian in $\bar{L}^2$. Notice that eq (3.3) is a unitary transformation from $L^2(\mathbf{R}^4, \sqrt{|g|}d^4q)$ to $\bar{L}^2(\mathbf{R}^4)$. The multiplicative operator Γ (and more generally any function of q) is hermitian in both $\bar{L}^2(\mathbf{R}^4)$ and $L^2(\mathbf{R}^4)$. Applying rule (3.5) to eq(3.1) we immediately find

$$2\tilde{K} = \Gamma^{-1} p_\mu \Gamma g^{\mu\nu} \Gamma p_\nu \Gamma^{-1} \quad (3.6)$$

which is manifestly flat-hermitian. We define an *effective* external potential F by setting

$$\tilde{K} = \frac{1}{2}p^2 + F \quad (3.7)$$

From its very definition, F is flat-hermitian. So finally, both $\tilde{K}$ and F are hermitian in $\bar{L}^2$.

Notice that we could write F down in closed form, by elementary commutator algebra, starting from (3.6) and splitting the contravariant metric tensor as $g^{\mu\nu} = \bar{g}^{\mu\nu} + h^{\mu\nu}$. Of course the contravariant components of $\bar{g}$ reduce to $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ in cartesian coordinates.

Let us now consider the behaviour of infinitesimal isometries.

If ξ generates infinitesimal isometries of $g_{\mu\nu}$, it follows that $\xi^\alpha \partial_\alpha$ commutes with K (both being considered as operators in $L^2(\mathbf{R}^4)$). After introduction of an auxiliary flat metric $\bar{g}$ we can write $X = -i\xi^\alpha \partial_\alpha = \xi \cdot p$ and $[\xi \cdot p, K] = 0$ where $\xi \cdot p$ and K are hermitian in $L^2(\mathbf{R}^4)$.

Henceforth we assume that $\bar{g}$ is isometrically admissible.

Proposition I

Assume that ξ is an isometry of both g and $\bar{g}$. Then transformation (3.5) leaves $X = \xi \cdot p$ unchanged, so $\tilde{X} = X$. In other words, in the Poincaré Lie algebra associated with the flat metric, any element which leaves also g invariant is unchanged by transformation (3.5).

Moreover $\tilde{K}$ as well as K commutes with X and X is hermitian in both L^2 and $\bar{L}^2$.

Proof

ξ being isometry of g and $\bar{g}$ implies that the Lie derivative of the bimetric scalar vanishes, say $\xi^\alpha \partial_\alpha \Gamma = 0$, thus the multiplicative operator Γ satisfies $[\xi \cdot p, \Gamma] = 0$. We have seen that $\xi^\alpha \partial_\alpha$ commutes with the Laplace Beltrami operator Δ . After transformation (3.5) we get $[\tilde{K}, \xi \cdot p] = 0$. We have already proved that $\xi \cdot p$ is hermitian in L^2 . But according to our assumption, there exist, *in cartesian coordinates*, a constant vector C^α and a skew-symmetric tensor $C^{\alpha\beta}$ such that

$$X = \xi^\alpha p_\alpha = C^\alpha p_\alpha + C^{\alpha\beta} m_{\alpha\beta} \quad (3.8)$$

In this form, it is manifest that $\xi \cdot p$ is hermitian also in the sense of $\bar{L}^2$.

Corollary II

If $\mathcal{L}_\xi g = \mathcal{L}_\xi \bar{g}$ then $[\xi \cdot p, F] = 0$. This is now obvious from equation (3.7) and shows the importance of having $\bar{g}$ isometrically admissible.

Example

We now consider a line element of the simple form

$$ds^2 = d(q^0)^2 - R^2(r)d\mathbf{q}^2 \quad (3.9)$$

where $d\mathbf{q}^2$ is a short-hand notation for $\delta_{ij}dq^i dq^j$ and r is $\sqrt{\mathbf{q}^2}$. This metric defines a particular case of static orthogonal spacetime. We assume that R is everywhere finite and differentiable as many times as necessary. Space sections are supposed to be isotropic but *not* homogeneous. They have a unique center of spherical symmetry, thus the form (3.9) of the metric is unique up to a change

$$q'^k = \Lambda q^k \quad R' = \Lambda^{-1} R$$

but we fix the space coordinates by requiring that $R(\mathbf{q} = 0) = 1$.

Our assumptions imply that

$$R^2 \neq \frac{const.}{(1 + \frac{1}{4}kr^2)^2}$$

This restriction excludes the particular case of an Einstein static spacetime.

A preferred background metric is of course $d\bar{s}^2 = d(q^0)^2 - d\mathbf{q}^2$.

Notice that the metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ coincide on the line $q^k = 0$, world line of the center of spherical symmetry.

Both metrics are invariant under time translation and space rotations, and $g_{\mu\nu}$ has no further symmetry. Each generator of these isometries (considered as a hermitian operator in L^2) commutes with K .

Alternatively, when the state of the particle is represented by $\tilde{\psi}$ and its motion described by eq (3.4), the infinitesimal rotations and the time translation keep being represented by the same operators m_{ij} , p_0 . Because Γ in (3.5) is rotational invariant, and doesnot depend on time, these operators are not affected by (3.5) and are hermitian in both Hilbert spaces $L^2(\mathbf{R}^4)$ and $\bar{L}^2(\mathbf{R}^4)$.

According to this particular form of g and $\bar{g}$, p_0 and m_{ij} are unchanged by (3.5). In other words $\tilde{p}_0 = p_0$ and $\tilde{m}_{ij} = m_{ij}$.

It is clear that $d\bar{s}^2$ defines $\bar{g}_{\mu\nu}$ as an isometrically admissible background metric. In the present example, eq (3.8) is satisfied with $C^\alpha = (1, 0, 0, 0)$ and $C^{\alpha\beta}$ reduces to C^{jk} .

Now our claim is that, owing to the assumptions made on the line element (3.9), both $[F, p_0]$ and $[F, q^0]$ vanish. According to the terminology introduced in a previous work [11], we formulate this statement as follows

Proposition III

Owing to the form (3.9) of the metric $g_{\mu\nu}$, the effective potential F defined by (3.7) is strongly invariant by time translation .

Proof

Commutation with p_0 is straightforward. Indeed Γ only depends on $\mathbf{q}$, thus $[\Gamma, p_0] = 0$; for a similar reason $[g^{\mu\nu}, p_0] = 0$, and obviously p_μ commutes with p_0 .

According to (3.7), in order to prove that also $[F, q^0]$ vanishes it will be sufficient to prove that $[2\tilde{K}, q^0]$ coincides with $[p^2, q^0]$.

We observe that

$$[p^2, q^0] = -2ip_0$$

Using (3.6) we easily compute

$$[2\tilde{K}, q^0] = -ig^{0\nu}\Gamma p_\nu \Gamma^{-1} - i\Gamma^{-1}p_\mu \Gamma g^{\mu 0}$$

But g^{0j} is zero, thus only g^{00} survives. We are left with

$$2[\tilde{K}, q^0] = -i\Gamma p_0 \Gamma^{-1} - i\Gamma^{-1}p_0 \Gamma$$

where $[\Gamma, p_0]$ vanishes, thus finally $2[\tilde{K}, p_0] = -2ip_0$ which completes the proof of our statement. Loosely speaking, F depends only on $\mathbf{q}$ and $\mathbf{p}$, we say that it is purely transverse.

Proposition IV

Owing to the form (3.9) of the metric, the effective potential F is (simply) invariant by space rotations.

Proof: From propo we know that K and $\tilde{K}$ commute with X , in particular when $X = C^{ij}m_{ij}$. Obviously, p^2 is invariant under space rotations. Finally, a glance at (3.7) ensures rotation invariance of F .

4. Two-body problem.

Assuming the unipotential simplification, we write

$$H_a = K_a + V \quad (4.1)$$

Our problem reduces to the determination of V by solving (2.1). Insofar as we consider gravity as an external field applied to the system, a natural approach consists in determining V by its no-field limit, say $V^{(0)}$. Indeed, in the absence of gravity, the most general form of $V^{(0)}$ allowing for compatibility and Poincaré invariance is well known [5]. Much care is needed however, because in general, for an arbitrarily specified metric $g_{\mu\nu}$, this point of view has no intrinsic meaning.

Anyway, this point of view requires introduction of an auxiliary flat metric $\bar{g}_{\mu\nu}$, and leads to the following developments.

Define $\Gamma_a = |g(a)/\bar{g}(a)|^{1/4}$ and $\Gamma_1\Gamma_2 = \Gamma_{12}$

Since $L^2(\mathbf{R}^8, \sqrt{g_1 g_2} d^4 q_1, d^4 q_2) = L^2(\mathbf{R}^4, \sqrt{|g(1)|} d^4 q_1) \otimes L^2(\mathbf{R}^4, \sqrt{|g(2)|} d^4 q_2)$ there is no confusion if now $\tilde{A}$ denotes the product $\Gamma_{12} A \Gamma_{12}^{-1}$ for all A acting in $L^2(\mathbf{R}^8)$. Indeed, when A is a product $A_1 A_2$ of operators acting in $L^2(1)$ and $L^2(2)$ respectively, we can check that $\tilde{A} = \tilde{A}_1 \tilde{A}_2$.

The lack of Poincaré symmetry is partially redeemed by considering the operators induced in $\bar{L}^2(\mathbf{R}^8)$ by the isometries of g . Let the contravariant vector field ξ generate a one-parameter group of isometries of spacetime. In Section 2 d) we have introduced the operators J_1, J_2 respectively acting in $L^2(\mathbf{R}^4, d^4 q_1)$ and in $L^2(\mathbf{R}^4, d^4 q_2)$. Their extension to $L^2(\mathbf{R}^8, d^4 q_1 d^4 q_2)$ is straightforward.

In the presence of an auxiliary flat metric we can write $J_1 = \xi(1) \cdot p_1$ $J_2 = \xi(2) \cdot p_2$.

Provided the auxiliary metric $\bar{g}$ is isometrically admissible, we apply Proposition I and obtain

$$J_1 = C \cdot p_1 + C^{\mu\nu} m_{1\mu\nu} \quad (4.2)$$

$$J_2 = C \cdot p_2 + C^{\mu\nu} m_{2\mu\nu} \quad (4.3)$$

for some constant vector C^α and tensor $C^{\mu\nu}$.

Now consider the *total isometry* induced by ξ , say $J = J_1 + J_2$. Proposition I implies that $\tilde{J}_a = J_a$, hence $\tilde{J} = J$

From the properties of the motion in the absence of mutual interaction, or equivalently from Proposition I extended by tensor product, we obtain

$$[K_a, J_b] = [\tilde{K}_a, J_b] = 0 \quad a, b = 1, 2 \quad (4.4)$$

It is clear that

$$J = C \cdot P + C^{\alpha\beta} M_{\alpha\beta} \quad (4.5)$$

Existence of nonvanishing coefficients $C^\alpha, C^{\mu\nu}$ ensures that some elements of the Poincaré algebra survive as symmetries of the motion in the presence of curvature.

Axiom d) of Section 2 (isometric invariance) can be precisely understood as follows: For all isometry-generating vector field ξ , the corresponding operator of total isometry J must be a constant of the two-body motion.

Later on we shall apply the above formula in order to check that this condition is actually satisfied by two-body motion in the particular spacetime (3.9).

Let us now turn to the problem of compatibility. Condition (2.1) is transformed into

$$[\tilde{K}_1 - \tilde{K}_2, \tilde{V}] = 0 \quad (4.6)$$

In view of (3.7) we define effective external potentials $F_a = F(q_a, p_a)$ by the formulas

$$\tilde{K}_a = \frac{1}{2}p_a^2 + F_a \quad (4.7)$$

It is clear that K_a and F_b are hermitian in $\bar{L}^2(\mathbf{R}^8)$.

In general the decomposition (4.7) would have no intrinsic significance (assuming that we rely on General Relativity as the true theory of gravity). But we shall now specialize to the particular case of the line element (3.9). The assumptions made about the function $R^2(r)$ ensure without ambiguity a preferred flat background.

For the metric (3.9) we have seen that C has components $(1, 0, 0, 0)$ and C reduces to its space-by-space components. Any isometry of $g_{\mu\nu}$ satisfies

$$J = C^0 P_0 + C^{ij} M_{ij} \quad (4.8)$$

Extension of Propositions III, IV to the two-body sector is straightforward and implies that the effective potentials F_1, F_2 are strongly invariant under time translations, and simply invariant by space rotations. As a result, compatibility can now be solved by a flat-space technique, using the Bijtebier ansatz [10]. Closed form expressions will be written at the price of a change of representation. The *external-field representation* is formally given by

$$A' = e^{iB} A e^{-iB} \quad \forall A \quad (4.9)$$

where $B = TL$ and

$$T = y_T \cdot P_T + F_1 - F_2 \quad (4.10)$$

$$L = L_\alpha z^\alpha, \quad L_\alpha = \frac{P_L^\alpha}{P_L^2} \quad (4.11)$$

In order to keep contact with a generalized formulation of this method we use our notation of ref[11]. The space of four-vectors is decomposed along longitudinal and transverse directions. Each four-vector has a unique transverse (resp. longitudinal) part. Subscripts T and L refer to this splitting.

But here, the longitudinal part has only the time component. Longitudinal canonical variables commute with the transverse ones, F_1, F_2 depend only on transverse variables and $[T, L] = 0$.

The wave equations will be now written as

$$2\tilde{H}'_a \tilde{\Psi}' = m_a^2 \tilde{\Psi}' \quad (4.12)$$

where $\tilde{H}'_a = \tilde{K}'_a + \tilde{V}'$. In order to avoid cumbersome notation we systematically write $\tilde{K}'$ for $(\tilde{K})'$, etc. One must keep in mind that

$$\begin{aligned} \tilde{\Psi}' &= e^{iB} \Gamma \Psi \\ \tilde{H}'_a &= e^{iB} \Gamma H_a \Gamma^{-1} e^{-iB} \end{aligned} \quad (4.13)$$

and so on.

In the present notation, (4.7) can be identified as (2.7) of ref.[11], where K_a and G_a have been replaced by $\tilde{K}_a$ and F_a respectively. Along this line $\tilde{K}'_a$ can be explicitly computed; see eqs. (3.36) of [11], or eqs (2.24)(2.25) of [13]. One finds

$$\tilde{K}'_a = \tilde{K}_a - T(L \cdot y \pm \frac{1}{2}) + \frac{1}{2} T^2 L \cdot L \quad (4.14)$$

with $+1$ (resp. -1) for $a = 1$ (resp. $a = 2$). Hence, using the identity $p_1^2 + p_2^2 = \frac{1}{2}P^2 + 2y^2$, we get

$$\tilde{K}'_1 + \tilde{K}'_2 = \frac{P^2}{4} + y^2 + F_1 + F_2 - 2T(L \cdot y) + T^2 L \cdot L \quad (4.15)$$

$$\tilde{K}'_1 - \tilde{K}'_2 = y_L \cdot P_L \quad (4.16)$$

In fact B was tailored in order to obtain (4.16). Now define

$$\mu = \frac{1}{2}(m_1^2 + m_2^2) \quad \nu = \frac{1}{2}(m_1^2 - m_2^2)$$

and replace the wave equations by their sum and difference. We get

$$(\tilde{H}'_1 + \tilde{H}'_2)\tilde{\Psi}' = \mu\tilde{\Psi}' \quad (4.17)$$

$$(\tilde{H}'_1 - \tilde{H}'_2)\tilde{\Psi}' = \nu\tilde{\Psi}' \quad (4.18)$$

According to the assumptions made about the metric, there is a unique "laboratory frame". Transverse and longitudinal decomposition simply refers to space and time splitting, thus B takes on the form given by Bijtebier in [10]. In other words

$$L \cdot L = \frac{1}{P_0^2}, \quad L \cdot y = \frac{y_0}{P_0} \quad (4.19)$$

$$T = -\mathbf{y} \cdot \mathbf{P} + F_1 - F_2 \quad L = \frac{z^0}{P^0} \quad (4.20)$$

These expressions are inserted into (4.15)(4.16) and one makes the ansatz

$$\tilde{V}' = f(\hat{Z}, P^2) \quad (4.21)$$

$$\hat{Z} = -z^2 P^2 - (\mathbf{z} \cdot \mathbf{P})^2 \quad (4.22)$$

We finally obtain

$$\tilde{H}'_1 + \tilde{H}'_2 = \frac{P^2}{4} + y^2 + F_1 + F_2 - 2T\frac{y_0}{P_0} + \frac{T^2}{P_0^2} + f(\hat{Z}, P^2) \quad (4.23)$$

whereas the difference of squared mass operators is simply

$$\tilde{H}'_1 - \tilde{H}'_2 = y_0 P_0 \quad (4.24)$$

In these formulas T is simply $F_1 - F_2 - \mathbf{y} \cdot \mathbf{P}$ and F_1, F_2 are given by (4.7).

At this stage, *invariance under time translation and space rotations is manifest, which demonstrates, according to (4.8), that axiom d) of Section 2 is actually satisfied*

Since P_0 and $y_0 P_0$ are constants of the motion, the wave eqs (4.17)(4.18) undergo a reduction. The dependence on the "relative time" is factorized out by diagonalization of these quantities. If E is eigenvalue of P^0 , we write

$$\tilde{\Psi}'(z, Q) = \exp i(EQ^0 + \frac{\nu}{E}z^0) \Phi(\mathbf{z}, \mathbf{Q})$$

We are left with 6 degrees of freedom.

Remark

The line followed from eq(4.10) to eqs (4.15)(4.24) was the most natural from a heuristic or pedagogic point of view. Let us however stress that the possibility of explicitly computing operators of the form $e^{iB} A e^{-B}$ without question about the self-adjointness of B is essentially founded on the fact that a multiple commutator

$$[B^{[n]}, A] = [B[B \dots [B, A] \dots]]$$

vanishes for some finite integer n . Beside the case of squared-mass operators, a trivial example is given by P_0 and M_{ij} . Obviously P'_0 and M'_{ij} exist and simply $P'_0 = P_0$, $M'_{ij} = M_{ij}$. It stems from (4.8) that J' exists and $J' = J$. But we have previously seen that $\tilde{J} = J$, thus finally $\tilde{J}' = J$. So the Lie algebra generated by the isometries of $g_{\mu\nu}$ is kept unchanged under transformations $A \rightarrow \tilde{A} \rightarrow \tilde{A}'$.

In our present knowledge, there is no evidence that B is essentially self-adjoint on some dense domain in $\overline{L}^2(\mathbf{R}^8)$. This difficulty is not specific of a curvature field. We have already pointed it out, in the case where the external field is electromagnetic. In the present state of the art, we cannot discard the possibility that A' may not exist or may be ambiguously defined, *when A is an operator other than those we explicitly consider here*.

For instance can we actually transform the generator of a Lorentz rotation mixing space and time ? Fortunately we are not concerned with such an operator because it does not leave the metric (3.9) invariant and therefore is not a symmetry of our system.

All we need is to represent the pieces of Poincaré algebra which survive the application of a curvature field. And in fact the Lie algebra generated by H_1, H_2 and the total isometries of $g_{\mu\nu}$ is respected by the transformation (4.9).

Notice that $\tilde{P}_0 = P_0$ and if $M = (q \wedge p)_1 + (q \wedge p)_2$ then $\tilde{M}_{ij} = M_{ij}$.

In view of these considerations, a rigorous exposition should start with (4.17)(4.18) from the outset, $\tilde{H}'_1$ and $\tilde{H}'_2$ being given by (4.23)(4.24). Then checking that $\tilde{H}'_1$ and $\tilde{H}'_2$ commute to each other is a mere exercise. In this approach we postulate the wave equation in the form (4.12). The squared mass operators are explicitly given by application of (4.23)(4.24).

For the sake of the principles formulated in Section 2, we may observe that it is always possible to construct a representation in terms of operators acting in L^2 , making the change of wave function $\tilde{\Psi} = \Gamma^{-1}\tilde{\Psi}'$ and the operator transformation $\hat{H}_a = \Gamma^{-1} \tilde{H}'_a \Gamma$ which maps $\overline{L}^2$ back onto L^2 . (Beware that $[\Gamma, B]$ doesnot vanish. H not coincide with $\hat{H}$).

But only the external-field representation is tractable.

No-field limit

In the absence of curvature, B doesnot vanish, but it reduces to

$$b = (y_T \cdot P_T) L = -\mathbf{y} \cdot \mathbf{P} \frac{z^0}{P^0}$$

which has been proved to be essentially self-adjoint in some dense domain of $\overline{L}^2$ [13]. Thus all operators of the form $e^{ib} A e^{-ib}$ are well-defined and the no-field limit of the external-field representation is unitarily equivalent to the customary representation describing two interacting particles in flat spacetime.

However, the no-field limits of $\tilde{H}'_1, \tilde{H}'_2$, say $\tilde{H}'^{(0)}_1, \tilde{H}'^{(0)}_2$ do not show up the familiar form one usually encounters for a two-body system isolated in Minkowski space. In particular their Poincaré invariance arises "in disguise" owing to the fact that the generators of (flat) spacetime displacements are not represented in the conventional fashion.

But one may easily revert to the primitive representation by the transformation

$$H_a^{(0)} = \Gamma^{-1} e^{-ib} \tilde{H}'^{(0)}_a e^{ib} \Gamma$$

This peculiarity is not characteristic of gravity. It already occurs when one turns off an external field in special relativity [10,13].

Independent-particle limit

The case of two independent particles moving in curved spacetime without mutual interaction is more tricky. The wave equations of the external-field representation are given in terms of $\tilde{K}'_1, \tilde{K}'_2$. But there is no rigorous evidence that these operators be unitarily equivalent to $\tilde{K}_1, \tilde{K}_2$. However, the important structure here is the

Lie algebra generated by K_1, K_2 and the *individual isometries* of the form $J_1 = -i\xi^\alpha \partial_{1\alpha}$, $J_2 = -i\xi^\alpha \partial_{2\alpha}$. We wish to stress that in (4.9) the transformed version of all the individual isometries is well-defined. In fact J_1, J_2 commute with Γ_{12} and with B , thus they keep their form unchanged under the transformations generated by these operators.

Finally the commutation relations among $\tilde{K}_1, \tilde{K}_2, P^0, M_{ij}$ are not altered when going over to $\tilde{K}'_1, \tilde{K}'_2, P^0, M_{ij}$.

5. Conclusion

The equations of motion proposed here fully take into account the two-body structure of the system. Hence some questions raised by Fishbach *et al.* are answered. In principle they could permit to formulate as well the scattering of a bound state *by* the gravitational field (with some reservations about the intrinsic meaning of this concept, we may return to this question in future work) as the $1+1$ scattering of two particles *in the presence of* curvature. They also should allow for investigating changes in the spectrum of bound states.

Unfortunately it is generally not possible to write these equations explicitly; exact solutions of the compatibility condition are exceptional, they require special symmetry of the gravitational field. But this complication is by no means a characteristic peculiarity of the gravitational field; it arises as well in special relativity when we consider two bodies interacting in an external field, as soon as recoil effects are taken into account.

Whereas, in the absence of curvature, an essential structure was the Lie algebra spanned by the squared-mass operators and the generators of Poincaré group, here we have the algebra spanned by H_1, H_2 and the total isometries (if any).

In addition to compatibility, a list of physically reasonable conditions has been discussed. In order to implement these requirements in a tractable model, we have considered a special case of static orthogonal spacetime. In this example, the existence of a preferred flat background allows for regarding the curvature field as an external field in Minkowski spacetime. In a representation using Hilbert space $\bar{L}^2$, we have applied standard methods of relativistic quantum mechanics of particles, and explicitly solved the compatibility problem in a way which respects the symmetries of the Riemannian metric. Axioms a) b) d) are strictly satisfied. Axiom c) is rigorously satisfied insofar as the no-field limit is concerned, and (at least) formally satisfied in the case of the independent-particle limit.

Of course, the example given in Section 4 is not very useful for practical applications because no reasonable stress-energy tensor is expected to support a line element of the form (3.9) through Einstein equations. For this reason we prefer to speak of a curvature field rather than of a gravitational field.

Nevertheless one must realize that no other solution of the compatibility condition (1.2) is available in closed form at the present time, except in the trivial case of Minkowski space.

The present example is at least nontrivial as it has a nonzero curvature. Our model, being defined in closed form, ensures that the general framework developed in Section 2 is not empty. Moreover it is an illustration of the role of isometries.

If one wishes to consider realistic spacetimes, it is possible to seek for solutions in the form of series expansions. In this article, for simplicity, we have considered scalar particles only. The general framework can be set forth also for fermions, provided K_1, K_2 are replaced by Dirac operators and the right-hand side of (1.1) become linear in the masses. But adapting to fermions the method displayed in Section 4 requires further work.

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Appendix

In standard notations of differential geometry, the four-dimensional Laplace- Beltrami operator is

$$\Delta = \delta d + d\delta$$

Claim:

If ξ is an isometry and f a differential form, then $\mathcal{L}_\xi \Delta f = \Delta \mathcal{L}_\xi f$.

For a p -form, the general proof rests on the identities $\delta = *^{-1}d*$ and $*\mathcal{L} = (-1)^p \mathcal{L}*$. When f is a function, that is a 0-form, we simply have

$$2Kf = \Delta f = \delta df$$

Claim: If ξ is an isometry, then $\xi^\alpha \partial_\alpha$ is skew-hermitian in $L^2(\mathbf{R}^4)$.

Proof

$$\int_{\Omega} (\xi^\alpha \partial_\alpha \phi^*) \psi \sqrt{|g|} d^4 = \int_{\Omega} \partial_\alpha (\xi^\alpha \phi^* \psi) \sqrt{|g|} d^4 q - \int_{\Omega} (\xi^\alpha \partial_\alpha \psi) \phi^* \sqrt{|g|} d^4 q - \int_{\Omega} (\partial_\alpha \xi^\alpha) \phi^* \psi \sqrt{|g|} d^4 q$$

The first term gets transformed into the flux of $\xi^\alpha \phi^* \psi$ accross the three dimensional surface $\partial\Omega$. In the limit where Ω extends to infinity *in all spacetime directions* then ϕ and ψ vanish on $\partial\Omega$ (they are supposed to be squared integrable), therefore this first term yields no contribution. The last term also vanishes since $\partial\xi$ is obviously zero. We are left with

$$\int (\xi \cdot \partial \phi^*) \psi \sqrt{|g|} d^4 q = - \int \phi^* (\xi \cdot \partial \psi) \sqrt{|g|} d^4 q$$

in other words, $\xi^\alpha \partial_\alpha$ is skew-symmetric in L^2 .

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