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Dilepton production in a chemically equilibrating, expanding and hadronizing quark-gluon plasma

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Abstract

Dilepton production is considered within a complete dynamical framework for thermalized matter assumed to be formed in ultra-relativistic heavy-ion collisions. Our model includes (i) chemical equilibration processes in the initially gluon enriched plasma, and (ii) longitudinal and transversal expansion, and (iii) the hadronization through a mixed phase. Besides the basic electromagnetic quark - antiquark annihilation process we also take into account the QCD Compton like and annihilation processes for calculating the dilepton rate in the deconfined phase, while in the hadronic stage we employ a parametrization of the effective form factor which is based on the complete set of meson decays and reactions. We find that, due to the transverse expansion of the matter, the dilepton yield from the hadron gas is strongly reduced and, therefore, the deconfined matter gives the dominant contribution in case of initial conditions which are expected to be achieved at RHIC. This provides the basis for the $M_1$ scaling restoration of the dilepton spectra from thermalized matter.
I. INTRODUCTION

The current ultra-relativistic heavy-ion experiments at CERN - SPS and the planned future experiments at Brookhaven RHIC and CERN - LHC are permanently stimulating the interest in studying processes which can provide penetrating probes from the hot and dense reaction zone. Intriguing questions concern the possibility to create thermalized deconfined matter in these heavy-ion collisions and the identification of such a novel matter state. In this respect the emission of lepton pairs (and real photons) in the high invariant mass region is particularly interesting since these messengers are thought to measure directly the initial temperature of the thermalized quark-gluon matter [1,2]. Also well known is the fact that the use of electromagnetic probes for getting reliable information on the initial stage of the quark-gluon plasma needs to solve at least two theoretical problems, namely (i) to clarify the space - time evolution of the matter which is intimately related to the kinetical equilibration under the specific conditions of high-energy heavy-ion collisions, and (ii) to find out the competing contributions from other sources of dileptons, such as the hard Drell Yan processes and the soft hadronic processes in confined matter, to the full spectrum.

The theoretical investigations in both directions have achieved considerable progress during the last few years. Based on parton cascade models [3,4] the decisive rôle of mini-jets in producing a highly excited medium of quarks and gluons has been realized. In particular, from these studies one learned that the dense deconfined matter is strongly gluon dominated [5]. Relying on a kinetic rate equation approach the subsequent chemical equilibration of the initially gluon enriched plasma has been analyzed [6-8]. The importance of very early pre-equilibrium states of parton matter for the electromagnetic probes has been shown in the framework of various kinetic models [9,10] and by simulations [11,12]. At the same time the detailed consideration of the light meson decays and reactions with dilepton production in a hot hadron gas [13] including the in-medium effects [14] seems to show a substantial change in the dilepton rate in comparison with the previous simple $\rho$ pole approximation of the electromagnetic form factor. All these developments give rise to the necessity to refine the standard estimates of the dilepton probe of deconfined matter, which, as a rule, relied on chemical equilibrium in the framework of scaling hydrodynamics and did not take into account the above quoted improvements of a detailed understanding.

The aim of the present paper is to study the emission of lepton pairs with high invariant mass $M$ within a complete dynamical model for the thermalized matter formed in ultra-relativistic heavy-ion collisions. The space - time evolution of the matter starts with the chemical equilibration process of the initially gluon enriched plasma accompanied by the collective longitudinal and transversal expansion and proceeds to the hadronic stage through a mixed quark - hadron phase. Since in the gluon plasma the $\alpha^2 \alpha_s$ order processes of dilepton creation might become important we include them in our consideration together with the basic electromagnetic $q\bar{q}$ annihilation. The relation of thermal dileptons
from deconfined matter to such ones from the hadron gas is quantified in order to find out the most favorable window in the dilepton spectra to search for signals of the quark-gluon plasma.

We pay especially attention to the transverse momentum \( (q_\perp) \) dependence of the dilepton spectra at fixed, high transverse mass \( M_\perp = \sqrt{M^2 + q_\perp^2} \). Under certain conditions the dilepton yield from a local equilibrium quark-gluon plasma is known to depend only on \( M_\perp \) [15,16], i.e., the thermal spectrum from the plasma exhibits the so-called \( M_\perp \) scaling. This scaling is now widely discussed as a possible signal of the quark-gluon plasma formation [17]. In particular the influence of the non-thermal parton matter on the \( q_\perp \) dependence of the dilepton spectra has been considered in Refs. [18,19], where it has been shown that the early off-equilibrium stage might affect the \( M_\perp \) scaling. On the other hand the \( M_\perp \) scaling is also not valid for the dilepton emission from hadron matter because of the specific dependence of the electromagnetic form factor on the invariant mass. Recently [20], the relative contributions from deconfined and confined matter have been analyzed by using the above mentioned new findings on the light meson decays and reactions with dilepton production [13,14]. However, these analyses are performed within a simplified scenario where both the transverse expansion and chemical evolution of the matter are not taken into account. As we shall show below the transverse expansion plays a crucial rôle for the relative weights of different matter stages. While the deconfined matter stage is hardly affected by transverse expansion, it reduces drastically the life time of the hadron stage and consequently the hadronic contributions to the thermal dilepton spectrum. This provides the basis of a \( M_\perp \) scaling restoration effect, at least for dileptons from thermalized matter expected to be produced in heavy-ion collisions at RHIC energies.

Our paper is organized as follows. In section II we consider the dilepton production processes. The rate in deconfined matter includes QCD Compton like and annihilation processes. To regularize infra-red divergencies in these rates we employ a cut-off parameter which is determined by the effective thermal parton mass. The dilepton rate in the high invariant mass region appears to depend on the cut-off parameter only weakly; this seems to justify our approximation. Also the dilepton rate from hot hadron matter is discussed in this section, and the space-time integrated rate, suitable for numerical calculations, is presented. In section III we describe our dynamical model of the chemically equilibrating gluon-enriched plasma accompanied by the longitudinal and transverse hydrodynamical expansion. The latter one is treated within a new global hydrodynamical scheme developed recently [7]. The results are used in section IV to analyze the relative contributions from different stages of thermalized matter to the thermal dilepton yield. The transverse momentum dependence of the dilepton spectrum at fixed \( M_\perp \) is also considered here in order to clear up whether the \( M_\perp \) scaling can be realized for dileptons emitted from the hot matter. Our conclusions are drawn in section V.
II. DILEPTON RATES

This section provides the presentation of dilepton producing processes which are expected to dominate in thermalized, strongly interacting matter.

A. Dileptons from deconfined matter

The majority of earlier estimates of the dilepton emission from equilibrium quark-gluon plasma has been based on the lowest order process $q\bar{q} \rightarrow \gamma^* \rightarrow ll$, i.e., the electromagnetic annihilation of quarks and antiquarks, which gives the rate per space-time volume $d^4x$ and four-momentum volume $d^4Q$ of the pair by

$$ \frac{dN_{ll}}{d^4x \, d^4Q} = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} f_1(p_1, x) f_2(p_2, x) \, \sigma(M^2) \, \delta^{(4)}(p_1 + p_2 - Q). \quad (1) $$

The pair four-momentum is $Q^u = (M_{\perp}Y, M_{\perp}ShY, l_\perp)$ (with $M_{\perp}^2 = M^2 + q_{\perp}^2$ as the transverse mass, and $M$ is the invariant mass; we focus on midrapidity pairs with $Y = 0$). The quantities $f_\pm(p_\pm, x)$ denote the quark (antiquark) distribution functions with four-momentum of the partons $p_\pm^u = (E_\pm, \vec{p}_\pm)$. The relative quark-antiquark velocity reads $v = M/2E_1E_2$. In case of two light quark flavors $u,d$ the annihilation cross section takes the form $\sigma(M^2) = \frac{4\pi\alpha^2}{3M^2}$ (with $\alpha = 1/137$). Since in the gluon enriched plasma the processes of dilepton creation with gluons can become important we include in our approach also the QCD processes (i) annihilation $q + \bar{q} \rightarrow g + ll$ and (ii) the Compton like reaction $q + g \rightarrow q + l\bar{l}$ (and the corresponding process with $\bar{q}$). It is convenient to express the contributions from each of these processes to the total rate in the form

$$ \frac{dN_{ll}^{(i)}}{d^4x \, d^4Q} = \frac{1}{(2\pi)^8} \int \int \frac{d^3p_1 \, d^3p_2 \, d^3p_3}{2E_1 \, 2E_2 \, 2E_3} f_1(p_1, x) f_2(p_2, x) \left[ 1 \pm f_3(p_3, x) \right] \times $$

$$ \delta^{(4)}(p_1 + p_2 - p_3 - Q) \, |M_i|^2, \quad (2) $$

where $M_i$ represent the amplitudes for the corresponding processes. The factor $[1 \pm f_3]$ accounts for the Pauli suppression or Bose enhancement related to the phase space occupation seen by the outgoing parton. In case of massless partons the QCD perturbative calculations up to order $\alpha^2\alpha_s$ yield [21,22]

$$ |M_{\text{ann}}|^2 = \frac{5}{9} \frac{2^9\pi\alpha^2\alpha_s}{3M^3} \frac{u^2 + t^2 + 2sM^2}{ut}, \quad (3) $$

$$ |M_{\text{Com}}|^2 = \frac{5}{9} \frac{2^9\pi\alpha^2\alpha_s}{3M^3} \frac{u^2 + s^2 + 2tM^2}{-us}. \quad (4) $$

Here $s, t, u$ are the usual Mandelstam variables with $s + u + t = M^2$. The summation over the spin and color states of the $u,d$ quarks is included in the expressions for the amplitudes.

Integration over the momenta of the incoming particles in eq. (1) gives for the basic process the rate [23,24]
\[
\frac{dN_{\gamma\gamma}^{\text{abs}}}{d^4 x d^4 Q} = \frac{\alpha^2}{4\pi^4} F_q \exp \left(-\frac{E}{T}\right) \lambda_q \lambda_{\bar{q}}
\] (5)

with \(F_q = 5/9\). \(E = Q_u u^\mu\) is the energy of the lepton pair in the local rest frame of a volume element which flows with four-velocity \(u_\mu\). We restrict ourselves to high-energy dileptons with \(E \gg T\) and utilize the Boltzmann approximation for all partons, i.e.

\[
f_\alpha(p_\alpha, x) = \lambda_\alpha(x) \exp \left(-\frac{p_\alpha^2 u_\mu}{T}\right).
\] (6)

This means that the parton distribution is assumed to be in thermal (i.e., momentum space) equilibrium, but not necessarily in chemical equilibrium. Only for chemical equilibrium the fugacities \(\lambda_\alpha\) become 1.

The rates for the annihilation and Compton like processes (2) can be simplified in the same way as has been done in Ref. [25] for the photon production rate. As result one gets for the high-energy dileptons

\[
\frac{dN_{\ell\ell}^{(i)}}{d^4 x d^4 Q} = \frac{1}{(2\pi)^6} \frac{T}{16Q} \exp \left(-\frac{E}{T}\right) \int \frac{ds}{s-M^2} \times
\] (7)

\[
\left[ \exp \{-(s-M^2)\beta_+\} - \exp \{-(s-M^2)\beta_-\} \right] \int dt |M_i(s, t)|^2 \bar{L}^{(i)}
\]

where \(Q = \sqrt{E^2 - M^2}, \beta^{\pm} = 2T(E \pm Q), \) and \(L^{\text{com}} = \lambda_q(\lambda_q + \lambda_{\bar{q}}), \) and \(L^{\text{ann}} = \lambda_q \lambda_{\bar{q}}.\)

Here we also employ the Boltzmann approximation for the parton distribution functions and consider charge symmetric matter with \(\lambda_q = \lambda_{\bar{q}}.\)

For massless particles the \(t\) integration in eq. (7) is infrared divergent. In an equilibrium quark-gluon plasma one might expect that the Kinoshita Lee Nauenberg theorem is valid. That is the self-energy and vertex corrections lead to cancellations of divergencies, and the full rate up to order \(\alpha^2 \alpha_s\) is finite [26–29]. For a chemical off-equilibrium plasma and for arbitrary invariant dilepton masses this problem needs further separate investigations. In particular the application of the Braaten Pisarski perturbation technique [30] has been used to regularize the photon rate in Ref. [25]; however, in case of dileptons one seems to be faced with additional difficulties in the high invariant dilepton mass region [31]. Otherwise, many-body effects in the QCD plasma certainly give rise to the appearance of a finite effective thermal parton mass which appears to have an interrelation to non-perturbative interaction effects [32]. Such an effective mass can also play a rôle as cut-off parameter \(k_c\) which regularizes the dilepton rate. Since we focus on high-energy lepton pairs with \(E \gg T\) their emission turns out to depend on \(k_c\) not too sensitively. Therefore, we do not need to know the precise value of \(k_c\). With this in mind we employ for our purposes a variant of Ref. [25] with \(k_c^2 = \frac{4}{3}\pi \alpha_s T^2\), which approximately coincides with the thermal mass extracted from lattice data [32]. Throughout the present work we utilize \(\alpha_s = 0.3\). Inserting this cut-off parameter \(k_c^2\) in the \(t\) channel and integrating in eq. (7) within the boundaries \(s \geq M^2 + 2k_c^2\) and \(- (s - M^2) + k_c^2 \leq t \leq -k_c^2\) we find in the limit \(k_c^2 \ll M^2\)
\[
\frac{dN_{\text{Com}}^{C}}{dx \, dQ^2} = \frac{2\alpha_s \, T^2}{\pi^5 \, 27 \, M^2} \exp \left( -\frac{E}{T} \right) L_{\text{Com}} \, K_{\text{Com}}(E, M, k_c^2),
\]

\[
K_{\text{Com}} = \frac{E}{Q} \log \left( \frac{E + Q}{E - Q} \right) + 1 - C_E + 2 \log \left( \frac{2MT}{k_c^2} \right) + \frac{3 \, M^2}{4 \, TQ} \, I_{1}^{\text{Com}} + \frac{M^2}{2TQ} \, I_{2}^{\text{Com}},
\]

\[
I_{1}^{\text{Com}} = \int_{2k_c^2}^{\infty} \frac{dx}{x + M^2} \left[ \exp(-\beta_+ x) - \exp(-\beta_- x) \right],
\]

\[
I_{2}^{\text{Com}} = \int_{2k_c^2}^{\infty} \frac{dx}{x + M^2} \log \left( \frac{k_c^2}{x - k_c^2} \right) \left[ \exp(-\beta_+ x) - \exp(-\beta_- x) \right],
\]

\[
\frac{dN_{\text{ann}}^{\text{ann}}}{dx \, dQ^2} = \frac{2\alpha_s \, T^2}{\pi^5 \, 27 \, M^2} \exp \left( -\frac{E}{T} \right) L_{\text{ann}} \, K_{\text{ann}}(E, M, k_c^2),
\]

\[
K_{\text{ann}} = \frac{E}{Q} \log \left( \frac{E + Q}{E - Q} \right) - 2(1 + C_E) - \frac{M^2}{TQ} \, I_{1}^{\text{ann}},
\]

\[
I_{1}^{\text{ann}} = \int_{2k_c^2}^{\infty} \frac{dx}{x^2} \log \left( \frac{k_c^2}{x - k_c^2} \right) \left[ \exp(-\beta_+ x) - \exp(-\beta_- x) \right],
\]

for the QCD Compton and annihilation rates, respectively (\(C_E\) is the Euler constant).

**B. Dileptons from hadron matter**

The dilepton emission rate from the hadron gas is usually given in a way which is analog to eq. (5) [23]

\[
\frac{dN_{\text{had}}^{\text{had}}}{dx \, dQ^2} = \frac{\alpha^2}{48\pi^2} \left[ 1 + \frac{2m_l^2}{M^2} \right] \left[ 1 - \frac{4m_\pi^2}{M^2} \right] F_h \exp \left( -\frac{E}{T} \right)
\]

with the \(\rho\) pole approximation in the pion electromagnetic form factor \(F_h(M^2)\) (and \(m_{l,\pi}\) stand for the lepton and pion mass, respectively). This approximation has been recently re-examined by a detailed consideration of the complete set of light mesons which are expected to contribute to the dilepton rate in hot hadronic matter [13,14]. Such a modification of the dilepton rate can be accumulated in an effective form factor \(F_{\text{eff}}(M^2)\) [20] by means of the replacement \(F_h \rightarrow F_{\text{eff}}\) in eq. (10). Our parametrization of the effective form factor is taken from Fig. 1 in Ref. [20]. Whenever needed we continue this form factor as constant into the regions \(M < 0.3\) GeV and \(M > 3\) GeV. Please notice the tremendous difference of the current effective form factor and the \(\rho\) pole approximation in the region \(M > 1\) GeV. We shall include the effective form factor \(F_{\text{eff}}\) in our calculations, however, a few remarks are in order. First, it should be stressed that \(F_{\text{eff}}\) in Refs. [13,14,20] originally accounts for the electron - positron production. The muon pair production, in particular in the high invariant mass region, can be forbidden in some cases due to obvious kinematical reasons. Second, the influence of the in-medium effects on the dilepton yield in dense and hot hadronic matter is still an quantitatively unsettled issue (cf. [14] for a recent discussion of the state of the art). Therefore, the reliability of any model needs to be confirmed by future experiments. Nevertheless, it is useful to take
into account the effective meson form factor described above as a rather extreme variant as compared to the standard $\rho$ pole approximation.

**C. The total yield**

Our calculations below are based on a space-time evolution model for the transverse expansion superimposed on the longitudinal boost-invariant scaling expansion within a hydrodynamical framework. The four-velocity then reads

$$u^\mu = \gamma_\perp (\text{ch}\eta, \text{sh}\eta, v_\perp \cos \varphi, v_\perp \sin \varphi), \quad \gamma_\perp = \frac{1}{\sqrt{1 - v_\perp^2}}, \quad (11)$$

which gives for the dilepton energy

$$E = Q^\mu u_\mu = M_\perp \gamma_\perp \text{ch}(Y - \eta) - q_\perp \gamma_\perp \cos \varphi. \quad (12)$$

In the case of high invariant masses of the dileptons, $M \gg T$, it is convenient to use the approximation

$$\exp \left( -\frac{M_\perp}{T} \gamma_\perp \text{ch}(Y - \eta) \right) \approx \sqrt{\frac{2\pi T}{\gamma_\perp M_\perp}} \exp \left( -\frac{M_\perp \gamma_\perp}{T} \right).$$

Then the space-time integrated rate can be expressed as

$$\frac{dN_{\ell\ell}}{dM^2_\perp dq_\perp^2 dY} = \frac{\alpha^2}{\pi^3} \int d\tau dTdr \, d\varphi \left\{ \frac{1}{8} F + \frac{5}{2 \pi^2} \frac{T^2}{M^2} x \left[ L^{\text{ann}} K^{\text{ann}} + L^{\text{Com}} K^{\text{Com}} \right] \right\} \quad (13)$$

$$\times \sqrt{\frac{2\pi T}{\gamma_\perp M_\perp}} \exp \left( -\frac{\gamma_\perp}{T} \left[ M_\perp - q_\perp v_\perp \cos \varphi \right] \right),$$

with the replacement $E \to \gamma_\perp (M_\perp - q_\perp v_\perp \cos \varphi)$ in the expressions for $K^{\text{Com,ann}}(E, M, k_c^2)$. The form factor $F$ in eq. (13) reads

$$F = \begin{cases} \frac{\hat{F}_q}{x} + (1 - x) \frac{1}{12} F_{\text{eff}} & \text{for } T > T_c, \\ \frac{1}{12} F_{\text{eff}} & \text{for } T = T_c, \\ \frac{1}{12} F_{\text{eff}} & \text{for } T < T_c, \end{cases} \quad (14)$$

where $x$ is as usual the relative weight of the deconfined matter in the mixed phase ($x = 1$ in the pure deconfined phase), and $\hat{F}_q = F_q \lambda^2$. The numerical integration in eq. (13) is based on the global hydrodynamical scheme developed previously [7] and discussed below.

**III. CHEMICAL EQUILIBRATION AND SPACE-TIME EVOLUTION**

The space-time evolution of the matter is closely connected with the kinetic processes of its chemical equilibration. In our approach the charge-symmetric chemical off-equilibrium parton matter obeys the rate equations for the gluon and quark densities [6,7]
\[ \partial_\mu (n_\rho u^\mu) = \frac{1}{2} \tilde{\sigma}^{(3)} (n_\rho^2 (1 - \lambda_\rho) - \tilde{\sigma}^{(2)} (n_\rho^2 - n_\rho^2 b)), \]  
\[ \partial_\mu (n_\sigma u^\mu) = \frac{1}{2} \tilde{\sigma}^{(2)} (n_\rho^2 - n_\rho^2 b), \]  

(15)  
(16)

where \( n_{\rho(\sigma)} \) denotes the gluon (quark) density obtained according to eq. (6), and \( b = (n_\rho^2/n_\rho^2)^2 \) is the squared ratio of chemical equilibrium densities when \( \lambda_{\rho(\sigma)} = 1 \). The rate equations (15, 16) contain the thermally averaged cross sections \( \tilde{\sigma}^{(2)} \) and \( \tilde{\sigma}^{(3)} \) for the reactions \( gg \to q\bar{q} \) and \( gg \to ggg \), respectively. We employ the estimates of Ref. [6], i.e.,

\[ \tilde{\sigma}^{(2)} = \frac{\pi \alpha_s^2 N_f}{48 T^2} \left( \log \frac{81}{2\pi \alpha_s \lambda_\rho} - \frac{7}{4} \right)^2 \Lambda_\eta, \]  
\[ \tilde{\sigma}^{(3)} = 4.2 \alpha_s T \sqrt{2\lambda_\rho - \lambda_\rho^2 n_\rho^{-1}} \Lambda_\eta, \]  

(17)  
(18)

\((N_f = 2\) is the number of flavors\) with the additional factors \( \Lambda_{\eta,\rho} \) which we shall vary in order to mimic higher order processes [33] and possible non-perturbative contributions, and to check how important they are for observables.

The chemical rate equations (15, 16) need to be supplemented by the hydrodynamical equations \( \partial_\mu T^{\mu\nu} = 0 \), where the energy-momentum tensor \( T^{\mu\nu} = \int d^3p E^{-1} p^\mu p^\nu f(p, x) \) is defined by the parton distribution function \( f(p, x) = \sum_{a=\rho,\sigma} d_a f_a(p, x) \) with respect to eq. (6) \( (d_a \) are the corresponding the degeneracies\). The complete set of the kinetic rate equations and the hydrodynamical evolution equations have been solved in our approach under the assumption of boost-invariant longitudinal expansion superimposed on global transverse expansion [7]. The global transverse expansion scheme rests on a linear transverse velocity profile but radially constant intensive quantities. The equations then reduce to two coupled, ordinary differential equations for the energy density and the acceleration of the transverse radius. The chemical evolution equations (15, 16) also become two ordinary differential equations. The detailed derivation can be found in Ref. [7]. In case of photon emission this scheme gives the same results as the full hydrodynamics [34], which is not useful when solving additionally rate equations.

As result the set of equations delivers the temperature \( T(\tau) \), the fugacities \( \lambda_{\rho(\sigma)}(\tau) \), and the transverse velocity \( v_\perp(r, \tau) \) as function of proper time \( \tau = \sqrt{t^2 - z^2} \). Some examples for different initial conditions are displayed in Fig. 1 and discussed in more detail below. In case of the chemical non-equilibrium parton plasma we assume for the initial fugacities \( \lambda_\rho = \frac{1}{3} \lambda_\sigma \). This is in line with the hot glue scenario [5] and is supported by the parton cascade calculations [3, 4, 6, 8]. For the absolute values of the initial gluon fugacities we choose here \( \lambda_\rho = 0.25 \) and \( 0.5 \) which is in between the extreme cases of possible oversaturation [3] and the very dilute glue system in Ref. [6]. The unique initial temperature \( T_0 = 550 \text{ MeV} \) fulfills also approximately the relation \( \tau_0 \sim 1/T_0 \) for \( \tau_0 = 0.32 \text{ fm/c} \). The general conclusion that can be drawn from our considerations of the space-time evolution of the chemically equilibrated or non-equilibrated parton matter is that the effect of the transverse expansion is not very large for the short-living deconfined matter down to temperatures \( T \sim 200 \text{ MeV} \) (only about 10% radius increase; we consider an
initial transverse radius of 7 fm corresponding to central Au + Au collisions). This result is in agreement with the early hydrodynamical calculations [23] including the transverse expansion of the matter. At the same time it should be stressed that the transverse expansion drastically reduces the life time of the hadron (and partially also of the mixed) phase compared to the pure longitudinal hydrodynamics. In the case of our present choice of the initial temperature $T_0 = 550$ MeV the life time of the system up to freeze-out at $T_f = 120$ MeV is shortened up to a factor $1/10$ and the transverse radius increases up to a factor 3, see Table 1. Due to the work of the pressure in transverse direction the cooling is therefore much faster. (At midrapidity $\eta = 0$ the work, done by the pressure in longitudinal direction, would only slightly speed up the cooling in case of non-flat rapidity distributions.)

We use here the standard ideal gas parametrization of the equation of state for the pure deconfined, mixed quark-hadron, and pure hadron phase by the bag model with the bag constant $B^{1/4} = 235$ MeV and massless $u,d$ quarks and gluons and 3 pions. Since the recent lattice calculations indicate that the bag model parametrization gives too large latent heat, we apply here also our previous procedure [7] to keep $T_c$ fixed but reduce the latent heat by scaling up the effective number of degrees of freedom in the hadron phase from 3 to 6.68. This corresponds to reducing the bag constant to $B^{1/4} = 229$ MeV. While in this way the latent heat is reduced by a factor 0.9, the life time of the system up to freeze-out is shortened by a factor 1/2.22 if there would be no transverse expansion. One also observes in Table 1 that the initial phase space undersaturation causes diminished life times too.

IV. NUMERICAL RESULTS

A. Dilepton yields and chemical equilibration

The eqs. (13, 14) for the dilepton rate have been numerically integrated over the space-time evolution of the matter to obtain the different contributions to the spectrum $dN_{ll}/dM^2 d\sigma_{ll}^2 dY$. Integrating over $q_\perp$ results in the invariant mass spectrum $dN_{ll}/dM^2 dY$. Let us first consider the contribution of deconfined matter to this invariant mass spectrum. Since the phase space of the initially gluon enriched plasma is undersaturated, it is important to investigate the influence of the chemical equilibration process on the dilepton spectra. In Fig. 2 the dilepton spectra from deconfined matter are displayed for the same initial conditions as in Fig. 1. The chemical equilibration velocity depends, of course, sensitively on the reaction cross sections $\sigma^{2,3}$ in eqs. (15, 16) [6–8]. We vary $\Lambda_{q,g}$ in a wide range to cover a variety of possible equilibration scenarios, see also Fig. 1. With respect to this variations of $\Lambda_{q,g}$ the temperature dependence of $\alpha_s$ is of minor importance. Faster equilibration causes faster cooling due to the work to be done to populate the parton phase space. While for $\Lambda_{q,g} < 20$ in case of $\lambda_y(\tau_0) \leq 0.5$ the equilibration velocity is too small to achieve chemical equilibrium at confinement temperature,
values of $\Lambda_{\tau, g} \geq 20$ allow for reaching $\lambda_{\tau, g} = 1$ at $T_\tau$ or even earlier (cf. Fig. 1; for more detailed presentations of the chemical evolution scenarios see Refs. [6-8]). In the former case the confinement transition would proceed by a strong off-equilibrium mechanism such as discussed in Ref. [35]. In the latter case a standard model of a quasi-equilibrium phase transition is applicable. Interestingly, at given $\lambda_{\tau, g}(\tau_0)$ these drastically different evolution scenarios result in rather similar dilepton spectra in the high invariant mass region as one can see in Fig. 2 (for photons this has been observed in Ref. [7]). For example, in case of the very undersaturated gluon plasma with $\lambda_g(\tau_0) = 0.25$ the fast equilibration with $\Lambda_{\tau, g} = 20$ gives at $M = 2.5$ GeV a yield which is only by a factor 1.5 larger than the yield in the slow chemical evolution case with $\Lambda_{\tau, g} = 2$. For larger invariant masses the differences are even smaller. The reason for this insensitivity against variations in $\Lambda_{\tau, g}$ is that the heavy dileptons are created in early stages and are less affected by the later evolution, while the low-mass dileptons carry also information on the later low-temperature stage. Comparing the slopes of the spectra in Figs. 2a and b one observes that, despite the same initial temperatures, the spectra for $\lambda_g = 0.5$ appear hotter than the ones for $\lambda_g = 0.25$. The undersaturation of the quark-gluon phase space obviously also reduces the dilepton yield: for the considered electromagnetic annihilation the rate is proportional to $\lambda_g^2$ and, in addition, depends on the temperature history. At $M = 2.6$ GeV the dileptons yields in Figs. 2a and b differ by a factor 6 (5) in case of fast (slow) equilibration.

B. Dileptons from deconfined and confined matter

In what follows we choose the fast equilibration scenario to model the evolution of the matter through the hadron gas up to freeze-out. The separate contributions from the deconfined and confined stages of the thermalized matter to the total invariant mass spectrum are shown in Fig. 3a. The initial conditions are still $T_0 = 550$ MeV, $\tau_0 = 0.32$ fm/c, and $\lambda_g(\tau_0) = 0.5$ as in Figs. 1b and 2b. Since we now include the mixed phase, the yield in the low-mass region from deconfined matter is somewhat increased. Even if the undersaturation of the initial phase space of the gluon enriched plasma leads to a diminished dilepton yield, the contribution from deconfined matter dominates over the one from the hadron gas which is based on the effective electromagnetic form factor $F_{eff}(M)$. This is the direct consequence of the transverse expansion of the matter that reduces strongly the life time of the hadron stage (cf. Table 1). One has to stress that the deconfined matter contribution displayed in Fig. 3a reflects a lower limit for this stage at given initial conditions since we do not include here the contribution from the $\alpha^2\alpha_s$, QCD dilepton production processes that we shall discuss below separately. We utilize here the equation of state with reduced latent heat. If the standard parametrization with full latent heat were employed then the hadron contribution would be stronger by a factor 3.3, i.e., at $M > 2.3$ GeV the deconfined matter yield would exceed the hadron gas yield.

The full yields from thermalized matter are displayed in Fig. 3b for the initially chemical equilibrium quark-gluon plasma (i.e., $\lambda_{\tau, g} = 1$) and the initially undersaturated
gluon enriched plasma with \( \lambda_g(\tau_0) = 0.25 \) or 0.5, and \( \lambda_q(\tau_0) = 0.05 \) or 0.1 together with the Drell Yan contribution estimated in the standard way [36] with \( K \) factor 2. Even keeping in mind that the Drell Yan background probably will be suppressed due to the nuclear shadowing effect [37] one can expect a competition between the dilepton yield from thermalized matter and the Drell Yan background in the region \( M \approx 2 - 3 \) GeV if the parton matter is initially undersaturated. The spectra for \( \lambda_g(\tau_0) = 0.25 \) and 0.5 differ by a factor 2 (6, 7) at \( M = 1.1 \) (3, 3.5) GeV. For both initial conditions there are hadron gas contributions up to \( M \sim 2.3 \) GeV seen as weak structures. Such hadron gas contributions become negligible for \( \lambda_{q,g} = 1 \). The yields from initially fully saturated matter are larger by a factor 47 (15) for \( M = 3.5 \) (2) GeV, compared to the most likely initial condition \( \lambda_g(\tau_0) = 0.5 \). Due to the involved interplay of chemical evolution and the later transverse expansion the spectra have significantly different slopes and absolute normalizations.

C. Transverse momentum spectra

The existence of various sources of the dilepton production in heavy-ion collisions makes it difficult to identify unambiguously the quark-gluon plasma formation when using only the invariant mass spectrum. The detailed information of the transverse momentum spectrum \( dN_{\text{th}}/dM_{\perp}^2 dq_\perp^2 dY \) is expected to allow for a more sensible diagnostic. Under the reasonable assumptions of the local thermalization and predominant longitudinal boost-invariant expansion and the absence of another scale than the temperature, the dileptons from the quark-gluon plasma are known to exhibit the above mentioned \( M_{\perp} \) scaling, i.e., at fixed value of \( M_{\perp} \) the spectrum does not depend on \( q_\perp \). However, the last two conditions are not fulfilled for the hadron gas because of both the complicated mass dependence of the form factor \( F_{\text{eff}}(M^2) \) and the strong transverse expansion in the late hadronic stages. As a result, the \( M_{\perp} \) scaling is in general not valid for the dileptons from the hadron matter. To see whether the \( M_{\perp} \) scaling can be realized for dileptons emitted from thermalized matter, evolving from deconfined matter to the hadron gas, we compare the relative contributions of them to the \( q_\perp \) spectrum at fixed \( M_{\perp} \) in Figs. 4 and 5. Following Ref. [17] we choose \( M_{\perp} = 2.6 \) GeV which is high enough to provide the favorable conditions for testing the early stage and, at the same time, is not too close to the specific \( J/\psi \) pole.

In Fig. 4 the \( q_\perp \) dependence of dileptons is displayed which are emitted from thermalized matter being initially in a chemical equilibrium state with \( T_0 = 550 \) MeV and \( \lambda_g(\tau_0) = \lambda_q(\tau_0) = 1 \). Keeping in mind the importance of the transverse expansion effects we plot separately in Fig. 4b the results of calculations in which the matter obeys only the boost-invariant longitudinal expansion, while Fig. 4a is for our full scenario. One observes in Fig. 4a that, with regard of the transverse flow, the dominant contribution in the spectrum in a wide \( q_\perp \) region stems from the deconfined matter. This effect can be easily explained since the hadron gas, undergoing a strong transverse expansion, cools
much faster compared to the case of a longitudinal expansion alone; therefore, the life time is considerably reduced. At the same time the quark-gluon plasma turns out to be almost insensitive to the transverse flow, as indicated by the nearly $q_\perp$ independent behavior of the transverse momentum spectrum in Figs. 4a and b. As consequence the wholly differential spectrum exhibits an approximate $M_\perp$ scaling in the wide $q_\perp$ region up to 2.3 GeV. The situation is quite different in the case of switching off the transverse expansion (Fig. 4b). Here, for the standard bag model parametrization the hadron gas contribution, which does not exhibit a $M_\perp$ scaling in such a large $q_\perp$ range, dominates the spectrum and, therefore, the memory of the previous scaling property of the deconfined matter contribution is lost.

To check the stability of our results we compare in Figs. 4a and b results for the modified equation of state with reduced latent heat and the standard bag model parametrization with only 3 pions in the hadron phase. As displayed in Fig. 4a the $q_\perp$ dependence of the wholly differential spectrum weakly depends on the change in latent heat. Of course, in reducing the latent heat the mixed phase and the hadronic phase life times are diminished as their contributions to the spectrum. This would become important in case of no transverse expansion: here the reduction of the hadron gas contribution would restore the approximate scaling behavior, see Fig. 4b. (As emphasized in Ref. [20] the apparent flatness of the hadron spectrum below $q_\perp = 1.5$ GeV is accidental and does not reflect the QCD $M_\perp$ scaling.)

As a measure of the $M_\perp$ scaling one can employ, as in refs. [17,18], the ratio

$$\mathcal{R}(q_\perp) = \frac{dN_{\text{dif}} / dM_\perp^2 dq_\perp^2 dY}{dN_{\text{dif}} / dM_\perp^2 dq_\perp^2 dY} |_{q_\perp = 0}$$

(19)

at fixed value of $M_\perp = 2.6$ GeV. For the longitudinally expanding quark-gluon plasma one has ideally $\mathcal{R} = 1$. A value of $\mathcal{R} < 3$ for the approximate $M_\perp$ scaling might be considered still as a signal of deconfined matter formation [17], see also Fig. 4. The simplified scenario with pure longitudinal flow can result in a considerable scaling violation because the mass dependence of the electromagnetic form factor in the dominating hadron phase becomes operative, while our complete dynamical model gives $\mathcal{R} \approx 1 \cdots 2$ for $q_\perp$ up to 2.3 GeV. Considering the former approach as reasonable approximation which is valid such a case, where the initial temperature of the matter is not very much above the deconfinement temperature and the transverse expansion is not too important [20], one expects a $M_\perp$ scaling restoration (i.e., a decreasing value of $\mathcal{R}$ towards unity) with increasing initial temperature of the matter. Such an effect, of course, concerns only the dileptons emitted from thermalized matter. The contributions of the non-thermal sources are considered in Refs. [17,19,18] and we comment it briefly below. Here one can point out only that the high initial temperatures, related to events with high rapidity density of secondary hadrons, are known to be also favorable to overwhelm the hard Drell Yan background.

Fig. 5 displays the same $q_\perp$ dependence of the spectra as Fig. 4 does, but now for the initially undersaturated phase space, i.e., $\lambda_\rho(\tau_0) = 0.5$ and $\lambda_\gamma(\tau_0) = 0.1$. This dilute
parton system leads evidently to a decreasing dilepton yield from deconfined matter. Nevertheless, also in this case the deconfined matter still shines out the hadron gas if the transverse expansion is properly taken into account. As displayed in Fig. 5a the whole dilepton spectrum of the initially off-equilibrium matter exhibits approximately the $M_\perp$ scaling as well. Without transverse expansion the spectrum is dominated by the hadron gas contribution (Fig. 5). The spectrum for the initial conditions $\lambda_\tau = 0.25$ is not displayed separately but we find a hadron gas yield which is increased relatively to the deconfined matter yield. This might be summarized as follows: A stronger initial phase space population significantly increases the contribution from deconfined matter. Due to transverse expansion the latter yield shines out for $\lambda_\tau(\tau_0) \geq 0.25$ and $\lambda_\tau(\tau_0) = 5\lambda_\tau(\tau_0)$ at not too large values of $q_\perp$.

While the transverse expansion reduces the full yield of the dileptons from the hadron gas, also the very shape of the corresponding $q_\perp$ dependence is modified. One observes in Figs. 4 and 5 that the dileptons with large values of $q_\perp$ are not so strongly suppressed as those ones with small transverse momenta. As consequence one can see the persistence of the $\rho$ peak at the kinematical boundary $q_\perp \to M_\perp$. This has been discussed in detail in pioneering investigations within the hydrodynamical model [23] and is attributed to the fact that the energy of the lepton pairs (cf. eq. (12)) for $v_\perp > 0$ at large values of $q_\perp$ and fixed $M_\perp$ can be even less than the pair energy in case of pure longitudinal expansion. Within the thermal approach this means a stronger dilepton production rate. Here we extend this result to initially non-equilibrium matter. The persistence of the rho peak violates, indeed, the $M_\perp$ scaling but (see Figs. 4 and 5) this is only the case for $q_\perp$ near the kinematical boundary and for high transverse mass, while the approximate $M_\perp$ scaling is valid in a rather wide $q_\perp$ region.

At the end of this subsection the following comments can be made on the experimental observation of the $M_\perp$ scaling in the dilepton spectra in ultrarelativistic heavy-ion collisions. As we have shown above the formation of thermal equilibrium deconfined matter provides the approximate scaling behavior that can not be strongly destroyed by the chemical equilibration and subsequent hadronization process if the initial temperature of the matter is high enough but still achievable in RHIC experiments. A plausible way to get such high initial temperature is to choose events with high rapidity density of secondaries [2,23,24]. One can also hope to separate in these events the dileptons emitted from thermalized matter from such ones which are emitted in charmed hadron decays [36]. In this case the $M_\perp$ scaling might be only affected on the parton level by the early non-equilibrium parton sources such as the hard Drell Yan mechanism and the pre-equilibrium parton matter with anisotropic parton distribution [18,19]. Anyway the resulting $q_\perp$ dependence of dileptons should reflect the dominating parton mechanism of the dilepton production. Due to this an experimental study of the $M_\perp$ scaling properties of dilepton spectra at RHIC is useful for the understanding of the early parton evolution in heavy-ion collisions.
D. Contributions from QCD processes

The contributions from the QCD Compton like and annihilation processes of dilepton production to the transverse momentum spectrum are displayed in Fig. 6 together with the basic electromagnetic annihilation process. Since we are interested here mainly in the realization of the $M_\perp$ scaling in the dilepton spectrum emerging from the deconfined matter we choose as above $M_\perp = 2.6$ GeV and consider the $q_\perp$ dependence of these contributions. While the rates of the QCD Compton like and annihilation processes contain the additional scale $k_c$ (i.e., the cut-off parameter, cf. eqs. (8, 9)) the full $q_\perp$ spectra turn out to depend weakly on this parameter in the sense that the $M_\perp$ scaling property is not strongly violated. This is indicated in Fig. 6 for $\lambda_g(\tau_0) = 0.5$ by the plateau in the wide $q_\perp$ region up to 2.2 GeV. Comparing the relative contributions of the basic and the QCD processes we conclude that the account of the last ones will increase the rate roughly by a factor 2 (the actual value depends, of course, on $\alpha_s$). This approximation is also valid for the invariant mass spectrum and appears to be useful for a brief comparison of the deconfined and confined matter contributions to the whole yield. Going into details one notices that, in spite of the fact that for the gluon enriched plasma the quark phase space is much more (i.e., by a factor 5) undersaturated than the gluon phase space, the QCD annihilation process dominates still over the Compton like process unless the kinematical boundary at larger values of $q_\perp$, where the Compton like process becomes strongly scaling violating, see Fig. 6. As immediately evident from eqs. (5, 8, 9), the three discussed contributions from deconfined matter scale as $\lambda_g^2(N^{bas} + \cdots N^{ann} + \cdots (\lambda_g/\lambda_q) N^{Com})$. Indeed, for $\lambda_g(\tau_0) = 0.25$ the curves (not displayed) are shifted down by a factor 1/4.6, in accordance with this estimate. For $\lambda_{g,3} = 1$ the QCD processes are stronger by a factor 16, while the basic annihilation process increases by a factor 21. We consider here again the fast equilibration scenario with $\Lambda_{g,3} = 20$ and allow for transverse expansion and include also the mixed phase.

In case of a very slow equilibration (e.g., $\Lambda_{g,3} = 0.2$, and no phase transition, of course) the yields from the electromagnetic annihilation scale as $100 : 1 : 1/4$ for $\lambda_q = 1, 0.1$ and 0.05. The QCD Compton contribution turns out then to be larger than the QCD annihilation yield, and already at $q_\perp \sim 1.9$ GeV the Compton like process exceeds the basic annihilation process, while the QCD annihilation process is in the average by a factor 1/3 smaller than the electromagnetic annihilation. The scaling violation of the separate contributions is not larger than in the fast equilibration scenario.

V. SUMMARY

In summary we analyze the emission of lepton pairs with high invariant mass within a complete dynamical model of thermalized matter, which is expected to be formed in high-energy heavy-ion collisions. The space-time evolution of the matter starts with the chemical equilibration process of the initially gluon enriched plasma which is accompanied
by both the longitudinal and transversal expansion. After some time the expanding, strongly interacting matter hadronizes. We find that in the deconfined stage the dilepton yield in the high invariant mass region is not very sensitive to the details of the chemical equilibration processes, in particular to the time interval needed for achieving the quark-gluon phase space saturation. Otherwise the absolute yields and the slopes of the invariant mass spectra are rather sensitive to the initial phase space occupancy. The dileptons from the deconfined stage are shown to exhibit the approximate $M_\perp$ scaling even if the deconfined matter undergoes a transverse expansion and QCD processes are included. Keeping in mind the significant reduction of the hadron gas contribution to the dilepton spectrum due to the transverse expansion we conclude that dileptons from thermalized matter should show the approximate $M_\perp$ scaling behavior in a wide $q_\perp$ region if the initial temperature is high enough but still achievable at RHIC conditions. Choosing the events with high rapidity density of secondaries one can hope to subtract the hard Drell Yan background and pairs from charm decays. In this case the $M_\perp$ scaling might be violated only on the parton level and its experimental observation depends on the dominating mechanism of the early parton creation. In this respect we find it useful to perform a dedicated experimental study of the scaling properties of dileptons with high transverse mass to get insight in the early parton matter dynamics.

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TABLE I. Approximate life times $\tau$ and transverse radii $R$ for various initial conditions of phase space saturation but unique initial time $\tau_0 = 0.32$ fm/c and temperature $T_0 = 550$ MeV. The subscripts 1 (2) mean begin (end) of the mixed phase, and $f$ denotes the freeze-out at $T = 120$ MeV.

<table>
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<td>$R_f$ [fm]</td>
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Fig. 1: The time evolution of the fugacities and temperature in the deconfined phase (full lines: $\Lambda_{q,g} = 20$, dashed lines: $\Lambda_{q,g} = 2$. The effect of transverse expansion is negligible.

a (upper panel): $\lambda_g(\tau_0) = 0.25$,
b (lower panel): $\lambda_g(\tau_0) = 0.5$. 
Fig. 2: The dilepton spectrum as function of invariant mass (only the yield from deconfined matter evolving as in Fig. 1 is considered; full lines: $\Lambda_{q,\phi} = 20$, dashed lines: $\Lambda_{q,\phi} = 2$).

a (upper panel): $\lambda_g(\tau_0) = 0.25$,
b (lower panel): $\lambda_g(\tau_0) = 0.5$. 
Fig. 3: The dilepton spectrum as function of invariant mass. The equation of state with reduced latent heat is utilized.

a (upper panel): Contributions from deconfined matter (dashed line) and hadron matter (full line) for $\lambda_2 (\tau_0) = 0.25$.

b (lower panel): The full thermal spectrum for $\lambda_2 (\tau_0) = 1$ (dot-dashed line), 0.5 (full line), and 0.25 (dashed line). The dotted line depicts the Drell Yan background.
Fig. 4: The different contributions to the transverse momentum spectrum (heavy full line) from deconfined matter (dotted line) and hadron matter (dashed line) at $M_\perp = 2.6$ GeV. The initial gluon fugacity is $\lambda_g, q = 1$.

a (upper panel): The bag model equation of state with reduced latent heat as discussed in the text is used, while the dot-dashed line depicts the whole spectrum in case of the standard bag model.

b (lower panel): The same as in (a), but without transverse expansion. The standard bag model is utilized; the dot-dashed line depicts the whole spectrum for the equation of state with reduced latent heat.
Fig. 5: The same as in Fig. 4 but for $\lambda_s(\tau_0) = 0.5$ and $\lambda_q(\tau_0) = 0.1$. 
Fig. 6: The different contributions to the transverse momentum spectrum from deconfined matter including the mixed phase at $M_\perp = 2.6$ GeV (thin line: basic electromagnetic quark - antiquark annihilation, dashed line: QCD quark - antiquark annihilation, dotted line: QCD Compton like processes, heavy full line: sum of these contributions). The system undergoes transverse expansion. The initial fugacities are $\lambda_3(\tau_0) = 0.5$, $\lambda_q(\tau_0) = 0.1$; fast equilibration ($\Lambda_{\pi,\eta} = 20$) is considered.