Valley Instanton in the Gauge-Higgs System

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Abstract

The instanton configuration in the SU(2)-gauge system with a Higgs doublet is constructed by using the new valley method. This method defines the configuration by an extension of the field equation and allows the exact conversion of the quasi-zero eigenmode to a collective coordinate. It does not require ad-hoc constraints used in the current constrained instanton method and provides a better mathematical formalism than the constrained instanton method. The resulting instanton, which we call “valley instanton”, is shown to have desirable behaviors. The result of the numerical investigation is also presented.

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1 Introduction

The instantons in the gauge-Higgs system play important roles in the investigation of the nonperturbative aspects of the theory. One of such nonperturbative phenomena is the baryon number violation process in the electroweak theory [1, 2]. The tunneling between different vacua, which may not be negligible in the TeV region, results in the violation of the baryon number and lepton number conservation via anomalies in the respective currents. This process is best evaluated in the instanton method [3, 4] and its variations [5]. The other is the investigation of the SUSY theories, where the instanton effects are essential in the symmetry breaking [6, 7, 8, 9].

Construction of instanton in these theories is not trivial due to the well-known scaling argument, which shows that among all the configurations of finite Euclidean action, only the zero-radius configuration can be a solution of equation of motion. Consequently, all of the evaluation mentioned above used the so-called constrained instanton formalism [10]. In this formalism, one introduces a constraint in the system so that the finite-radius configuration can be a solution of the equation of motion under the constraint. By integrating over the constraint parameter, one hopes to recover the original functional integral. One problem about this method is that its validity depends on the choice of the constraint: Since in practice one only does the Gaussian integration around the solution under the constraint, the degree of approximation depends on the way constraint is introduced. Unfortunately, no known criterion guarantees the effectiveness of the approximation.

This situation could be remedied once one realizes that what we have near the point-like (true) instanton is the valley. The trajectory along the valley bottom should correspond to the scaling parameter, or the radius parameter of the instanton. As such, the finite-size instanton can be defined as configurations along the valley trajectory. Furthermore, in practical applications we need to incorporate the contribution of the instanton-anti-instanton valley [8, 11]. Thus treating a single instanton as a configuration on the valley provides a means of unifying the approximation scheme. We dub these configurations “valley instanton”.

One convenient way to define the valley trajectory is to use the new valley method [12, 13]. Denoting all the bosonic fields in the theory by \( \phi_\alpha(x) \), the new valley equation for the theory with action \( S[\phi_\alpha] \) is written as follows:

\[
\sum_\beta \int d^4y \frac{\delta^2 S}{\delta \phi_\alpha(x) \delta \phi_\beta(y)} F_\beta(y) = \lambda F_\alpha(x), \quad F_\alpha(x) = \frac{\delta S}{\delta \phi_\alpha(x)},
\]  

(1)
where $\lambda$ is the smallest eigenvalue of $\delta^2 S/\delta \phi\delta \phi$. The eigenvalue $\lambda$ also plays the role of the parameter on the valley trajectory. The field $F_\alpha(x)$ is an auxiliary field that can be eliminated, but is convenient for later analysis.

Unlike the streamline method [14] used before, the new valley equation (1) is a local equation in the functional space, and does not require any boundary condition for the valley itself. Also, it is an extension of the equation of motion, which can be analyzed by using the conventional methods. The strength of the method lies in the fact that it defines the valley trajectory such that the eigenvalue used in the new valley equation is completely removed from the Gaussian integration. In other words, by choosing the eigenvalue in the new valley equation one can remove the unwanted negative, zero or pseudo zero modes from the Gaussian integral. In this sense, it defines a method of extending the ordinary collective coordinate method to the case of negative or psuedo-zero modes.

In the following, we will write down the new valley equation for the gauge-Higgs system and present the analytical and numerical evaluation of the solutions to reveal the properties of the valley instanton.

## 2 New-valley equation in the gauge-Higgs system

We consider the SU(2) gauge theory with one scalar Higgs doublet, which has the following action $S = S_g + S_h$:

$$S_g = \frac{1}{2g^2} \int d^4x \ tr F_{\mu\nu} F_{\mu\nu},$$

$$S_h = \frac{1}{\lambda} \int d^4x \left\{ (D_{\mu}H)^\dagger (D_{\mu}H) + \frac{1}{8} (H^\dagger H - v^2)^2 \right\},$$

where $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - i [A_{\mu}, A_{\nu}]$ and $D_{\mu} = \partial_{\mu} - i A_{\mu}$. The masses of the gauge boson and the Higgs boson are given by,

$$m_w = \sqrt{\frac{g^2}{2\lambda}} v, \quad m_H = \frac{1}{\sqrt{2}} v. \quad (4)$$

The valley equation for this system is given by,

$$\frac{\delta^2 S}{\delta A_{\mu} \delta A_{\nu}} F_{\nu}^A + \frac{\delta^2 S}{\delta A_{\mu} \delta H^\dagger} F_{\mu}^H + \frac{\delta^2 S}{\delta A_{\mu} \delta H} F_{H}^H = \lambda e F_{\mu}^A,$$

$$\frac{\delta^2 S}{\delta H^\dagger \delta A_{\mu}} F_{\mu}^A + \frac{\delta^2 S}{\delta H^\dagger \delta H} F_{H}^H + \frac{\delta^2 S}{\delta H^\dagger \delta H} F_{H}^H = \lambda e F_{H}^H \quad (5),$$

$$F_{\mu}^A = \frac{\delta S}{\delta A_{\mu}}, \quad F_{H}^H = \frac{\delta S}{\delta H},$$
where the integration over the space-time is implicit. The valley is parametrized by the eigenvalue $\lambda_e$ which is identified with the zero mode corresponding to the scale invariance in the massless limit, $v \rightarrow 0$.

To simplify the valley equation, we adopt the following ansatz;

$$A_\mu(x) = \frac{x_\nu \partial_\mu \nu}{x^2} \cdot 2a(r), \quad H(x) = v (1 - h(r)) \eta,$$  \hspace{1cm} (6)

where $\eta$ is a constant isospinor, and $a$ and $h$ are real dimensionless functions of dimensionless variable $r$, which is defined by $r = \sqrt{x^2}/\rho$. We have introduced the scaling parameter $\rho$ so that we adjust the radius of valley instanton as we will see later. The tensor structure in (6) is the same as that of the instanton in the singular gauge. Inserting this ansatz to (5), the structure of $F^A_\mu$ and $F^H_\tau$ is determined as the following;

$$F^A_\mu(x) = \frac{x_\nu \partial_\mu \nu}{x^2} \cdot \frac{2v^2}{\lambda} f^a(r), \quad F^H_\tau(x) = -\frac{v^3}{\lambda} f^h(r)\eta.$$  \hspace{1cm} (7)

By using this ansatz, (6) and (7), the valley equation (5) is reduced to the following;

$$-\frac{1}{r^3} \frac{d}{dr} \left( r \frac{da}{dr} \right) + \frac{3}{r^2} a(a - 1)(2a - 1) + \frac{g^2}{2\lambda} (\rho v)^2 a(1 - h)^2 = \frac{g^2}{\lambda} (\rho v)^2 f^a,$$  \hspace{1cm} (8)

$$-\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{dh}{dr} \right) + \frac{3}{r^2} (h - 1)a^2 + \frac{1}{4} (\rho v)^2 h(h - 1)(h - 2) = (\rho v)^2 f^h,$$  \hspace{1cm} (9)

$$-\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{df^a}{dr} \right) + \frac{4}{r^2} (6a^2 - 6a + 1) f^a + \frac{g^2}{2\lambda} (\rho v)^2 h(h - 1)^2 f^a$$

$$+ \frac{g^2}{\lambda} (\rho v)^2 a(h - 1) f^h = \frac{g^2}{\lambda} (\rho v)^2 \nu f^a,$$  \hspace{1cm} (10)

$$-\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{df^h}{dr} \right) + \frac{3a^2}{r^2} f^h + \frac{1}{4} (\rho v)^2 (3h^2 - 6h + 2) f^h$$

$$+ \frac{6a}{r^2} (h - 1) f^a = (\rho v)^2 \nu f^h,$$  \hspace{1cm} (11)

where $\nu$ is defined as $\lambda_e = v^2 \nu/\lambda$.

In the massless limit, $\rho v \rightarrow 0$, (8) and (9) reduce to the equation of motion and (10) and (11) to the equation for the zero-mode fluctuation around the instanton solution. The solution of this set of equations is the following;

$$a_0 = \frac{1}{1 + r^2}, \quad h_0 = 1 - \left( \frac{r^2}{1 + r^2} \right)^{1/2},$$

$$f^a_0 = \frac{2Cr}{(1 + r^2)^2}, \quad f^h_0 = \frac{Cr}{(1 + r^2)^3/2},$$  \hspace{1cm} (12)

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where $C$ is an arbitrary function of $\rho v$. Note that $a_0$ is an instanton solution in the singular gauge and $h_0$ is a Higgs configuration in the instanton background [1]. We have adjusted the scaling parameter $\rho$ so that the radius of the instanton solution is unity. The mode solutions $f^a_0$ and $f^h_0$ are obtained from $\partial a_0/\partial \rho$ and $\partial h_0/\partial \rho$, respectively.

3 Analytic construction of the valley instanton

In this section we will construct the valley instanton analytically. When $\rho v = 0$, it is given by the ordinary instanton configuration $a_0$, $h_0$, $f^a_0$ and $f^h_0$. When $\rho v$ is small but not zero, it is expected that small $\rho v$ corrections appear in the solution. On the other hand, at large distance from the core of the valley instanton, this solution is expected to decay exponentially, because gauge boson and Higgs boson are massive. Therefore, the solution is similar to the instanton near the origin and decays exponentially in the asymptotic region. In the following, we will solve the valley equation in both region and analyze the connection in the intermediate region. In this manner we will find the solution.

In the asymptotic region, $a$, $h$, $f^a$ and $f^h$ become small and the valley equation can be linearized;

\[ -\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{d}{dr} \right) \frac{d}{dr} a + \frac{4}{r^2} a + \frac{g^2}{2\lambda} (\rho v)^2 a = \frac{g^2}{\lambda} (\rho v)^2 f^a, \]  

(13)

\[ -\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{d}{dr} \right) + \frac{1}{2} (\rho v)^2 h = (\rho v)^2 f^h, \]  

(14)

\[ -\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) f^a + \frac{4}{r^2} f^a + \frac{g^2}{2\lambda} (\rho v)^2 f^a = \frac{g^2}{\lambda} (\rho v)^2 \nu f^h, \]  

(15)

\[ -\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{d}{dr} \right) f^h + \frac{1}{2} (\rho v)^2 f^h = (\rho v)^2 \nu f^h. \]  

(16)

The solution of this set of equations is

\[ a(r) = C_1 r \frac{d}{dr} G_{\mu \nu w} (r) + \frac{1}{\nu} f^a(r), \]  

(17)

\[ h(r) = C_2 G_{\mu \nu} (r) + \frac{1}{\nu} f^h(r), \]  

(18)

\[ f^a(r) = C_3 r \frac{d}{dr} G_{\rho \nu w} (r), \]  

(19)

\[ f^h(r) = C_4 G_{\rho \nu w} (r), \]  

(20)
where $C_i$ are arbitrary functions of $\rho v$ and $\mu_{w,H}$ are defined as $\mu_{w,H} = m_{w,H}\sqrt{1-2\nu}$. The function $G_\mu(r)$ is

$$G_\mu(r) = \frac{\mu K_1(\mu r)}{(2\pi)^2 r},$$

(21)

where $K_1$ is a modified Bessel function. As was expected above, these solutions decay exponentially at infinity and when $r \ll (\rho v)^{-1}$ they have the series expansions;

$$a(r) = \frac{C_1}{(2\pi)^2} \left[ -\frac{2}{r^2} + \frac{1}{2}(\rho m_w)^2 + \cdots \right] + \frac{C_3}{\nu(2\pi)^2} \left[ -\frac{2}{r^2} + \frac{1}{2}(\rho \mu_w)^2 + \cdots \right],$$

(22)

$$h(r) = \frac{C_2}{(2\pi)^2} \left[ \frac{1}{r^2} + \frac{1}{2}(\rho m_w)^2 \ln(\rho m_w rc) + \cdots \right]$$

$$+ \frac{C_4}{\nu(2\pi)^2} \left[ \frac{1}{r^2} + \frac{1}{2}(\rho \mu_w)^2 \ln(\rho \mu_w rc) + \cdots \right],$$

(23)

$$f^a(r) = \frac{C_3}{(2\pi)^2} \left[ -\frac{2}{r^2} + \frac{1}{2}(\rho \mu_w)^2 + \cdots \right],$$

(24)

$$f^h(r) = \frac{C_4}{(2\pi)^2} \left[ \frac{1}{r^2} + \frac{1}{2}(\rho \mu_w)^2 \ln(\rho \mu_w rc) + \cdots \right],$$

(25)

c being a numerical constant $e^{\gamma-1/2}/2$, where $\gamma$ is the Euler’s constant.

Near the origin, we expect that the valley instanton is similar to the ordinary instanton. Then the following replacement of the field variables is convenient; $a = a_0 + (\rho v)^2 \hat{a}$, $h = h_0 + (\rho v)^2 \hat{h}$, $f^a = f_0^a + (\rho v)^2 \hat{f}^a$, $f^h = f_0^h + (\rho v)^2 \hat{f}^h$. If we assume $a_0 \gg (\rho v)^2 \hat{a}$, $h_0 \gg (\rho v)^2 \hat{h}$, $f_0^a \gg (\rho v)^2 \hat{f}^a$ and $f_0^h \gg (\rho v)^2 \hat{f}^h$, the valley equation becomes

$$-\frac{1}{r} \frac{d}{dr} \left( r \frac{d\hat{a}}{dr} \right) + \frac{4}{r^2} (6a_0^2 - 6a_0 + 1)\hat{a} + \frac{g^2}{2\lambda} a_0(h_0 - 1)^2 = \frac{g^2}{\lambda} f_0^a,$$

(26)

$$-\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{d\hat{h}}{dr} \right) + \frac{3}{r^2} a_0^2 \hat{h} + \frac{6}{r^2} (h_0 - 1) a_0 \hat{a} + \frac{1}{4} h_0(h_0 - 1)(h_0 - 2) = f_0^h,$$

(27)

$$-\frac{1}{r} \frac{d}{dr} \left( r \frac{d\hat{f}^a}{dr} \right) + \frac{4}{r^2} (6a_0^2 - 6a_0 + 1)\hat{f}^a + \frac{24}{r^2} (2a_0 - 1) f_0^a \hat{a}$$

$$+ \frac{g^2}{2\lambda} (h_0 - 1)^2 f_0^a + \frac{g^2}{\lambda} a_0(h_0 - 1)f_0^h = \frac{g^2}{\lambda} \nu f_0^a,$$

(28)

$$-\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{d\hat{f}^h}{dr} \right) + \frac{3}{r^2} a_0^2 \hat{f}^h + \frac{6}{r^2} a_0 f_0^h \hat{a} + \frac{1}{4} (3h_0^2 - 6h_0 + 2) f_0^h$$

$$+ \frac{6}{r^2} a_0(h_0 - 1) \hat{f}^a + \frac{6}{r^2} (h_0 - 1) f_0^a \hat{a} + \frac{6}{r^2} a_0 f_0^h \hat{h} = \nu f_0^h.$$

(29)
To solve this equation, we introduce solutions of the following equations;

\[-\frac{1}{r} \frac{d}{dr} \left( r \frac{d\varphi_a}{dr} \right) + \frac{4}{r^2}(6a_0^2 - 6a_0 + 1)\varphi_a = 0, \tag{30}\]

\[-\frac{1}{r^3} \frac{d}{dr} \left( r^3 \frac{d\varphi_h}{dr} \right) + \frac{3}{r^2}a_0^2\varphi_h = 0.\]

They are given as,

\[\varphi_a = \frac{r^2}{(1 + r^2)^2}, \quad \varphi_h = \left( \frac{r^2}{1 + r^2} \right)^{1/2}. \tag{31}\]

Using these solutions, we will integrate the valley equation. We multiply (26) and (28) by \(r\varphi_a\), and multiply (27) and (29) by \(r^3\varphi_h\) then integrate them from 0 to \(r\). Integrating by parts and using (30), we obtain

\[-\varphi_a r \frac{da}{dr} + \frac{d\varphi_a}{dr} r \dot{a} = \frac{g^2}{\lambda} \int_0^r dr' r' \varphi_a \left[ f_0^a - \frac{1}{2}a_0(h_0 - 1)^2 \right], \tag{32}\]

\[-\varphi_h r^3 \frac{d\hat{h}}{dr} + h r^3 \frac{d\varphi_h}{dr} = \int_0^r dr' r'^3 \varphi_h \left[ f_0^h - \frac{1}{4}a_0(h_0 - 1)(h_0 - 2) - \frac{6}{r^2}(h_0 - 1)a_0 \dot{a} \right], \tag{33}\]

\[-\varphi_a r \frac{d\hat{a}}{dr} + \frac{d\varphi_a}{dr} r \dot{a} = \int_0^r dr' r' \varphi_a \left[ \frac{g^2}{\lambda} \nu f_0^a - \frac{24}{r^2}(2a_0 - 1)f_0^a \dot{a} - \frac{g^2}{2\lambda}(h_0 - 1)^2f_0^a - \frac{g^2}{\lambda}a_0(h_0 - 1)f_0^h \right], \tag{34}\]

\[-\varphi_h r^3 \frac{d\hat{a}}{dr} + h r^3 \frac{d\varphi_h}{dr} = \int_0^r dr' r'^3 \varphi_h \left[ \nu f_0^h - \frac{6}{r^2}a_0 f_0^a \dot{a} - \frac{1}{4}(3h_0^2 - 6h_0 + 2)f_0^h \right.
\]

\[\left. - \frac{6}{r^2}a_0(h_0 - 1) f_0^a \dot{a} - \frac{6}{r^2}a_0 f_0^h \right]. \tag{35}\]

First we will find \(\dot{a}\). The right-handed side of (32) is propositional to \((C - 1/4)\) and when \(r\) goes to infinity this approaches a constant. At \(r \gg 1\), (32) becomes

\[-\frac{1}{r} \frac{da}{dr} - \frac{2}{r^2} \dot{a} = \frac{g^2}{3\lambda} \left( C - \frac{1}{4} \right). \tag{36}\]

Then at \(r \gg 1\), \(\dot{a}(r)\) is proportional to \((C - 1/4)r^2\) and \(a(r)\) becomes

\[a = \frac{1}{r^2} - \frac{(\rho v)^2 g^2}{12\lambda} \left( C - \frac{1}{4} \right) r^2 + \cdots. \tag{37}\]

To match this with (22), it must be hold that \(C = 1/4\) when \(\rho v = 0\). When \(C = 1/4\), the right-handed sides of (32) vanishes and \(\dot{a}\) satisfy \(-\varphi_a \dot{a}/dr + \dot{a} \varphi_a/dr = 0\). Hence \(\dot{a}\) is
\( \dot{a} = D \varphi_a \), where \( D \) is a constant. Identifying \( a_0 + (\rho \nu)^2 \dot{a} \) with (22) again at \( r \gg 1 \), we find that \( C_1 + C_3/\nu = -2\pi^2 \) and \( C_3 = -\pi^2 \) at \( \rho \nu = 0 \). In the same manner, \( \dot{h} \), \( \dot{f}^a \) and \( \dot{f}^h \) are obtained. At \( r \gg 1 \), we find

\[
\hat{h} = \text{const.} + \cdots, \\
\hat{f}^a = \frac{g^2}{48\lambda} \left( \frac{1}{4} - \nu \right) r^2 - \frac{g^2}{16\lambda} (1 - 2\nu) + \cdots, \\
\hat{f}^h = \frac{1}{16} (1 - 2\nu) \ln r + \cdots.
\]

(38)

Here const. is a constant of integration. Comparing (23)-(25) with them, we find that \( C_2 + C_4/\nu = 2\pi^2 \), \( C_4 = \pi^2 \) and \( \nu = 1/4 \) at \( \rho \nu = 0 \).

Now we have obtained the solution of the new valley equation. Near the origin of the valley instanton, it is given as \( a = a_0 \), \( h = h_0 \), \( f^a = f_0^a \) and \( f^h = f_0^h \). As \( r \) becomes large, the correction terms become important;

\[
a = \frac{1}{r^2} + \cdots, \\
h = \frac{1}{2r^2} - \frac{(\rho \nu)^2}{16} \ln 2 + \cdots, \\
f^a = \frac{1}{2r^2} - \frac{g^2(\rho \nu)^2}{32\lambda} + \cdots, \\
f^h = \frac{1}{4r^2} - \frac{(\rho \nu)^2}{32} \ln r + \cdots,
\]

(39)

and finally it is given by (22)-(25), where \( C_1 = 2\pi^2 \), \( C_2 = -2\pi^2 \), \( C_3 = -\pi^2 \), \( C_4 = \pi^2 \) and \( \nu = 1/4 \) at \( \rho \nu = 0 \). Let us make a brief comment about the consistency of our analysis. Until now, we have implicitly assumed that there exists an overlapping region where both (13)-(16) and (26)-(29) are valid. Using the above solution, it is found that (13)-(16) are valid when \( r \gg (\rho \nu)^{-1/2} \) and (26)-(29) are valid when \( r \ll (\rho \nu)^{-1} \). If \( \rho \nu \) is small enough, there exists the overlapping region \( (\rho \nu)^{-1/2} \ll r \ll (\rho \nu)^{-1} \). Then our analysis is consistent.

The action of the valley instanton can be calculated using the above solution. Rewriting the action in terms of \( a \) and \( h \), we find

\[
S_g = \frac{12\pi^2}{g^2} \int_0^\infty \frac{dr}{r} \left\{ \left( r \frac{da}{dr} \right)^2 + 4a^2(a - 1)^2 \right\},
\]

(40)

\[
S_h = \frac{2\pi^2}{\lambda}(\rho \nu)^2 \int_0^\infty r^3 dr \left\{ \left( \frac{dh}{dr} \right)^3 + \frac{3}{r^2} (h - 1)^2 a^2 + \frac{1}{8}(\rho \nu)^2 h^2(h - 2)^2 \right\}.
\]

(41)

Substituting the above solution for \( S \), we obtain

\[
S = \frac{8\pi^2}{g^2} + \frac{2\pi^2}{\lambda}(\rho \nu)^2 + O((\rho \nu)^4 \ln(\rho \nu)).
\]

(42)

The leading contribution \( 8\pi^2/g^2 \) comes from \( S_g \) for \( a_0 \), which is the action of the instanton, and the next-to-leading contribution comes from \( S_h \) for \( a_0 \) and \( h_0 \).
4 Numerical analysis

In this section, we solve the valley equation (8)-(11) numerically. We need a careful discussion for solving the valley equation (8)-(11): Since the solution must be regular at the origin, we assume the following expansions;

\[ a(r) = \sum_{n=0}^{\infty} a_n r^n, \quad h(r) = \sum_{n=0}^{\infty} h_n r^n, \]
\[ f^a(r) = \sum_{n=0}^{\infty} f^a_n r^n, \quad f^h(r) = \sum_{n=0}^{\infty} f^h_n r^n, \]

for \( r \ll 1 \). Inserting (43) to (8)-(11), we obtain

\[ a(0) = 1, h(0) = 1, f^a(0) = 0, f^h(0) = 0, \]
\[ a(1) = 0, f^a(1) = 0, \]
\[ h(2) = 0, f^h(2) = 0. \]  

The coefficients \( a(2) \), \( h(1) \), \( f^a(2) \) and \( f^h(1) \) are not determined and remain as free parameters. The higher-order coefficients \( (n \geq 3) \) are determined in terms of these parameters. Four free parameters are determined by boundary conditions at infinity. The finiteness of action requires \( a, h \to 0 \) faster than \( 1/r^2 \) at infinity. This condition also requires \( f^a, f^h \to 0 \).

We have introduced \( \rho \) as a free scale parameter. We adjust this parameter \( \rho \) so that \( a(2) = -2 \) to make the radius of the valley instanton unity. As a result we have four parameters \( h(1), f^a(2), f^h(1) \) and \( \rho v \) for a given \( \nu \). These four parameters are determined so that \( a, h, f^a, \) and \( f^h \to 0 \) at infinity.

A numerical solution is plotted in Fig.1 and Fig.2 for \( \rho = 0.1 \) and \( \lambda/g^2 = 1 \). Fig.1 is for the region near the origin and Fig.2 is for the asymptotic region. This behavior of the numerical solution agrees with the result of the previous section. In fact, if we plot the numerical solution and the instanton solution (12) with \( C = 1/4 \) near the origin, these two lines completely overlap with each other at the present scale. In the same way, the solution in Fig.2 and the analytic solution (22)-(25) also overlap with each other.

The values of the action at \( \rho v = 0.001, 0.005, 0.01, 0.02, 0.05 \) and 0.1 are plotted in Fig.3. The solid line shows the behavior of the action of the analytical result (42). This figure shows that our numerical solutions are quite consistent with (42).

The relation between the scaled eigenvalue \( \nu \) and \( \rho v \) is plotted in Fig.4. We find the tendency that \( \nu \to 1/4 \) as \( \rho v \to 0 \). (For very small \( \rho v \), the values of \( \nu \) are different from 1/4.
This, however, should not be taken seriously due to the numerical difficulties.) This result also confirms the analysis of section 3.

We summarize all the numerical data in the Table 1.

5 Conclusions

In this letter we have presented a construction of the finite size instanton in the SU(2) gauge-Higgs system. We used the new valley method for this purpose. From the analysis both analytic and numerical, we have established the existence of the solution, which has the same tensor structure as the point-like instanton, but has a finite radius.

We have found that the deformation of the valley instanton is smaller than that of the constraint instanton. For example, there appears a \( O(\rho^2) \) correction to \( a(x) \) in the constraint instanton [10], while there is no \( O(\rho^2) \) correction in the valley instanton. This infers that the behavior of the large-radius valley instanton is quite different from that of the constrained instanton. We expect that the valley instanton has a much smaller action than the constrained instanton, although analysis of such a configuration is difficult both analytically and numerically. Instead, we have carried out analysis of the scalar field case studied in [10] and found that in fact the valley instanton has much more desirable behaviors. Thus we expect that overall our valley instanton has larger contribution to the functional integral and gives us a better approximation scheme.

We note here that there is a way to incorporate the new valley equation itself in the constrained instanton formalism: As noted in [13], the new valley equation can be understood as the equation of motion under the constraint \( \int d^4x \sum_\alpha (\delta S/\delta \phi_\alpha(x))^2. \) (The eigenvalue \( \lambda \) plays the role of the constraint parameter.) In this sense, the new valley method gives us the definition of the constraint: There is no arbitrariness in the constraint as we mentioned in the introduction. The constraint determined by the new valley method guarantees the effectiveness of the approximation scheme, for the new valley method has many virtues as a means of determining the dominant configurations.

In this letter, we have not incorporated fermions in the theory. Its introduction is straightforward, however. We need to solve the mode equations for fermion fields in the background of the valley instanton, as we have no reason to take into account the fermion contribution to the new valley equation itself. Analysis along this line is in progress and will be reported in future publications.
Acknowledgment

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References


Figure captions

Fig. 1 Shapes of the numerical solution of $a(r)$, $f^a(r)$, $h(r)$ and $f^h(r)$ for $\rho v = 0.1$ near the origin.

Fig. 2 Shapes of the numerical solution of $a(r)$, $f^a(r)$, $h(r)$ and $f^h(r)$ for $\rho v = 0.1$ in the asymptotic region.

Fig. 3 The action $S$ (in units $g^2 S/8\pi^2$) of the numerical solution of the valley equation, at $\lambda/g^2 = 1$, as a function of the parameter $\rho v$. The solid line is the behavior of the analytical result that $g^2 S/8\pi^2 = 1 + \rho^2/4$.

Fig. 4 Numerical results of the scaled eigenvalues $\nu$ at $\rho v = 0.001$, 0.005, 0.01, 0.02, 0.05 and 0.1.
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<th>$f_{(2)}^a$</th>
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Table 1: The numerical data of $\rho v$, $\nu$, $h_{(1)}$, $f_{(2)}^a$, $f_{(1)}^h$ and $g^2 S/8\pi^2$.
Figure 1: Shapes of the numerical solution of $a(r)$, $f^a(r)$, $h(r)$ and $f^h(r)$ for $\rho v = 0.1$ near the origin.
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Figure 4: Numerical results of the scaled eigenvalues \( \nu \) at \( \rho \nu = 0.001, 0.005, 0.01, 0.02, 0.05 \) and 0.1.