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**QUANTUM CRITICAL PHENOMENA AND CONFORMAL INVARIANCE** [1]

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ABSTRACT

We show that the Abelian bosonization of continuum limit of the 1D Hubbard model corresponds to the 2D explicitly conformal invariant Gaussian model at weak coupling limit. A universality argument is used to extend the equivalence to an entire segment of the critical line of the strongly correlated electron system. An integral equation satisfied by the mapping function between critical lines of the 1D Hubbard model and 2D Gaussian model is obtained and then solved in some limiting cases. By making use of the fact that the free Hubbard system reduces to four fermions and each of them is related to a $c = \frac{1}{2}$ conformal field theory, we present exactly the partition function of the Hubbard model on a finite 1D lattice.

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# 1 Introduction

The properties of strongly correlated electron systems, such as heavy-fermion compounds, high temperature superconductors and Kondo systems have been studied intensively. Although it is not clear how well the Hubbard Hamiltonian[1] can describe these systems,the single-band Hubbard model is strongly argued[2] as the appropriate model for the high $T_c$ superconductivity[3]. Despite remarkable efforts the understanding of the model is still far from complete. Even in one spatial dimension where the Hubbard model can be solved exactly by Bethe ansatz[4], although the spectrum has been known for many years, the calculation of the correlation functions and partition function proved to be a delicate problem.

Ever since Baxter's solution of his eponymmous vertex model[5], it has been known that certain two-dimensional statistical models and one-dimensional quantum ones exhibit a line of critical points along which at least some of the critical exponents vary in a continuous fashion. As noticed by Cardy[6], many of these systems are related to $c = 1$ conformal field theory[7]. By assuming that the spectrum of excitations be described in terms of a semi-product of two Virasoro algebras, conformal invariance can be used to determine the correlation functions of the Hubbard model[8]. However the conformal invariance of the Hubbard model is just a working hypothesis. A way to show conformal invariance of a model is to set a connection with an explicitly conformal invariant model[9]. As we know that there are several apparently different phase-transition problems which each have critical line along which the critical behavior varies continuously. In contrast with the Hubbard model, the Gaussian model in two dimension has a whole host of exactly known critical behavior[10]. We would like to demonstrate the relationship between critical behavior in the two different models by a universality argument.

In this paper, at weak coupling limit, we use Abelian bosonization transformation[11] to map the continuum limit of one-dimensional Hubbard model into two-dimensional Gaussian model. Bethe ansatz solution of one dimensional Hubbard model gives critical exponent of charge density. By making use of a universality argument, we show conformal invariance of the Hubbard model along an entire segment of the critical line parametrized by the on-site Coulomb interaction strength $U$. A systematic study of partition function of the model is presented.

## 2 Weak Coupling Limit

The Hamiltonian of the Hubbard model is of the form

$$\begin{aligned} H &= H_0 + H_{int} , \\ H_0 &= -\epsilon \sum_{j,\sigma} \left( c_{j\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j\sigma} \right) , \\ H_{int} &= U \sum_j c_{j\uparrow}^\dagger c_{j\uparrow} c_{j\downarrow}^\dagger c_{j\downarrow} , \end{aligned} \tag{1}$$

where $\epsilon$ is the hopping matrix element, $U$ $(>0)$ is the Coulomb repulsion and the operators $c_{j\sigma}$ $(c_{j\sigma}^\dagger)$ $(j = 1, 2, \cdots, L)$ describe electrons on lattice and satisfy the anti-commutation relation,

$$\{c_{j\sigma}, c_{k\sigma'}\} = 0 , \qquad \{c_{j\sigma}, c_{k\sigma}^\dagger\} = \delta_{jk}\delta_{\sigma\sigma'} , \qquad \{c_{j\sigma}^\dagger, c_{k\sigma}^\dagger\} = 0 . \tag{2}$$

In taking the continuum limit, it is necessary to take account of the fact that the time derivative of fields on even (odd) sites is related to the values of the fields themselves on odd (even) sites. This property may be preserved by introducing two functions $\psi_e(x,t)$ and $\psi_o(x,t)$ such that

$$\begin{aligned} c_{2j,\sigma}(t) &\longrightarrow i^{2j}\sqrt{2a}\psi_e(x,t) , \\ c_{2j+1,\sigma}(t) &\longrightarrow i^{2j+1}\sqrt{2a}\psi_o(x,t) , \end{aligned} \tag{3}$$

where $a$ denotes lattice spacing and $2ja \to x$.

Under such a continuum limiting process, we have the following correspondence

$$\begin{aligned} \{c_{j\sigma}, c_{l\sigma'}^\dagger\} = \delta_{jl}\delta_{\sigma\sigma'} &\longrightarrow \{\psi_{\alpha\sigma}(x), \psi^\dagger(y)_{\beta\sigma'}\} = \delta_{\alpha\beta}\delta_{\sigma\sigma'}\delta(x-y) , \\ c_{j+1,\sigma} + c_{j-1,\sigma} &\longrightarrow i^{2j}\left(\psi_{\alpha\sigma}(ja) - \psi_{\alpha\sigma}((j-1)a)\right) . \end{aligned} \tag{4}$$

Defining

$$\psi_1(x,t) = \frac{\psi_e(x,t) + \psi_o(x,t)}{\sqrt{2}} , \qquad \psi_2(x,t) = \frac{\psi_e(x,t) - \psi_o(x,t)}{\sqrt{2}} .$$

we can write the equations of motion of the free system as follows

$$\begin{aligned} \frac{\partial\psi_1(x,t)}{\partial t} &= -v_F\frac{\partial\psi_1(x,t)}{\partial x} , \\ \frac{\partial\psi_2(x,t)}{\partial t} &= v_F\frac{\partial\psi_2(x,t)}{\partial x} , \end{aligned} \tag{5}$$

where $v_F = 2\epsilon a$ is the Fermi velocity. By making use of the Fourier transformation

$$\begin{aligned} a_{k\sigma} &= \frac{1}{\sqrt{L}}\int dx e^{-ikx}\psi_{1\sigma}(x) , \\ b_{k\sigma} &= \frac{1}{\sqrt{L}}\int dx e^{-ikx}\psi_{2\sigma}(x) , \end{aligned}$$



we can write the free Hamiltonian (up to a chemical potential term) of the form

$$H_0 = \sum_{k>0} v_F(k-k_F)a_{k\sigma}^\dagger a_{k\sigma} + \sum_{k<0} v_F(-k-k_F)b_{k\sigma}^\dagger b_{k\sigma} . \tag{6}$$

It should be noticed that the free Hamiltonian (6) describes four free fermions and each of them corresponding to a $c = \frac{1}{2}$ conformal field theory[7].

At weak coupling limit, the perturbation form of the interaction Hamiltonian $H_{int}$ can be cast into the form

$$\begin{aligned} H_{int} &= Ua\sum_\sigma \int dx \left( \psi_{o\sigma}^\dagger\psi_{o\sigma}\psi_{o,-\sigma}^\dagger\psi_{o,-\sigma} + \psi_{e\sigma}^\dagger\psi_{e\sigma}\psi_{e,-\sigma}^\dagger\psi_{e,-\sigma} \right) \\ &= H_{LM} + H_{BS} + H_{US} , \\ H_{LM} &\equiv \frac{Ua}{4}\sum_\sigma\int dx \left( \psi_{1\sigma}^\dagger\psi_{1\sigma}\psi_{1,-\sigma}^\dagger\psi_{1,-\sigma} + \psi_{2\sigma}^\dagger\psi_{2\sigma}\psi_{2,-\sigma}^\dagger\psi_{2,-\sigma} + 2\psi_{1\sigma}^\dagger\psi_{1\sigma}\psi_{2,-\sigma}^\dagger\psi_{2,-\sigma} \right) , \\ H_{BS} &\equiv \frac{Ua}{4}\sum_\sigma\int dx \left( \psi_{1\sigma}^\dagger\psi_{2\sigma}\psi_{2,-\sigma}^\dagger\psi_{1,-\sigma} + h.c. \right) , \\ H_{US} &\equiv \frac{Ua}{4}\sum_\sigma\int dx \left( \psi_{2\sigma}^\dagger\psi_{1\sigma}\psi_{2,-\sigma}^\dagger\psi_{1,-\sigma}e^{iG} + h.c. \right) , \end{aligned} \tag{7}$$

where $G$ is a reciprocal lattice vector, for a half filled band $G = 4k_F$.

By making use of Abelian bosonization transformation[11], we express the dynamical processes represented by electron field operators, $\psi_{i\sigma}$ $(i = 1, 2)$ by those of a boson field $\phi_{i\sigma}$, as follows

$$\begin{aligned} \psi_{1\sigma}(x,t) &\longrightarrow \frac{1}{\sqrt{2\pi\alpha}}\exp\left(-ik_Fx - \phi_{1\sigma}(x,t)\right) \equiv O_{1\sigma}(x,t) , \\ \psi_{2\sigma}(x,t) &\longrightarrow \frac{1}{\sqrt{2\pi\alpha}}\exp\left(ik_Fx + \phi_{2\sigma}(x,t)\right) \equiv O_{2\sigma}(x,t) , \end{aligned} \tag{8}$$

where $\alpha$ is the cutoff parameter to be set $\alpha \to 0$ whenever possible and

$$\phi_{i\sigma} \equiv \frac{2\pi}{L}\sum_q \frac{1}{q}\exp\left(-\frac{1}{2}\alpha|q| - iqx\right)\rho_{i\sigma}(q) , \tag{9}$$

here

$$\begin{aligned} \rho_{1\sigma}(q) &= \sum_{k>0} a_{k+q,\sigma}^\dagger a_{k\sigma} , \qquad & \rho_{1\sigma}(-q) &= \sum_{k>0} a_{k\sigma}^\dagger a_{k+q,\sigma} , \\ \rho_{2\sigma}(q) &= \sum_{k<-q} b_{k+q,\sigma}^\dagger b_{k\sigma} , \qquad & \rho_{2\sigma}(-q) &= \sum_{k<-q} b_{k\sigma}^\dagger b_{k+q,\sigma} . \end{aligned} \tag{10}$$

It is convenient to introduce the phase variables $\Phi_i(x,t)$,

$$
\begin{aligned}
\Phi_1(x,t) &= \frac{i}{\sqrt{4\pi}}\sum_q \frac{2\pi}{Lq}\exp\left(-\frac{1}{2}\alpha|q| - iqx\right)(\rho_1(q)+\rho_2(q)) \\
&= \frac{i}{2\sqrt{2\pi}}\left((\phi_{1\uparrow}(x)+\phi_{1\downarrow}(x)) + (\phi_{2\uparrow}(x)+\phi_{2\downarrow}(x))\right) , \\
\Phi_2(x,t) &= \frac{i}{\sqrt{4\pi}}\sum_q \frac{2\pi}{Lq}\exp\left(-\frac{1}{2}\alpha|q| - iqx\right)(\sigma_1(q)+\sigma_2(q)) \\
&= \frac{i}{2\sqrt{2\pi}}\left((\phi_{1\uparrow}(x)-\phi_{1\downarrow}(x)) + (\phi_{2\uparrow}(x)-\phi_{2\downarrow}(x))\right) .
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
\rho_i(q) &= \frac{1}{\sqrt{2}}\left(\rho_{i\uparrow}(q)+\rho_{i\downarrow}(q)\right) , \\
\sigma_i(q) &= \frac{1}{\sqrt{2}}\left(\rho_{i\uparrow}(q)-\rho_{i\downarrow}(q)\right) .
\end{aligned}
$$

The canonical conjugations of the phase variables $\Phi_i(x,t)$ are

$$
\begin{aligned}
\Pi_1(x,t) &= \frac{\partial\Phi_1(x,t)}{\partial t} \\
&= \frac{-v_F}{\sqrt{4\pi}}\sum_q \frac{2\pi}{L}\exp\left(-\frac{1}{2}\alpha|q| - iqx\right)(\rho_1(q)-\rho_2(q)) , \\
\Pi_2(x,t) &= \frac{\partial\Phi_2(x,t)}{\partial t} \\
&= \frac{-v_F}{\sqrt{4\pi}}\sum_q \frac{2\pi}{L}\exp\left(-\frac{1}{2}\alpha|q| - iqx\right)(\sigma_1(q)-\sigma_2(q)) .
\end{aligned} \tag{12}
$$

In terms of the phase variables $\Phi_i$ and $\Pi_i$, the Hubbard Hamiltonian is of the form

$$
\begin{aligned}
H_0 + H_{LM} &= \int dx\left(\left(\frac{v_F}{2}+\frac{Ua}{4\pi}\right)\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_1}{\partial x} + \frac{1}{2v_F}\frac{\partial\Phi_1}{\partial t}\frac{\partial\Phi_1}{\partial t}\right) \\
&\quad + \int dx\left(\left(\frac{v_F}{2}-\frac{Ua}{4\pi}\right)\frac{\partial\Phi_2}{\partial x}\frac{\partial\Phi_2}{\partial x} + \frac{1}{2v_F}\frac{\partial\Phi_2}{\partial t}\frac{\partial\Phi_2}{\partial t}\right) , \\
H_{BS} &= \frac{Ua}{4(2\pi\alpha)^2}\int dx\cos\left(2\sqrt{2\pi}\Phi_2(x,t)\right) , \\
H_{US} &= \frac{Ua}{4(2\pi\alpha)^2}\int dx\cos\left((G-4k_F)x + 2\sqrt{2\pi}\Phi_1(x,t)\right) .
\end{aligned} \tag{13}
$$

$H_{\mathrm{BS}}$ and $H_{\mathrm{US}}$ introduce non-linearity into the system. However, for repulsive interaction ($U > 0$) $H_{\mathrm{BS}}$ is renormalized to zero in the long-wavelength limit. And at off half-band $H_{\mathrm{US}}$ do not appear[12]. So that the only interested Hamiltonian is

$$
\begin{aligned}
H &= \int dx\left(\left(\frac{v_F}{2}+\frac{Ua}{4\pi}\right)\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_1}{\partial x} + \frac{1}{2v_F}\frac{\partial\Phi_1}{\partial t}\frac{\partial\Phi_1}{\partial t}\right) \\
&\quad + \int dx\left(\left(\frac{v_F}{2}-\frac{Ua}{4\pi}\right)\frac{\partial\Phi_2}{\partial x}\frac{\partial\Phi_2}{\partial x} + \frac{1}{2v_F}\frac{\partial\Phi_2}{\partial t}\frac{\partial\Phi_2}{\partial t}\right) .
\end{aligned} \tag{14}
$$

After a proper renormalizing, we write the corresponding action into the familiar form



$$
L = -\frac{K_{wcl}}{2}\int d^2x\frac{\partial\Phi_1}{\partial x_\mu^+}\frac{\partial\Phi_1}{\partial x^{+\mu}} - \frac{1}{2}\int d^2x\frac{\partial\Phi_2}{\partial x_\mu^-}\frac{\partial\Phi_2}{\partial x^{-\mu}}, \quad (\mu = 0,\ 1) , \tag{15}
$$

where $K_{wcl} \equiv \sqrt{\frac{1+\frac{Ua}{2\pi v_F}}{1-\frac{Ua}{2\pi v_F}}}$, $x_0^\pm \equiv iv_F\sqrt{1\pm\frac{Ua}{2\pi v_F}}t$ and $x_1^\pm = x$.

This is to say that the field theoretic description of the 1D Hubbard system at weak coupling limit consisting of a pair of 2D decoupling Gaussian models. And the coupling constants of the corresponding Gaussian models are dependent on the strength of the on-site Coulomb repulsion. It should be noticed that the result is only valid for weak coupling limit because it is obtained by a perturbation method. A universality argument should be used to extend it to an entire segment of the critical line of the Hubbard model.

## 3 Universality Class

By making use of the Bethe ansatz, Lieb and Wu[4] have solved the Hubbard model exactly. The eigenstates of the model with $N_c = N_\uparrow + N_\downarrow$ electrons and $N_s = N_\downarrow$ down spins are characterized by the momenta $k_j$ of charges and the rapidities $\lambda_\alpha$ of spin waves. Imposing periodic boundary conditions on the eigenstates leads to a familiar called Bethe-Ansatz equations,

$$
\begin{aligned}
Lk_j &= 2\pi I_j - \sum_{\beta=1}^{N_s} 2\arctan\left(\frac{\sin k_j - \lambda_\beta}{u}\right) , \\
\sum_{j=1}^{N_c} 2\arctan\left(\frac{\lambda_\alpha - \sin k_j}{u}\right) &= 2\pi J_\alpha + \sum_{\beta=1}^{N_s} 2\arctan\left(\frac{\lambda_\alpha - \lambda_\beta}{2u}\right) ,
\end{aligned} \tag{16}
$$

where $u \equiv \frac{U}{4}$ denotes the interaction strength in units of the bandwidth and the quantum numbers $I_j$ and $J_\alpha$ are integers or half-odd- integers, depending on the parities of the number $N_c$ and $N_s$,

$$
\begin{aligned}
I_j &= \frac{N_s}{2} \bmod 1 , \\
J_\alpha &= \frac{N_c + N_s + 1}{2} \bmod 1 .
\end{aligned} \tag{17}
$$

In the thermodynamic limit ($L \to \infty$, with $N_c/L$, $N_s/L$ kept constant) the ground state of the model is a Fermi sea, characterized by the distribution function of charges with momentum $k$, $\rho_c(k)$, defined by the integral equation

$$
\rho_c(k) = \frac{1}{2\pi} + \frac{\cos k}{2\pi}\int_{-k_0}^{k_0} dk' K(\sin k - \sin k')\rho_c(k') , \tag{18}
$$

where $K(x) = \int_0^\infty d\omega \frac{e^{u\omega}}{\cosh u\omega} \cos\omega x$ and $k_0$ is fixed by

$$\int_{-k_0}^{k_0} dk \rho_c(k) = \frac{N_c}{L} = n_c ,$$

here $n_c$ denotes the total density of electrons.

The critical behavior of the Hubbard model is described by the dressed charge $\xi(k)$.

$$\xi(k) = 1 + \frac{1}{2\pi}\int_{-k_0}^{k_0} dk' \cos k' K(\sin k - \sin k')\xi(k') . \tag{19}$$

We may rewrite the above integral equation as

$$\xi(z) = 1 + \frac{1}{2\pi}\int_{-z_0}^{z_0} dz' \int_0^\infty d\omega \frac{e^{-\omega}}{\cosh\omega} \cos(\omega(z-z'))\xi(z') , \tag{20}$$

where $z = \sin k/u$. For half-filled band, $n_c = 1$, only the spin excitations are gapless, the Hubbard model reduces to the spin-1/2 Heisenberg model. In the case of less than half-filling, $n_c = 1-\delta$ ($\delta$ ($>0$) is the doping), both charge and spin excitations are gapless. Here the critical exponents of the model in terms of the dressed charge $\xi(k)$ are of the form[13]

$$\begin{aligned} \Delta_c^\pm &= \frac{1}{2}\xi^2(z_0)(D_c + \frac{1}{2}D_s)^2 + \frac{1}{8\xi^2(z_0)}(\Delta N_c)^2 \pm \frac{1}{4}\Delta N_c(2D_c + D_s) + N_c^\pm , \\ \Delta_s^\pm &= \frac{1}{4}(D_s)^2 + \frac{1}{4}(\Delta N_s - \frac{1}{2}\Delta N_c)^2 \pm \frac{1}{4}(2\Delta N_s - \Delta N_c)D_s + N_s^\pm . \end{aligned} \tag{21}$$

where $D_c$, $D_s$ and $\Delta N_c$, $\Delta N_s$ are characteristic numbers of the excited state under consideration. $N_c^\pm$ and $N_s^\pm$ are positive numbers.

Equation (20) shows clearly that $\xi(z_0)$ depend on $z_0 = \sin k_0/u$ only. Then Eq.(21) tells us that there is a critical line along which the critical exponents vary continuously as the on-site Coulomb interaction strength $U$ changes. The critical exponent of the charge density with momentum around $2k_F$ takes the form

$$x_{nn} = 1 + \frac{\xi^2(z_0)}{2} . \tag{22}$$

From Eqs.(20)-(22), we know that the critical behavior of the Hubbard model is dependent on $z_0 = \sin k_0/U$ only.

The concept of universality and the existence of a marginal operator should be used to argue that the connection of the 1D Hubbard model and 2D Gaussian model holds not only at the weak coupling limit but along an entire segment of the critical lines of the Hubbard model and Gaussian model, i.e., two different problems will have critical



lines being in the same universality class. We imagine that we start from a knowledge of an infinite set of correlation functions of the Hubbard model at weak coupling limit of a critical line parameterized by the on-site Coulomb repulsion strength $U$ and then develop the extra information which is derivable about the critical line.

It is well known that in a problem involving a critical line there is one very special operator – which has a property of generating motion along the critical line. In our presently interested case, the marginal operator is

$$E_j = c_{j\uparrow}^\dagger c_{j\uparrow} c_{j\downarrow}^\dagger c_{j\downarrow} .$$

Local density fluctuations of charge and spin varying slowly in space are given by

$$\begin{aligned} n(x,t) &= \sqrt{\frac{2}{\pi}}\nabla\Phi_1(x,t) , \\ S_z(x,t) &= \sqrt{\frac{1}{2\pi}}\nabla\Phi_2(x,t) . \end{aligned} \tag{23}$$

On the other hand the electron density operator with wave number around $\pm 2k_F$ is given by

$$\rho_{\pm 2k_F,\sigma}(x,t) = \frac{1}{2\pi\alpha}\exp\left(\pm i\left(2k_F x - \sqrt{2\pi}\Phi_1(x,t) - \sigma\sqrt{2\pi}\Phi_2(x,t)\right)\right) . \tag{24}$$

Thus the marginal operator takes of the form

$$\begin{aligned} E(x,t) &= \frac{1}{2\pi}\left((\nabla\Phi_1)^2 - (\nabla\Phi_2)^2\right) \\ &+ \frac{1}{(\pi\alpha)^2}\left(\cos^2(2k_F x - \sqrt{2\pi}\Phi_1)\cos^2(\sqrt{2\pi}\Phi_2)\right. \\ &\left. - \sin^2(2k_F x - \sqrt{2\pi}\Phi_1)\sin^2(\sqrt{2\pi}\Phi_2)\right) . \end{aligned} \tag{25}$$

To see this motion, imagine that an extra term is added to the Hubbard Hamiltonian of the form

$$\delta H = \delta U \sum_{x,t} E(x,t) . \tag{26}$$

Then this extra term can move one along the critical line by an amount proportional to $\delta U$. In particular, then, any derivative of a physical quantity $O$ along the critical line can be generated via $O$, $E(x,t)$ correlation function according to

$$\frac{d}{dU}\langle O\rangle_U = \sum_{x,t}\langle OE(x,t)\rangle_U . \tag{27}$$

Higher derivatives are generated by high-order correlation functions involving more factors of $E(x,t)$. A knowledge of all correlation functions of powers of $E(x,t)$ with $O$ generates via a Taylor series – the behavior all along the critical line. This line of logic enables us to make a very useful statement about the connection between two models which have the same asymptotic form of correlation functions at their critical points.

It is well known that the Gaussian model with action

$$L = -\frac{K}{2}\int d^2x^+ \frac{\partial\Phi}{\partial x^+_\mu}\frac{\partial\Phi}{\partial x^{+\mu}} - \frac{1}{2}\int d^2x^- \frac{\partial\Phi}{\partial x^-_\mu}\frac{\partial\Phi}{\partial x^{-\mu}} \tag{28}$$

has a critical line parameterized by the coupling constant $K$. The basic operators of the system are the exponentials of the free field $O_{(e^+,e^-)} = e^{i(e^+\sqrt{2\pi}\Phi_1 + e^-\sqrt{2\pi}\Phi_2)}$, satisfying

$$\langle O_{(e^+,e^-)}(\mathbf{r})O_{-(e^+,e^-)}(\mathbf{r}')\rangle \to |\mathbf{r}-\mathbf{r}'|^{-\left(\frac{(e^+)^2}{K}+(e^-)^2\right)}, \tag{29}$$

and the dual operators $O_{(m^+,m^-)}$, the correlation functions of which are obtained by imposing a discontinuity of $2\pi m^+$ and $2\pi m^-$ on the field $\Phi_1$ and $\Phi_2$ respectively when one crosses a line connecting $\mathbf{r}$ to $\mathbf{r}'$,

$$\langle O_{(m^+,m^-)}(\mathbf{r})O_{-(m^+,m^-)}(\mathbf{r}')\rangle \to |\mathbf{r}-\mathbf{r}'|^{-\left(K(m^+)^2+(m^-)^2\right)}. \tag{30}$$

Combining Eqs.(29) and (30), one gets a general operator $O_{(e^+,e^-)(m^+,m^-)}$, satisfying

$$\begin{aligned}&\langle O_{(e^+,e^-)(m^+,m^-)}(\mathbf{r})O_{(-e^+,-e^-)(-m^+,-m^-)}(\mathbf{r}')\rangle\\ &\longrightarrow |\mathbf{r}-\mathbf{r}'|^{-\left(\frac{(e^+)^2}{K}+(e^-)^2\right)-\left(K(m^+)^2+(m^-)^2\right)}\exp(-2i(e^+m^+ + e^-m^-)\alpha(\mathbf{r}-\mathbf{r}')),\end{aligned} \tag{31}$$

where $\alpha(\mathbf{r}-\mathbf{r}')$ is the angle of the $\mathbf{r}-\mathbf{r}'$ vector with an arbitrary direction. Thus $O_{(e^+,e^-)(m^+,m^-)}$ has a pair of conformal dimensions $h^\pm_{e^\pm m^\pm}$ and $\bar{h}^\pm_{e^\pm m^\pm}$ given by

$$\begin{aligned} h^+_{e^+m^+} &= \frac{(e^+)^2}{4K} + \frac{K(m^+)^2}{4} + \frac{e^+m^+}{2}, & \bar{h}^+_{e^+m^+} &= h^+_{e^+,-m^+},\\ h^-_{e^-m^-} &= \frac{(e^-)^2}{4} + \frac{(m^-)^2}{4} + \frac{e^-m^-}{2}, & \bar{h}^-_{e^-m^-} &= h^-_{e^-,-m^-}. \end{aligned} \tag{32}$$

The correlation functions of a list of fluctuating operators $O_{(e^+,e^-)(m^+,m^-)}(x_0^\pm, x_1^\pm)$ are known all along the line

$$\begin{aligned}&G_K((e_1^+,e_1^-),(m_1^+,m_1^-),(x_0^\pm)_1,(x_1^\pm)_1;\cdots;(e_n^+,e_n^-),(m_n^+,m_n^-),(x_0^\pm)_n,(x_1^\pm)_n)\\ &\qquad = \langle\prod_{i=1}^n O_{(e_i^+,e_i^-),(m_i^+,m_i^-)}((x_0^\pm)_i,(x_1^\pm)_i)\rangle_K.\end{aligned} \tag{33}$$



At the weak coupling limit, as we know, the correlation functions of the Hubbard model chain problem are of the form

$$g_{wcl}(\alpha_1,x_1,t_1;\cdots;\alpha_n,x_n,t_n) = \langle\prod_{i=1}^n s_{\alpha_i}(x_i,t_i)\rangle_{H_{wcl}}. \tag{34}$$

We also know the fact that the asymptotic form for large separation of $G_K$ and $g_{wcl}$ agree for $K = K_{wcl}$. This asymptotic agreement can be indicated by writing

$$g_{wcl} \Leftrightarrow G_{K=K_{wcl}}. \tag{35}$$

Since $G_K$ has a fixed line, it must have a marginal operator in its complete set. Because there is one and only one such marginal operator, the problem with Hamiltonian $H_{wcl}$ must have a line of critical points identical in structure to that described by $G_K$. In particular there exists a line of critical points with Hamiltonian

$$H(U) = H_0 + U\int dx E(x,t) \tag{36}$$

and a set of critical operators $s_\alpha(x,t;U)$ which have identical critical properties to those in the Gaussian model. This identity is the statement that if we form

$$g_U(\alpha_1,x_1,t_1;\cdots;\alpha_n,x_n,t_n) = \langle\prod_{i=1}^n s_{\alpha_i}(x_i,t_i,\lambda_i)\rangle_{H_U}, \tag{37}$$

then there exists a mapping $K = F(U)$ which makes the Hubbard model problem identical to the Gaussian model problem

$$g_U \Leftrightarrow G_{K=F(U)}. \tag{38}$$

This identity will hold over the full range of $U$ and of $K$ such that both problems have precisely one marginal operator. From the analysis it follows that an entire segment of Hubbard model critical line must be in the same universality class as the Gaussian model.

We have not as yet determined the parameterization of the critical line, i.e., we have not found the coupling constant $K$ of the corresponding Gaussian model as a function of the Coulomb interaction strength $U$. We can make use of Bethe ansatz solution of the Hubbard model to determine the explicit form of the mapping function $F(U)$. The critical exponent of the electron density $n(x,t)$ with wave number around $2k_F$ corresponding to the Gaussian action (28) is of the form

$$x_{nn}^{\text{Gaussian}} = 1 + \frac{1}{K}. \tag{39}$$

Comparing Eqs.(22) and (39), we obtain

$$K = \frac{2}{\xi^2(z_0)} \ . \tag{40}$$

Thus the mapping function $F(U)$ satisfies the following integral equation,

$$\sqrt{\frac{2}{F\left(\frac{\sin k}{u}\right)}} = 1 + \frac{1}{2\pi}\int_{-\frac{\sin k_0}{u}}^{\frac{\sin k_0}{u}} d\left(\frac{\sin k'}{u}\right)\int_0^\infty d\omega \frac{e^{-\omega}}{\cosh\omega}\cos\left(\omega\left(\frac{\sin k}{u} - \frac{\sin k'}{u}\right)\right)\sqrt{\frac{2}{F\left(\frac{\sin k'}{u}\right)}} \ . \tag{41}$$

The integral equation can be solved in some limiting cases.

For $0 < u << \sin k_0$, the perturbative scheme based on the Wiener-Hopf method can be used. The solution is

$$F\left(\frac{\sin k_0}{u}\right) = 1 + \frac{u}{\pi \sin k_0} \ . \tag{42}$$

This result gives the decoupling point of the mapping function

$$K = F(U = 0) = 1 \ . \tag{43}$$

For $u >> \sin k_0$, we have

$$F\left(\frac{\sin k_0}{u}\right) = 2\left(1 - \frac{2\sin k_0}{\pi u}\ln 2\right) \ . \tag{44}$$

For arbitrary repulsion $U > 0$ and density $0 < n_c < 1$ the coupling $K$ varies in the interval $1 \le K \le 2$.

From the above discussion, we conclude that there is indeed a universality connection between the critical lines of the 1D Hubbard model and 2D Gaussian model. In particular, there exists a mapping function $F(U)$, which is determined by an integral function and can be explicitly written down in some limiting cases, such that when $K = F(U)$ the Hubbard model lies in the same universality class as the Gaussian model with coupling $K = F(U)$.

## 4 Partition Function

It should be noticed that we are now discussing a finite size system and in a finite geometry, the boundary conditions generate various constraints on the successive transformations,



resulting in a modified free-field theory. The consistency of conformal theories defined on a torus restricts severely their operator content. The partition function of the 1D Hubbard system on a torus may written as

$$\begin{aligned} Z(\tau) &= Z^+(\tau)\cdot Z^-(\tau) \\ &= \mathrm{tr}\left(q^{L_0^+ - \frac{1}{24}}\bar{q}^{\bar{L}_0^+ - \frac{1}{24}}\right)\cdot \mathrm{tr}\left(q^{L_0^- - \frac{1}{24}}\bar{q}^{\bar{L}_0^- - \frac{1}{24}}\right) \ , \end{aligned} \tag{45}$$

where $L_0^+$ ($\bar{L}_0^+$) and $L_0^-$ ($\bar{L}_0^-$) are the dilation generators of the Virasoro algebras on the planes $x^+$ and $x^-$, respectively, $q = e^{2i\pi\tau}$, and $\tau(\tau_R + i\tau_I) = \omega_2/\omega_1$ is the modular ratio of the torus.

This trace which runs over all states of the Hilbert space may be decomposed on the various irreducible representations of the semi-direct product of the two Virasoro algebras (on the planes $x^+$ and $x^-$, respectively) in the form[14]

$$Z(\tau) = \sum_{h_+,\bar{h}_+} N_{h_+\bar{h}_+}\chi_{h_+}(q)\chi_{\bar{h}_+}(\bar{q})\cdot\sum_{h_-,\bar{h}_-} N_{h_-\bar{h}_-}\chi_{h_-}(q)\chi_{\bar{h}_-}(\bar{q}) \ , \tag{46}$$

where the non-negative integers $N_{h_\pm\bar{h}_\pm}$ represent the multiplicity of the operators of dimensions $(h_\pm, \bar{h}_\pm)$ in the partition function and the character $\chi_{h_\pm}$ read

$$\chi_{h_\pm}(\tau) = \mathrm{tr}_{h_\pm}\left(q^{L_0^\pm - \frac{1}{24}}\right)$$

in each irreducible representation of highest weight $h_\pm$.

Imposing that $Z(\tau)$ is a modular invariant function of $\tau$ then puts strong constraints on the $N$'s, hence on the operator content of the theory, leading ultimately to a classification of the possible partition function.

The action (28) is characterized by the ($c = 1$) $Vir \otimes \overline{Vir} \otimes Vir \otimes \overline{Vir}$ algebra. And the corresponding partition function takes the following form

$$\begin{aligned} Z(K,\tau) &= Z^+(K,\tau)\cdot Z^-(\tau) \\ &= \int[D\Phi_1]\exp\left(-\frac{K}{2}\int d^2x^+ \frac{\partial\Phi_1}{\partial x_\mu^+}\frac{\partial\Phi_1}{\partial x^{+\mu}}\right) \\ &\cdot\int[D\Phi_2]\exp\left(-\frac{1}{2}\int d^2x^- \frac{\partial\Phi_2}{\partial x_\mu^-}\frac{\partial\Phi_2}{\partial x^{-\mu}}\right) \ . \end{aligned} \tag{47}$$

It is easy to see from the above equation that

$$Z^+(K = 1, \tau) = Z^-(\tau) \equiv Z^{(+-)}(\tau) \ . \tag{48}$$

The $x^\pm$ integration is performed on a torus defined by the ratio $\tau$ which lies in the upper half-plane. To evaluate the partition function, we first perform the integral over fields $\Phi_1$ and $\Phi_2$ periodic around the torus.

For a free field on a torus $\mathcal{T}$, with the action (28) integrated over $\mathcal{T}$, a properly renormalized expression for the partition function

$$Z_1(K,\tau) = \int_{\Phi_1,\Phi_2 \text{ periodic}} [D\Phi_1][D\Phi_2] \exp\left(-\frac{K}{2}\int d^2x^+ \frac{\partial\Phi_1}{\partial x^+_\mu}\frac{\partial\Phi_1}{\partial x^{+\mu}} - \frac{1}{2}\int d^2x^- \frac{\partial\Phi_2}{\partial x^-_\mu}\frac{\partial\Phi_2}{\partial x^{-\mu}}\right) \tag{49}$$

is

$$Z_1(K,\tau) = \frac{\sqrt{K}}{\tau_I \eta^2(q)\eta^2(\bar q)}\,, \tag{50}$$

where $\eta$ is the Dedekind function

$$\eta(q) = q^{\frac{1}{24}} \prod_{N=1}^{\infty}\left(1-q^N\right)\,. \tag{51}$$

For a variation of angle equation to $2\pi m_\pm$ $(2\pi m'_\pm)$ along $\omega_1$ $(\omega_2)$, the corresponding partition function is readily evaluated using the classical solution (such that $\Delta\Phi_1 = \Delta\Phi_2 = 0$),

$$\begin{aligned} Z_{m'_\pm,m_\pm}(K,\tau) = Z_1(K,\tau)\ \ &\exp\left(-\pi K\frac{m'^2_+ + m^2_+(\tau_R^2+\tau_I^2) - 2\tau_R m_+ m'_+}{\tau_I}\right)\\ &\cdot\exp\left(-\pi\frac{m'^2_- + m^2_-(\tau_R^2+\tau_I^2) - 2\tau_R m_- m'_-}{\tau_I}\right)\,. \end{aligned} \tag{52}$$

The physical partition function is obtained by summing over $Z_{m'_\pm,m_\pm}$. After a Poisson transformation, one finds

$$\begin{aligned} Z(K,\tau) &= Z^+(K,\tau)\cdot Z^-(\tau)\\ &= \frac{1}{\eta(q)\bar\eta(\bar q)}\sum_{e_+,m_+\in Z} q^{\frac{(e_+/\sqrt{K}+m_+\sqrt{K})^2}{4}}\bar q^{\frac{(e_+/\sqrt{K}-m_+\sqrt{K})^2}{4}}\\ &\cdot\frac{1}{\eta(q)\bar\eta(\bar q)}\sum_{e_-,m_-\in Z} q^{\frac{(e_-+m_-)^2}{4}}\bar q^{\frac{(e_--m_-)^2}{4}}\,. \end{aligned} \tag{53}$$

The partition function enjoys the important property of modular invariance, i.e., invariance under the modular group

$$\tau \longrightarrow \frac{a\tau+b}{c\tau+d}\,,\qquad a,b,c,d\in Z,\quad ad-bc=1\,. \tag{54}$$

The partition function of the Hubbard model takes the form

$$Z^{Hub}(K,\tau) = \left(c_1 Z^+(K,\tau)+S(\tau)\right)\left(c_1 Z^-(\tau)+S(\tau)\right)\,, \tag{55}$$



where $c_1$ is a constant not yet fixed and the modular invariant function $S(\tau)$ should not dependent on the coupling constant because $Z^+(K,\tau)$ carries all the moving primary operators.

To fixed the constant term $S(\tau)$, we note a special property which applies to a decoupling point $K=1$, where the Hubbard model reduces to four decoupled free fermions. We get the crucial condition

$$\begin{aligned} Z^{Hub}(K=1,\tau) &= \left(aZ^{(+-)}(\tau)+S(\tau)\right)^2\\ &= \left(Z^F(\tau)\right)^4\,, \end{aligned} \tag{56}$$

where the fermion partition function on a torus is known

$$Z^F(\tau) = |\chi_0(\tau)|^2 + |\chi_{\frac{1}{2}}(\tau)|^2 + |\chi_{\frac{1}{16}}(\tau)|^2\,. \tag{57}$$

Here $\chi_h(\tau)$ are the characters of the $c=\frac{1}{2}$ Virasoro algebra

$$\chi_{h(r/s)} = \frac{1}{\eta(\tau)}\sum_{k\in Z}\left(q^{\frac{(24k+4r-3s)^2}{48}} - q^{\frac{(24k+4r+3s)^2}{48}}\right)\qquad r,s\in Z\,.$$

It is useful to introduce the Hamiltonians (include zero-point energy) of Neveu-Schwarz (NS) and Ramond (R) Majorana fermions,

$$\begin{aligned} H_{NS} &= \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}b_r - \frac{1}{48}\,,\\ H_R &= \sum_{n=1}^{\infty} n b_{-n}b_n - \frac{1}{24}\,, \end{aligned} \tag{58}$$

where $b_r$ $(r\in Z+\frac{1}{2})$ and $d_n$ $(n\in Z)$ are normal mode oscillators of NS and R fields, respectively.

Then we can rewrite the free fermion partition function as

$$Z^F(\tau) = \frac{1}{2}\left(|Z_{NS}(\tau)|^2 + |Z_{\tilde{NS}}(\tau)|^2 + 2|Z_R(\tau)|^2\right)\,, \tag{59}$$

where

$$\begin{aligned} Z^{NS}(\tau) &= \mathrm{tr}q^{H_{NS}} = q^{-\frac{1}{48}}\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}\right)\,,\\ Z^{\tilde{NS}}(\tau) &= \mathrm{tr}(-1)^F q^{H_{NS}} = q^{-\frac{1}{48}}\prod_{n=1}^{\infty}\left(1-q^{n-\frac{1}{2}}\right)\,,\\ Z^R(\tau) &= \mathrm{tr}q^{H_R} = q^{\frac{1}{24}}\prod_{n=1}^{\infty}\left(1+q^n\right)\,. \end{aligned}$$

Here $(-1)^F$ denotes the fermion chirality operator.

It is easy to verify the identity

$$\left(Z^F(\tau)\right)^2 = \frac{1}{4}Z^V(\tau) + Z^W(\tau)\ , \tag{60}$$

where

$$Z^V(\tau) = \left|\frac{1}{\eta(\tau)}\sum_{n\in Z} q^{(n+\frac{1}{2})^2/2}\right|^2 + \left|\frac{1}{\eta(\tau)}\sum_{n\in Z} q^{\frac{n^2}{2}}\right|^2 + \left|\frac{1}{\eta(\tau)}\sum_{n\in Z}(-1)^n q^{\frac{n^2}{2}}\right|^2\ ,$$
$$Z^W(\tau) = \frac{1}{2}\left|\frac{1}{\eta(\tau)}\sum_{n\in Z}(-1)^n q^{\frac{n^2}{2}}\right|^2 + \left|\frac{1}{\eta(\tau)}\sum_{n\in Z} q^{(n+\frac{1}{4})^2}\right|^2 + \left|\frac{1}{\eta(\tau)}\sum_{n\in Z}(-1)^n q^{(n+\frac{1}{4})^2}\right|^2\ ,$$

are modular invariant respectively.

On the other hand, at decoupling point, we have

$$Z(K=1,\tau) = \left(\frac{1}{2}Z^V(\tau)\right)^2\ . \tag{61}$$

Thus we obtain

$$Z^{Hub}(K,\tau) = \left(c_1 Z^+(K,\tau) + c_2 Z^V(\tau) + Z^W(\tau)\right)\left(Z^F(\tau)\right)^2\ . \tag{62}$$

where the coefficients $c_1$ and $c_2$ obey

$$\frac{1}{2}c_1 + c_2 = \frac{1}{4}$$

to satisfy the decoupling condition.

The Hubbard partition function has the general form in terms of characters.

$$Z^{Hub}(K,\tau) = \left(\sum_{h_+,\bar{h}_+} N_{h_+\bar{h}_+}\chi_{h_+}(q)\chi_{\bar{h}_+}(\bar{q}) + \sum_{h_+,\bar{h}_+}\tilde{N}_{h_+\bar{h}_+}\tilde{\chi}_{h_+}(q)\tilde{\chi}_{\bar{h}_+}(\bar{q})\right)\left(Z^F(\tau)\right)^2 . \tag{63}$$

This requires $b=0$. We now find

$$Z^{Hub}(K,\tau) = \left(\frac{1}{2}Z^+(K,\tau) + Z^W(\tau)\right)\left(Z^F(\tau)\right)^2 \tag{64}$$

# 5 Concluding Remarks

We have thus succeeded in demonstrating the conformal invariance in quantum critical phenomena. The continuum limit of the free Hubbard system reduced to four decoupled free fermions. The four-particle on-site interaction which moves us along the critical line is dealt with Abelian bosonization transformation at weak coupling limit. Then we used a universality argument to extend the equivalence of the 1D Hubbard model



and 2D Gaussian model at weak coupling limit to an entire segment of the critical line parameterized by the on-site Coulomb interaction strength. After noticed the various constraints on the successive conformal transformations generated by boundary conditions of the finite size geometry, we gave a systematic study of partition function of the strongly correlated electron system by conformal field theory.

The correlation functions which describes long-distance behavior of the strongly correlated electron system can be read out easily at the scheme. For example, we know that the density operators of charge and spin, $n(x,t)$ and $S_z(x,t)$ are of the form

$$n(x,t) = \sqrt{\frac{2}{\pi}}\Delta\Phi_1(x,t) + \frac{2}{\pi\alpha}\cos(2k_F x - \sqrt{2\pi}\Phi_1)\cos(\sqrt{2\pi}\Phi_2)\ ,$$
$$S_z(x,t) = \sqrt{\frac{1}{2\pi}}\Delta\Phi_2(x,t) + \frac{1}{\pi\alpha}\sin(2k_F x - \sqrt{2\pi}\Phi_1)\sin(\sqrt{2\pi}\Phi_2)\ . \tag{65}$$

From Eq.(31), we have

$$\langle n(x,t)n(0,0)\rangle \sim A_{n1}\frac{1}{x^2} + A_{n2}\cos(2k_F x)\frac{1}{x^{1+\frac{1}{F(U)}}}\ ,$$
$$\langle S_z(x,t)S_z(0,0)\rangle \sim A_{s1}\frac{1}{x^2} + A_{s2}\cos(2k_F x)\frac{1}{x^{1+\frac{1}{F(U)}}}\ . \tag{66}$$

where $A_{n1}$ ($A_{s1}$) and $A_{n2}$ ($A_{s2}$) are irrelevant constants.
This result is in good agreement with ones of the numerical simulations[15] and Tomonaga-Littinger model[16].

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