The Two Phases of Topologically Massive Compact $U(1)$ Theory.

Ian I. Kogan *

*On leave of absence from ITEP, B.Cheremyshkinskaya 25, Moscow, 117259, Russia

and

Theoretical Physics Institute and Physics Department
University of Minnesota, Minneapolis, MN 55455, USA

and

Alex Kovner

Physics Department, University of Minnesota, Minneapolis, MN 55455, USA

PACS: 03.70, 11.15, 12.38

Abstract

The mean field like gauge invariant variational method formulated recently, is applied to a topologically massive QED in 3 dimensions. We find that the theory has a phase transition in the Chern Simons coefficient $n$. The phase transition is of the Berezinsky-Kosterlitz - Thouless type, and is triggered by the liberation of Polyakov monopoles, which for $n > 8$ are tightly bound into pairs. In our Hamiltonian approach this is seen as a similar behaviour of the magnetic vortices, which are present in the ground state wave functional of the compact theory. For $n > 8$, the low energy behavior of the theory is the...
same as in the noncompact case. For $n < 8$ there are no propagating degrees of freedom on distance scales larger than the ultraviolet cutoff. The distinguishing property of the $n < 8$ phase, is that the magnetic flux symmetry is spontaneously broken.
1. Introduction.

In recent years topologically massive gauge theories, (TMGT) i.e. three-dimensional gauge theories with a Chern-Simons term [1] have attracted a lot of attention.

In this paper we study the simplest model of this variety - the topologically massive $U(1)$ gauge theory. The Lagrangian of this theory in the naive continuum limit is

$$L = \frac{1}{4g^2} F_{\mu\nu} F_{\mu\nu} + \frac{n}{8\pi} \epsilon_{\mu\nu\lambda} F^{\mu\nu} A_\lambda \quad (1.1)$$

where the charge $g^2$ has a dimension of mass and the Chern-Simons coefficient is a number. In a noncompact case this Lagrangian describes free massive particles with mass $2\kappa = ng^2/2\pi$. We will be interested in the compact case. Our motivation to study this problem is twofold.

First, compact three-dimensional QED without Chern-Simons term is confining due to Polyakov’s monopoles-instantons [2]. The coexistence of these monopoles with Chern-Simons term is an interesting question. It was studied to some extent in several papers [3] where it was argued that the monopoles are irrelevant in the presence of the Chern-Simons term, the theory looses its confining properties and behaves in all aspects as a non-compact theory. On the other hands these arguments are not very rigorous and the question of the relevance of the monopoles is not completely settled.

The second issue arises from the connection between the $2+1$ topologically massive gauge theory with compact $U(1)$ group and an induced 2 dimensional $XY$ model on the space - time boundary of the $2+1$ dimensional manifold. This model arises because the Chern-Simons term in the TMGT Lagrangian (1.1) is not gauge invariant under the gauge transformations which do not vanish on the boundary. These gauge degrees of freedom
become physical and have nontrivial dynamics on the boundary. One can show that the induced two-dimensional action has the form

$$S_{XY} = \frac{n}{8\pi} \int d^2 x \, \partial_i \phi \partial_i \phi$$

(1.2)

This is the action for the massless two-dimensional scalar field $\phi$ which is a parameter of a gauge transformation in the original three dimensional theory. The question of whether the three dimensional theory is compact or not is of crucial importance. In the noncompact case the gauge parameter varies from $-\infty$ to $\infty$ and the theory on the boundary is the theory of free massless bosons - the conformal field theory with $c = 1$. In the compact case with gauge group $U(1)$, the gauge parameter lives on a circle $S^1$ and eq.(1.2) defines the $XY$ model which has the famous Berezinsky-Kosterlitz-Thouless (BKT) phase transition [4]. The difference between the $XY$ model and the free scalar field is the existence of singular field configurations (vortices). The $XY$ model has two phases. At large $n$ the vortices are bound into dipolar molecules and do not affect long distance physics. This phase is described by the conformal $c = 1$ theory. At small $n$ the vortices are in the plasma phase. They dominate the statistical sum and lead to emergence of finite correlation length. The critical value of $n$ for the action (1.2) is $n = 8$. Thus, due to the connection between the $XY$ model and the compact $U(1)$ TMGT one is lead to think that the three-dimensional theory should also have two phases. This phase transition and existence of a new phase in the topologically massive gauge theory as well as some related issues, for example $n \to 1/n$ duality in both theories ($R \to 1/R$ in a standard notation in string theory) have been discussed some time ago by one of the authors [5]. However the complete picture of the phase transition at small $n$ was not clear at that time.

In this paper we study the topologically massive compact $U(1)$ gauge theory using
the gauge invariant variational approach developed in [6], [7]. In ref. [7] this method was used to study compact QED\(_3\) without the Chern Simons term. These results are in agreement with the well known scenario of confinement due to magnetic monopoles. In the framework of the hamiltonian approach of [7] the monopoles appear as topologically nontrivial configurations of the gauge group when the trial vacuum wave functional is projected onto the gauge invariant subspace of the Hilbert space. The advantage of this approach is a straightforward generalization in the case of non-zero Chern-Simons term. As opposed to that, in the path integral approach in the presence of the non-zero Chern-Simons term, the monopoles are not solutions of the classical equations of motion anymore. One has to consider complex-valued solutions of the equations of motion and their meaning is not completely clear [3].

We show that the model has two phases depending on the value of the Chern-Simons coefficient \(n\). The existence of these two phases is related to the different behaviour of the monopoles. At \(n > 8\) the monopoles are irrelevant for the infrared physics and the theory behaves as in the noncompact case and induces conformal \(c = 1\) model on the boundary. This is also in a qualitative agreement with the results of ref. [3].

For \(n < 8\) the monopoles condense in a new ground state. In this phase they are important for the infrared physics and the induced theory on the boundary is a deformed \(c = 1\) model corresponding to the plasma phase. Let us note that even though there is a monopole condensate in this phase, there is no confinement for any nonzero \(n\).

The transition between these two phases indeed takes place precisely at \(n = 8\), which is a strong argument in support of the idea that all phenomena in induced two-dimensional dynamics have they counterparts in an original three-dimensional theory.

The organisation of the paper is the following. In the next section we consider the
Hamiltonian formalism for the topologically massive gauge theory. First we discuss the noncompact case and then modify it for the case of compact $U(1)$. We introduce the gauge invariant trial wave functional. In section 3 we discuss some properties of this wave functional and the interpretation of the calculations in terms of gases of magnetic and electric vortices. In section 4 we calculate the average energy in this trial functional and minimize it with respect to our variational parameters. We will demonstrate that for large $n$ the long distance properties of the ground state are the same as in the noncompact case and then will investigate the case of small $n$. In conclusion we discuss the properties of the new $n < 8$ phase.

2 The model and the setup.

2.1 The noncompact limit.

Let us start with setting up the Hamiltonian formalism for the theory. To do that consider first a noncompact theory. It is described by the following Hamiltonian

$$ H = \frac{1}{2}[E^2_i + B^2] $$

(2.3)

augmented by the Gauss' law constraint

$$ C(x) = \partial_i(E_i - 2\kappa \epsilon_{ij} A_j) = 0 $$

(2.4)

The commutation relations satisfied by the fields are

$$ [A_i(x), A_j(y)] = 0; \quad [E_i(x), A_j(y)] = i \delta_{ij} \delta(x - y); \quad [E_i(x), E_j(y)] = 2i\kappa \epsilon_{ij} \delta(x - y) $$

(2.5)

The electric field $E_i$ can be represented in terms of the momentum, canonically conjugate to the vector potential $A_i$ as

$$ E_i = -\Pi_i + \kappa \epsilon_{ij} A_j $$

(2.6)
This Hamiltonian formulation follows directly from the Lagrangian (1.1).

The Gauss’ law operator $C(x)$ generates time independent gauge transformation. The elements of the gauge group are the operators

$$U_\phi = \exp\left\{i \int d^2x \phi(x)C(x)\right\} = \exp\left\{i \int d^2x \partial_i \phi(\Pi_i(x) + \kappa \epsilon_{ij} A_j(x))\right\} \quad (2.7)$$

It is straightforward to check, that both $E_i$ and $B$ commute with the Gauss’ law, and are therefore gauge invariant operators. The physical Hilbert space of the theory contains only states which are invariant under the action of $U$

$$U_\phi |\Psi\rangle = |\Psi\rangle \quad (2.8)$$

It is most convenient for our purposes to work in the field basis. Then, defining

$$\Phi[A_i] =< A_i |\Phi\rangle \quad (2.9)$$

we find

$$< A|U_\phi |\Phi\rangle = \exp\left\{i \kappa \int d^2x \partial_i \phi(x)\epsilon_{ij} A_j(x)\right\} \Phi[A_i + \partial_i \phi] \quad (2.10)$$

A gauge invariant state can be then constructed from an arbitrary state $\Phi$ by averaging over the gauge group

$$\Psi[A_i] = \int D\phi \exp\left\{i \kappa \int d^2x \partial_i \phi(x)\epsilon_{ij} A_j(x)\right\} \Phi[A_i + \partial_i \phi] \quad (2.11)$$

In the case of noncompact theory the integral over $\phi$ can be performed, and one can give a more compact representation of a gauge invariant state as

$$\exp\{i \kappa \int \frac{\partial}{\partial^2} A_i B\} \Phi[B] \quad (2.12)$$

However, we are going to deal with the compact theory, in which case the representation eq.(2.11) is more helpful.
2.2 Compactifying the gauge group. The vortex operator.

The next step in our discussion is to compactify the gauge group. As discussed in some detail in [7] this means, that we have to enlarge the gauge group so that it includes operators which create pointlike magnetic vortices of integer strength. These operators satisfy the following commutation relation

\[ V_k^\dagger(x)B(y)V_k(x) = B(y) + \frac{2\pi k}{g} \delta^2(x - y) \]  

with \( g \) - a dimensional constant, determining the radius of compactness and \( k \) - arbitrary integer. Naively this just means including in the gauge group the gauge transformations of the form eq.(2.7), corresponding to singular functions

\[ \phi(x) = \frac{n}{g} \theta(x - x_0) \]  

where \( \theta \) - is the planar angle, and with the understanding that the derivative of \( \phi \) in eq. (2.7) should be taken modulo \( 2\pi \). For future convenience let us introduce a special symbol to denote this derivative

\[ \Delta_i \theta(x) = \partial_i [\theta(x)]_{\text{mod} 2\pi} = \frac{\epsilon_{ij} x_j}{x^2} \]  

That is, these derivatives do not feel quantized discontinuities in \( \phi(x) \).

In other words, to compactify the theory we must limit physical Hilbert space to states which are eigenstates of the operators

\[ V(x) = \exp \left\{ \frac{i}{g} \int d^2 y \Delta_i \theta(x - y) [E_i(y) - 2\kappa \epsilon_{ij} A_j] \right\} \]  

with a unit eigenvalue. For \( \kappa = 0 \) this procedure is straightforward and was described in [7]. However, for a nonzero \( \kappa \) one should be a little more careful. The point is, that the operators defined in eq. (2.16) do not commute with noncompact gauge transformations,
and also do not commute between themselves, but rather satisfy

\[ V(x)V(y) = V(y)V(x)e^{i\alpha(x,y)} \]  

(2.17)

where \( \alpha(x - y) \) is a c-number. Obviously, if \( \kappa \alpha \neq 2\pi m \), one can not impose the condition \( V(x)\Psi = |\Psi \rangle \) simultaneously for all points \( x \). Fortunately, eq. (2.13) does not define the operator \( V \) completely, and we can take advantage of this freedom. Let us consider the following operator

\[ V_C(x) = \exp \left\{ \frac{i}{g} \int d^2 y [\Delta_i \theta(x - y) E_i(y) - 2\kappa \epsilon_{ij} \partial_i \theta(x - y) A_j] \right\} \]  

(2.18)

Note, that the function multiplying \( A_j \) in the exponential is now a \textit{bona fide} derivative. Since the planar angle is defined relative to some direction, the function \( \theta \) has a discontinuity of \( 2\pi \) along some curve \( C \), which starts at the origin and goes to infinity. Its derivative therefore has the form

\[ \partial_i \theta(x) = \Delta_i \theta(x) + 2\pi \epsilon_{ij} \hat{n}_j(x) \delta(x - C) \]  

(2.19)

where \( \hat{n}_i(x) \) is the unit tangent vector to the curve \( C \) at the point \( x \). The operator \( V(x) \) defined in eq. (2.18) still satisfies eq. (2.13), but now is invariant under the noncompact part of the gauge group\(^*\), and satisfies

\[ V_{C_1}(x)V_{C_2}(y) = V_{C_2}(y)V_{C_1}(x)e^{i\kappa^2 g N(C_1C_2)} \]  

(2.20)

where \( N(C_1C_2) \) is the number of intersections between the curves \( C_1 \) and \( C_2 \). It is clear therefore, that if we choose the radius of compactness so that

\[ 4\pi \kappa = ng^2 \]  

(2.21)

\(^*\)To see this, note that the \( A \)-dependent term in the exponential, integrating by parts can be rewritten as \( \int \phi B \), which is explicitly gauge invariant. In principle one could worry about the surface term, which we omitted. In the present case however, it is harmless for two reasons. First, because the vector potential itself is massive, and therefore vanishes at infinity; and second, since for all practical purposes it is enough to consider transformations with nonzero number of vortices and antivortices, but with net zero vorticity, and for those the function \( \phi \) itself vanishes at infinity.
with arbitrary integer $n$, the operators $V_C(x)$ commute at all points and for arbitrary choice of the set of curves $C$. Only in this case the compact theory will not depend on the choice of the curves, and will therefore be rotationally and translationally invariant. We see, therefore that for nonzero $\kappa$ the radius of compactness must be quantized according to eq. (2.21). This is, of course in complete agreement with the fact that in nonabelian Yang Mills theories, where the gauge group is necessarily compact, the ratio of the coupling constant and the Chern - Simons coefficient is quantized in the same way.

The expression for the vortex operator $V$ is simpler on a subspace of the Hilbert space invariant under the noncompact part of the gauge group. On these states the Gauss’ law is implemented exactly. Integrating the second term in the exponential in eq. (2.18) by parts and using the Gauss’ law, we get

$$V_s = \exp\{i \int s_i(y) E_i(y)\} \tag{2.22}$$

with $s_i(x) = 2\pi/g\epsilon_{ij}\hat{n}_j^C(x)\delta(x - C)$. We find it more convenient to label the operator $V$ by the vector function $s_i$ rather than by the curve $C$. They are of course in one to one correspondence.

There is one subtlety related to the algebra of the operators $V_s$ for odd $n$. Although the vortex operators commute for any $n$, their group multiplication properties are not completely trivial.

$$V_{s_1}V_{s_2} = e^{i\pi n N(1,2)}V_{s_1 + s_2} \tag{2.23}$$

where $N(1,2)$ is again the intersection number of the two curves. For even $n$ the extra phase factor is always an integer of $2\pi$. In this case one can construct physical states by requiring $V_s\Psi = \Psi$ for all $s$. For odd $n$ the phase factor can be an odd integer of $\pi$. We therefore can not require that the wave functional be invariant under the action of all $V_s$. This however can be remedied in the following way. Let us define for future convenience
the densities $\rho$ and $\sigma$ by

$$\frac{g}{2\pi} \epsilon_{ij} \partial_i s_j = \rho, \quad \frac{g}{2\pi} \partial_i s_i = \sigma$$

(2.24)

With our definition of $s_i$, both $\rho$ and $\sigma$ satisfy

$$\int_S d^2 x \rho(x) = 2\pi n_1, \int_S d^2 x \sigma(x) = 2\pi n_2$$

(2.25)

with integer $n_1$ and $n_2$, for any finite area $S$. Then the field $\tau$

$$\partial^2 \tau = 2\pi \sigma$$

(2.26)

takes integer values.

Now, note that the phase factor in eq. (2.23) can be represented as

$$\exp\{i\pi n N(1, 2)\} = \exp\{i\kappa \int \epsilon_{ij} s_i^1 s_j^2\} = \exp\{\frac{2\pi \kappa}{g^2} \int [\tau^1 \rho^2 - \rho^1 \tau^2]\}$$

(2.27)

Due to the property eq. (2.25) this can be further rewritten as

$$\exp\{i \frac{n}{2} \int [\tau^1 \rho^2 + \rho^1 \tau^2]\} = \exp\{i \frac{n}{2} \int [(\tau^1 + \tau^2)(\rho^1 + \rho^2)]\}$$

(2.28)

It is clear now, that a modified vortex operator, defined by

$$V_s = \exp\{i \int s_i E_i + i \frac{n}{2} \int \tau(s) \rho(s)\}$$

(2.29)

has a simple group multiplication rule

$$V_{s_1} V_{s_2} = V_{s_1 + s_2}$$

(2.30)

One therefore can require consistently, that these operators for all $s_i$ leave physical states invariant. In the following we will, however disregard this subtlety and for simplicity use the vortex operators defined in eq. (2.22). Thus our calculations in the way they are presented below are valid only for even $n$. 11
Before we continue, we wish to remark that the densities $\rho$ and $\sigma$ as defined in eq. (2.25) have a very clear physical meaning. The vortex operator $V_s$ has the following commutation relations with electric and magnetic fields

\[
V_s^\dagger(x)B(y)V_s(x) = B(y) + \frac{2\pi}{g}\rho(y) \quad (2.31)
\]
\[
V_s^\dagger(x)\partial_iE_i(y)V_s(x) = \partial_iE_i(y) + ng\rho(y)
\]
\[
V_s^\dagger(x)\epsilon_{ij}\partial_iE_j(y)V_s(x) = \epsilon_{ij}\partial_iE_j(y) + ng\sigma(y)
\]

Therefore the operator $V_s$ when acting on any state creates magnetic vortices (with magnetic flux quantized in units of $2\pi/g$) with the density $\rho$, and also creates electric vortices (with vorticity quantized in units of $ng$) with the density $\sigma$.

### 2.3 The Hamiltonian.

As just noted, the operators $B$ and $E_i$ do not commute with $V$. The Hamiltonian of the noncompact theory should, however be invariant under the complete compact group. It must therefore be slightly modified from the simple form of eq. (2.3). The most natural way to do it, is to use gauge invariant operators which in the naive continuum limit reduce to the standard $B^2$ and $E^2$ terms in the continuum hamiltonian. For the magnetic part of the energy density we take therefore

\[
H_B(x) = \frac{a^{-4}}{g^2m^2}[1 - \text{Re} \sigma^2B(x)] \quad (2.32)
\]

Here $a$ is the ultraviolet regulator which has a dimension of distance, and has the meaning of the lattice spacing.\footnote{In general, in the following wherever ultraviolet regularization is needed we assume lattice regularization with lattice spacing $a$. Accordingly, all derivatives should be understood as symmetric lattice derivatives etc.} The constant $m$ is an arbitrary integer, $m < n$. 

1. The constant $m$ is an arbitrary integer, $m < n$. 

As opposed to the case $\kappa = 0$, the electric part of the Hamiltonian should also be modified. Again, guided by the naive continuum limit and gauge invariance, we write for the electric part

$$H_E(x) = \frac{g^2 l^2 a^{-2}}{2\pi} \sum_{i=1,2} [1 - \text{Re} e^{i\alpha_i(x)} E_i(x)]$$

(2.33)

The vector function $\alpha_i(x)$ is defined on the links of the lattice in terms of the unit vector $\hat{n}_i(x)$ parallel to the given link as

$$\alpha_i(x) = \frac{2\pi a}{g} \epsilon_{ij} \hat{n}_j(x)$$

(2.34)

In order for $H_E$ to commute with the vortex operator $V$, the integer $l$ must be a divisor of $n$: $n/l = \text{integer}$. Note, that for $l = 1$, the operator $H_E(x)$ is equal to the sum of two vortex operators. In that case, it is trivial on all physical states. The same is true for $H_B$ for $m = n$. In that case, using the Gauss’ law one can see that $H_B$ is given entirely in terms of a vortex operator $V_s$ where the function $s_i$ corresponds to the contour $C$ which is the boundary of the elementary plaquette on the lattice. Therefore it is obvious, that in order for the dynamics of the theory defined with this Hamiltonian to be the most nontrivial, one should choose the largest possible $l$ and the smallest possible $m$. However, it turns out, that to avoid subtleties similar to the ones encountered for odd $n$, it is convenient to choose both $m$ and $n/l$ to be even numbers. We will therefore take $m = 2$ and $l = n/2$ in the rest of this paper.
3 The Variational wave functional.

Now it is straightforward to write down a general wave functional, which is invariant under the whole compact gauge group\footnote{The last, c-number, term in the phase factor appears, since the two components of the electric field do not commute, and is readily obtained using Baker-Campbell-Hausdorff formula.}

\[ \Psi[A_i] = \int Ds_i D\phi \exp \left\{ i\kappa \int d^2 x \left[ \epsilon_{ij}(s_i(x) - \partial_i \phi(x))A_j(x) + \epsilon_{ij} \partial_i \phi(x)s_j(x) \right] \right\} \Phi[A_i - \partial_i \phi - s_i] \] (3.35)

Our purpose is to find a wave functional of the vacuum of this theory using a gauge invariant generalization of a gaussian variational approximation. We shall minimize the VEV of the energy on the following set of explicitly gauge invariant states:

\[ \Psi[A_i] = \int Ds_i D\phi \exp \left\{ i\kappa \left[ s_i \epsilon_{ij} A_j - \partial_i \phi \epsilon_{ij} A_j + \epsilon_{ij} \partial_i \phi s_j \right] - \frac{1}{2} A^\phi,s G^{-1} A_i^{\phi,s} \right\} \] (3.36)

For convenience we have introduced the following notation

\[ A_i^{\phi(x), s(x)} = A_i(x) - \partial_i \phi(x) - s_i(x) \] (3.37)

and have switched to the matrix notations, so that

\[ a_i M b_i = \int d^2 x d^2 y a_i(x) M(x - y) b_i(y) \] (3.38)

In the next section we are going to calculate the expectation value of the Hamiltonian on this set of states, and functionally minimize it with respect to \( G(x - y) \), which is our variational parameter.

3.1 The noncompact limit.

We start with solving the noncompact limit of the theory. This serves to illustrate the method, and also to get a clear idea of what to expect of the variational function \( G(x) \).
Our variational state now is given by eq.(3.36), but without the integration over $s_i$. The Hamiltonian is the simple quadratic Hamiltonian of eq.(2.3). The exact ground state wavefunctional in this case is therefore gaussian. So our variational calculation in this case will give the exact result. The expectation value of any gauge invariant operator is given by the following path integral

$$< O[A] > = \int D\phi DA_i \exp \left\{ -\frac{1}{2} A_i^\phi G^{-1} A_i^\phi + i\kappa \partial_i(\phi)\epsilon_{ij} A_j \right\} O[A] \exp \left\{ -\frac{1}{2} A_i G^{-1} A_i \right\}$$

(3.39)

The integrals over $A_i$ and $\phi$ are both Gaussian, and are easily performed. Calculating the relevant averages we obtain

$$< B^2(x) > = \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} b^2 [G^{-1}(p) + \kappa^2 G(p)]^{-1}$$

(3.40)

$$< E^2(x) > = \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \left[ G^{-1}(p) + \kappa^2 G(p) + 4\kappa^2 [G^{-1}(p) + \kappa^2 G(p)]^{-1} \right]$$

(3.41)

Here $G(k)$ is the Fourier transform of the variational function $G(x)$.

The expectation value of the energy is

$$2V^{-1} < H >= \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \left[ G^{-1}(p) + \kappa^2 G(p) + (p^2 + 4\kappa^2)[G^{-1}(p) + \kappa^2 G(p)]^{-1} \right]$$

(3.42)

Minimizing this expression with respect to $G(p)$ we obtain

$$G^{-1}(p) + \kappa^2 G(p) = \sqrt{p^2 + 4\kappa^2}$$

(3.43)

As is clear from this calculation, the function

$$D(p) = [G^{-1}(p) + \kappa^2 G(p)]^{-1}$$

(3.44)

plays the role of the propagator. For example, the equal time propagator of magnetic field is given by

$$< B(x)B(y) >= \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} e^{ip(x-y)}(2\pi)^2 p^2 [G^{-1}(p) + \kappa^2 G(p)]^{-1}$$

(3.45)
As expected, $D(p)$ in the noncompact case comes out as the propagator of a free massive particle with mass $m^2 = 4\kappa^2$.

3.2 The norm of the state in the compact theory.

As a preamble to calculating the energy expectation value, let us find the norm of the state eq. (3.36).

$$Z = \int Ds_i D\phi DA_i \exp \left\{ -\frac{1}{2}[A_i G^{-1}A_i + A_i^{\phi,s} G^{-1} A_i^{\phi,s}] - i\kappa[(s_i - \partial_i \phi)\epsilon_{ij}A_j + \epsilon_{ij}\partial_i s_j] \right\}$$

(3.46)

The integrals over $A_i$ and over the noncompact part of the gauge group, $\phi$ are gaussian. After performing them we are left with the integral over the discrete variable $s_i$.

$$Z = Z_\alpha Z_\phi Z_s$$

(3.47)

with

$$Z_\alpha = \det(\pi G)$$

(3.48)

$$Z_\phi = \int D\phi \exp \left\{ -\frac{1}{4} \int \partial_i \phi D^{-1} \partial_i \phi \right\} = \det \left[ \frac{\partial^2}{4\pi} D^{-1} \right]^{-1/2}$$

$$Z_s = \int Ds_i \exp \left\{ -\frac{1}{4} \int \frac{d^2p}{(2\pi)^2} \left[ s_i(p)\left[ \delta_{ij} - \frac{p_ip_j}{p^2} \right] D^{-1}(p)s_j(-p) + \kappa^2 s_i(p)\frac{p_ip_j}{p^2} D(p)s_j(-p) \right] \right\}$$

where $D(p)$ is defined in eq.(3.44).

A more convenient and physically intuitive representation of the integral over $s_i$ can be given in terms of the variables $\rho$ and $\sigma$ defined in eq.(2.24).

$$Z_s = Z_\rho Z_\sigma$$

(3.49)

with

$$Z_\rho = \int D\rho \exp \left\{ -\frac{\pi^2}{g^2} \int \frac{d^2p}{(2\pi)^2} \rho(p)\frac{1}{p^2} D^{-1}(p)\rho(-p) \right\}$$

(3.50)
\[ Z_\sigma = \int D\sigma \exp \left\{ -\pi n\kappa \int \frac{d^2p}{(2\pi)^2} \sigma(p) \frac{1}{p^2} D(p) \sigma(-p) \right\} \] (3.51)

One should keep in mind, that the measures \( D\rho \) and \( D\sigma \) are not the same as the measure for the functional integration over a free field. Rather, due to eq.(2.25) both these path integrals are equivalent to statistical sums for classical interacting gases:

\[
Z_\rho = \sum_{n_+,n_-=0}^{\infty} \prod_{\alpha=1}^{n_+} \prod_{\beta=1}^{n_-} (a^{-2})^{n_+ n_-} dx_\alpha dx_\beta \tag{3.52}
\]

\[
\exp \left\{ -\left[ \sum_{\alpha,\alpha'} u(x_\alpha - x_{\alpha'}) + \sum_{\beta,\beta'} u(x_\beta - x_{\beta'}) - \sum_{\alpha,\beta} u(x_\alpha - x_\beta) \right] + \left( \frac{n_+ + n_-}{2} \right) u(0) \right\}
\]

\[
Z_\sigma = \sum_{n_+,n_-=0}^{\infty} \prod_{\alpha=1}^{n_+} \prod_{\beta=1}^{n_-} (a^{-2})^{n_+ n_-} dx_\alpha dx_\beta \tag{3.53}
\]

\[
\exp \left\{ -\left[ \sum_{\alpha,\alpha'} v(x_\alpha - x_{\alpha'}) + \sum_{\beta,\beta'} v(x_\beta - x_{\beta'}) - \sum_{\alpha,\beta} v(x_\alpha - x_\beta) \right] + \left( \frac{n_+ + n_-}{2} \right) v(0) \right\}
\]

The interparticle interaction potentials \( u \) and \( v \) are given by

\[
u(x) = \frac{2\pi^2}{g^2} \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2} D^{-1}(p) \cos(px) \tag{3.54}
\]

\[
v(x) = 2\pi n\kappa \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2} D(p) \cos(px)
\]

Since \( u(0) \) and \( v(0) \) are singular, the last terms in the exponential in the equations (3.53,3.54) should be understood, as usual in the regularized sense, that is at finite UV cutoff: \( u(0) \) (\( v(0) \)) should be substituted by \( u(x = a) \) (\( v(x = a) \)).

Remembering the interpretation of \( \rho \) and \( \sigma \), discussed in the previous section, we shall refer to the statistical mechanical systems defined by eqs.(3.53) and (3.54) as gases of magnetic vortices and electric vortices respectively.

The calculation of any expectation value (such as the expectation value of the energy) in our trial wave functional should now proceed in the standard way. The integration over
the vector potential $A_i$ and the noncompact part of the gauge group is always Gaussian and therefore trivial. After performing this integral we will be invariably faced with the problem of calculating certain correlators in the two vortex ensembles, eq. (3.53, 3.54). It is therefore worthwhile to try and understand some general features of these ensembles.

First note, that the cases $\kappa = 0$ and $\kappa \neq 0$ are essentially different. For $\kappa = 0$ (or equivalently $n = 0$), the electric vortex partition function becomes completely trivial - the interaction potential vanishes. This is of course to be expected, since we know that the electric vortices do not play any role in the theory without the Chern - Simons term [7].

Another crucial difference is that the behaviour of the magnetic vortex interaction at large distances is significantly changed. At $n = 0$ the function $D^{-1}(p) = G^{-1}(p)$ for the best variational state vanishes at zero momentum [7]. Consequently, the interaction potential between the magnetic vortices is short range. For $\kappa \neq 0$ on the other hand this cannot happen, since the expression $D^{-1}(p) = G^{-1}(p) + \kappa^2 G(p)$ is bounded from below by $2\kappa$. Therefore the interaction between the vortices is logarithmic at large distances, and whatever $G$ is, we are dealing with the Coulomb gas. Obviously, for large $\kappa$ the gas will be in the molecular phase, and one expects that it will have no effect on the large distance physics. This is in agreement with the general arguments of ref.[3], that monopoles should be irrelevant in a Chern - Simons theory. However at smaller $\kappa$ the Coulomb gas will be in the plasma phase, and will certainly affect physics. The effective temperature of the Coulomb gas of magnetic vortices in our case is $T_m = 2g^2/2\kappa = 4\pi/n$, and one therefore expects this change in behaviour to set in at $n = 8$.

For nonzero $n$ the electric vortices behave in a similar way. Taking the noncompact expression $D^{-1}(p = 0) = 2\kappa$, we find that at large distances the interaction between the
electric vortices is also logarithmic, and in fact has the same asymptotics as the magnetic vortex potential. The effective temperature is also \( T_e = 4\pi/n \) and one again expects the Berezinsky-Kosterlitz-Thouless like phase transition at \( n = 8 \).

### 3.3 The magnetic vortex gas.

In the following we will need to calculate correlation functions of the vortex densities. To facilitate this we use the standard trick [2], [8] to rewrite the partition function of a classical gas in terms of a path integral over a scalar field. First consider the magnetic vortices. The exponential factor in eq. (3.53) can be rewritten as

\[
a^{-2(n_+ + n_-)} \int D\chi \exp\{-\frac{g^2}{4\pi^2} \chi p^2 D\chi + i \rho \chi\}
\]

(3.55)

The summation over the number of vortices (or integral over \( \rho \)) can be performed exactly

\[
\int D\rho \exp\{i \rho(x) \chi(x)\} = \delta(\exp\{i \chi(x)\} - 1)
\]

(3.56)

This gives

\[
Z_\rho = \int D\chi \exp\{-\frac{g^2}{4\pi^2} \chi p^2 D\chi\} \Pi_x \delta(\exp\{i \chi(x)\} - 1)
\]

(3.57)

\[
= \int D\chi D\alpha \exp\left\{-\frac{g^2}{4\pi^2} \chi p^2 D\chi + i \int_x 2\alpha(x)(\cos \chi(x) - 1)\right\}
\]

In general the integration over the lagrange multiplier field \( \alpha \) is nontrivial. However, when the density of vortices is small, one can perform the integration over \( \rho \) in the dilute gas approximation, that is sum only over those configurations which have only at most one vortex or antivortex in every point in space. This is equivalent to substitute for \( \alpha \) a constant

\[
Z_\rho = \int D\chi \exp\left\{-\frac{g^2}{4\pi^2} \chi p^2 D\chi + 2a^{-2} \cos \chi(x)\right\}
\]

(3.58)
To calculate the correlator of $\rho$ in the dilute gas approximation one can add $i\rho J$ to the vortex free energy, and calculate functional derivatives of the resulting partition function with respect to $J$ at zero $J$. A simple derivation gives

$$<\rho(x)\rho(y)> = u^{-1}(x-y) - <u^{-1}\chi(x)u^{-1}\chi(y)>$$

(3.59)

The propagator of $\chi$ is easily calculated. At weak coupling the interaction in eq. (3.58) is very small ($<\cos \chi> \ll 1$). To first order in the interaction the only contribution to the propagator comes from the tadpole diagrams. This is easily seen by rewriting the cosine potential in equation (3.58) in the normal ordered form

$$\cos \chi = <\cos \chi> : \cos \chi :$$

(3.60)

One should be a little careful in the definition of the normal ordering. Ordinarily the normal ordering would be performed relative to the free theory with the propagator $u(x-y)$. In the present case however the free propagator at small momentum behaves like a propagator of a massless particle $u(k) \rightarrow k \rightarrow 0 k^2$, and the bubble integral which enters the calculation of $<\cos \chi>$ is infrared divergent. This problem can be overcome by performing normal ordering relative to a massive theory. This can be done selfconsistently, by including the quadratic term in $:\cos(\chi):$ into the free propagator. In this approximation the propagator of $\chi$ is

$$\int d^2x e^{ikx} <\chi(x)\chi(0)> = \frac{1}{u^{-1}(k) + 2z}$$

(3.61)

with $z$ determined selfconsistently by

$$z = a^{-2} <\cos(\chi)> = a^{-2} \exp\{-\frac{1}{2} <\chi^2>\} =$$

$$a^{-2} \exp\left\{-\frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \left[ \frac{g^2}{2\pi^2} p^2 D(p) + 2z \right]^{-1}\right\}$$

(3.62)
The correlator of the vortex densities is then
\[
K(k) = \int d^2 x e^{i k x} < \rho(x) \rho(0) > = \frac{2z}{1 + 2zu(k)} \tag{3.63}
\]

The existence of the critical point \( n = 8 \) is straightforward to see in this approximation. Using the result for the noncompact theory \( D(0) = 1/2\kappa \), and anticipating the fact that the infrared asymptotics of the propagator is the same in the noncompact theory, we can rewrite eq. (3.63) as
\[
z = a^{-2}(za^2 e^{-c})^{n/8} e^{-\mu_\rho} \tag{3.64}
\]

Here the constant \( c \) is the chemical potential of the pure Coulomb gas. It’s exact value is not important, but one has to remember that it is of order one and positive. The chemical potential \( \mu_\rho \) measures how different the magnetic vortex gas is from the Coulomb gas and is defined by
\[
\mu_\rho = \frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left[ \left( \frac{g^2}{2\pi^2} p^2 D(p) + 2z \right)^{-1} - \left( \frac{1}{\pi np^2} + 2z \right)^{-1} \right] \tag{3.65}
\]

The chemical potential \( \mu_\rho \) depends on \( z \) very weakly for small \( z \). Also, at weak coupling the chemical potential is very big. Taking, for orientation the noncompact result for \( D(p) \), we find that \( \mu_\rho \propto \frac{a^{-1}}{g^2} \). Under these circumstances it is easily seen that for \( n < 8 \) the solution for eq. (3.65) exists, and is indeed parametrically much smaller than the ultraviolet cutoff \( a^{-2} \).
\[
z \propto a^{-2} e^{-\frac{8}{8-n} \mu_\rho} \tag{3.66}
\]

For \( n > 8 \) one can check that the selfconsistency equation eq. (3.63) has only the trivial solution \( z = 0 \).

\footnote{Even though eq. (3.66) for \( n > 8 \) seems to give \( z \propto O(a^{-2}) \) this is not correct, since this equation was derived under assumption that \( z \) is small.}
Importantly, the use of the noncompact expression for $D(p)$ is not at all crucial for this derivation. Remember, that quite generally $D(p) < 1/2\kappa$. The chemical potential for the magnetic vortices is therefore always larger than the chemical potential in the Coulomb gas that corresponds to the XY model. The dilute gas approximation is known to work reasonably well for the XY model [4], and we are assured therefore that the dilute vortex gas approximation for the magnetic vortex gas should be reliable for any variational function $G(p)$.

3.4 The electric vortex gas.

While the magnetic vortex gas can be consistently treated in the dilute gas approximation, the behaviour of the electric vortex gas is more complicated. A dilute electric gas approximation is easily formulated along the same lines. This results in a partition function:

$$Z_\sigma = \int D\psi \exp \left\{ -\frac{1}{4\pi n\kappa} \psi^2 D^{-1} \psi + \int d^2 x \frac{2a^{-2}}  \cos \psi(x) \right\}$$

(3.67)

The “bubble summation” then gives the propagator of the field $\psi$

$$\int d^2 x e^{ikx} < \psi(x)\psi(0) > = \frac{1}{v^{-1}(k) + 2\zeta}$$

(3.68)

with $\zeta$ determined selfconsistently by

$$\zeta = a^{-2} < \cos(\psi) > = a^{-2} \exp \left\{ -\frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left[ \frac{1}{2\pi n\kappa} p^2 D^{-1}(p) + 2\zeta \right]^{-1} \right\}$$

(3.69)

This can be recast into the form

$$\zeta = a^{-2}(\zeta a^2 e^{-c})^{n/8} e^{-\mu_\sigma}$$

(3.70)

with the chemical potential $\mu_\sigma$

$$\mu_\sigma = \frac{1}{2} \int \frac{d^2 p}{(2\pi)^2} \left[ \left( \frac{1}{2\pi n\kappa} p^2 D^{-1}(p) + 2\zeta \right)^{-1} - \left( \frac{1}{2\pi n} p^2 + 2\zeta \right)^{-1} \right]$$

(3.71)
One should however ask oneself, under what circumstances is the dilute gas approx-
imation valid. A necessary condition for this is that the interaction energy between the
vortices is not negligibly small even at very short distances of order of the ultraviolet
cutoff. It is clear therefore, that the diluteness of the electric vortex gas holds only for
some restricted set of the variational functions $D$. Let us take, for example the noncom-
 pact form for $D(p)$, so that for small momenta $D(p) = 1/2\kappa$ and in the ultraviolet region
$D(p) \propto (p^2)^{-1/2}$. In this case the interaction potential between the vortices is logarithmic
at large distances ($x > 1/\kappa$) but at short distances is very weak $v(x) \propto n\kappa x$. The vortex
gas will therefore be dilute on all distance scales only if $n > (\kappa a)^{-1}$, that is either for very
large values of $n$, or in a theory where the mass of the photon is close to the ultraviolet
cutoff. The former case corresponds to extremely weak coupling, while the latter to the
limit of pure Chern - Simons theory without the Maxwell term. On the other hand for
finite $n$ and $\kappa a \to 0$ (naive continuum limit), the electric vortices therefore hardly feel
the presence of each other if the distance between them is smaller than $1/\kappa$. At these
distance scales the gas will therefore not be dilute at all. To wit, in this case the chemical
potential $\mu_\sigma$ defined by eq. (3.71) is large and negative.

Another situation in which the dilute gas approximation is valid, is for those variational
functions $D(p)$ which have a constant value $D(p) = 1/2\kappa$ for all momenta larger than some
scale $\mu$, such that $\mu a << 1$. In this case $\mu_\sigma$ is close to zero, and one is back to the case
of pure Coulomb gas. For these variational functions the behaviour of the electric vortex
gas is very similar to the behaviour of the magnetic vortex gas. Both have BKT phase
transition at $n = 8$, both are in the molecular phase for $n > 8$ and in the plasma phase
at $n < 8$.

The question is however, is there any reason to expect, that the best variational
propagator will behave in this fashion. Surprising as it may seem at the first glance, the answer to this is positive. In fact, a little thought convinces one that it is almost unavoidable in a compact theory with finite radius of compactness $1/g$. The point is the following. As discussed in Section 2, imposing compact gauge invariance, among other things has an effect of imposing the following condition on the magnetic field

$$e^{ia^2g n B(x)}|\Psi> = |\Psi>$$

(3.72)

This means, that a gauge invariant state $|\Psi>$ has nonzero projection only on states with quantized eigenvalues of the magnetic field $B$: $B(x) = \frac{2\pi a^{-2}}{n}$. We know, however that in the noncompact vacuum $<B^2> \propto a^{-3}$. This means that the natural scale for the magnetic field in the vacuum state of the noncompact theory is $B \propto a^{-3/2}$. Therefore if one projects a state with the ultraviolet properties of the noncompact vacuum onto a gauge invariant state, only the contributions of $B = 0$ states will survive and the magnetic energy will vanish. Since the expectation value of the magnetic field in a noncompact state is just given by $<B^2> = \int \frac{dp}{4\pi^2} p^2 D(p)$, a nontrivial compact dynamics can be described only by states with a much larger value of the “propagator” $D(p)$. In fact, since $D(p)$ is bounded from above, nontrivial fluctuations of magnetic field can survive only if $D(p) = O(1/\kappa)$ for all momenta between the ultraviolet cutoff $a^{-1}$ and some intermediate scale $\mu$, which itself is much less than the cutoff. In this case one has $<B^2> \propto a^{-4}\kappa^{-1}$ and the scale of the fluctuations is just right to satisfy eq. (3.72) in a nontrivial way.

We see therefore that the emergence of the intermediate scale $\mu$ is mandatory in the compact theory, and the dilute gas approximation for electric vortices gas should be reliable. It turns out however, that one can not determine the value of the scale $\mu$ within the dilute approximation itself. The reason is that, at least in the molecular phase, the behaviour of the electric vortices is very insensitive to the exact value of the crossover
scale $\mu$, as long as it is much smaller than the ultraviolet cutoff. This is so because in the Coulomb gas, away from the critical point the vortices are bound in pairs of the characteristic size of order of the ultraviolet cutoff $a$. Therefore if the interaction potential is changed at distances $x > 1/\mu >> a$, the behaviour of the gas hardly changes at all.

Even though the precise value of $\mu$ can not be determined, we can establish reliably the important fact that $\mu >> \kappa$. This is of course crucial, since if this was not the case the propagator would be constant practically for all momenta, and the theory would have no propagating degrees of freedom in the continuum limit. The reason this can be established, is that, as will be shown in the next section within the dilute gas approximation the scale $\mu$ is pushed up all the way to the ultraviolet cutoff. Of course, as discussed earlier, one cannot believe the approximation for such large $\mu$. However, the dilute gas approximation is expected to be valid for values of $\mu$ which are much larger than $\kappa$ as long as they are much smaller than the ultraviolet cutoff.

To verify this picture and also estimate the value of the crossover scale $\mu$ we will also perform a calculation using a different approximation for the electric vortex gas. This approximation is valid for the propagator functions $D(p)$ which vanish at large momentum, so that functions with the ultraviolet asymptotics of the noncompact propagator can be studied reliably. To define this approximation we rewrite the partition function of the electric gas in terms of the integer valued field $\tau$ defined in eq. (2.26) as

$$Z_\sigma = \int D\tau D\beta \exp \left\{ -\pi n\kappa p^2 D\tau + i \int d^2 x 2\beta(x)(\cos(\tau(x)) - 1) \right\}$$ \hspace{1cm} (3.73)

Approximating the Lagrange multiplier field $\beta$ by a constant (the same kind of approximation which lead to eq. (3.58) for magnetic vortices) gives a tractable expression.

$$Z_\sigma = \int D\tau \exp \left\{ -\pi n\kappa p^2 D\tau + 2a^{-2} \int d^2 x (\cos(\tau(x)) - 1) \right\}$$ \hspace{1cm} (3.74)
It is again convenient to define an appropriate fugacity $\xi$ in analogy to (3.69).

$$\frac{n}{4\pi} \xi = a^{-2} < \cos(\tau) >= a^{-2} \exp \left\{ -\frac{\pi}{n} \int \frac{d^2 p}{(2\pi)^2} \left[ \kappa p^2 D(p) + \xi \right]^{-1} \right\}$$  \hspace{1cm} (3.75)

With the noncompact function $D$, this fugacity is exponentially small

$$\xi \propto \exp\left(-\frac{r}{n}(a\kappa)^{-1}\right)$$  \hspace{1cm} (3.76)

where $r$ is a numerical constant. For this type of variational functions therefore, electric vortices can be consistently treated in this “$\tau$ - approximation”.

4 Minimization of the energy and the phases of the model.

4.1 The expectation value of the energy.

Now we are ready to compute the expectation value of the Hamiltonian eqs. (2.32,2.33). We start with the magnetic part eq.(2.32). Straightforward integration over the vector potential and the noncompact part of the gauge group gives

$$< \exp\{i2ga^2 B(x)\} > = \exp\left\{ -a^4 g^2 \int \frac{d^2 p}{4\pi^2} p^2 D(p) \right\} \times$$  \hspace{1cm} (4.77)

$$< \exp\{i2a^2 \pi \rho(x)\} >_\rho < \exp\left\{ -2a^2 \kappa \int \frac{d^2 p}{4\pi^2} e^{ipx} D(p) \sigma(p) \right\} >_\sigma$$

Where $<>_\rho$ and $<>_\sigma$ mean average over the magnetic and electric vortex ensembles correspondingly. The magnetic vortex contribution is trivial, since $a^2 \rho(x)$ is an integer.

We calculate the electric vortex contribution in the dilute gas approximation, keeping the first order correction in $\zeta$. A straightforward calculation gives

$$< \exp\left\{ -2a^2 \kappa \int \frac{d^2 p}{4\pi^2} e^{ipx} D(p) \sigma(p) \right\} >_\sigma =$$  \hspace{1cm} (4.78)

$$\exp\left\{ \frac{8\pi \kappa}{n} a^4 \zeta \int \frac{d^2 p}{4\pi^2} \frac{p^2 D(p)}{2\pi \kappa p^2 D^{-1}(p) + 2\zeta} + 2\zeta \int d^2 x \left[ \cosh X(x) - 1 - \frac{1}{2} X(x) \right] \right\}$$

26
where the function $X(x)$ is defined by

$$X(x) = \frac{2}{n} a^2 \int \frac{d^2 p}{4\pi^2} e^{ipx} \frac{p^2}{1/2\pi n p^2 D^{-1}(p) + 2\zeta}$$

(4.79)

An analogous calculation for the electric energy part, eq. (2.33) gives

$$\langle \exp \left\{ \frac{i^2}{n} \alpha_i(x) E_i(x) \right\} \rangle = \exp \left\{ -\frac{n a^2}{2} \int \frac{d^2 p}{4\pi^2} (D^{-1}(p) + 4\kappa^2 D(p)) \right\}$$

(4.80)

$$\langle \exp \left\{ -\frac{\kappa}{\pi} \int \frac{d^2 p}{4\pi^2} e^{ipx} \rho(p) p^2 D^{-1}(p) \right\} \rangle < \langle \exp \left\{ -\frac{\pi a^2}{2n\kappa} \int \frac{d^2 p}{4\pi^2} \rho(p) p^2 D^{-1}(p) \right\} \rangle > \langle \exp \left\{ -\frac{\kappa}{\pi} \int \frac{d^2 p}{4\pi^2} e^{ipx} \sigma(p) D(p) \omega(p) \right\} \rangle$$

where we have defined

$$\epsilon = \frac{g}{2\pi} \epsilon_{ij} \partial_i \alpha_j; \quad \partial^2 \omega = g \partial_i \alpha_i$$

(4.81)

The electric vortex contribution is calculated as

$$\exp \left\{ \frac{4\pi \kappa a^2}{n} \int \frac{d^2 p}{4\pi^2} \frac{D(p)}{1/2\pi n p^2 D^{-1}(p) + \zeta} + 2\zeta \int d^2 x \left[ \cosh Y(x) - 1 - \frac{1}{2} Y^2(x) \right] \right\}$$

(4.82)

with

$$Y(x) = \frac{2}{n} \int \frac{d^2 p}{4\pi^2} e^{ipx} \frac{p^2 \omega(p)}{1/2\pi n p^2 D^{-1}(p) + 2\zeta}$$

(4.83)

The magnetic vortex contribution this time is nontrivial and is given by

$$\exp \left\{ \frac{\pi a^2}{\kappa n} \int \frac{d^2 p}{4\pi^2} \frac{D^{-1}(p)}{2\pi n p^2 D(p) + 2z} + 2z \int d^2 x \left[ \cosh L(x) - 1 - \frac{1}{2} L^2(x) \right] \right\}$$

(4.84)

with

$$L(x) = \frac{2}{n} \int \frac{d^2 p}{4\pi^2} e^{ipx} \frac{\epsilon(p)}{2\pi n p^2 D(p) + 2z}$$

(4.85)
Here we also give the expressions for the electric vortex contributions calculated in the “τ”-approximation:

\[
< \exp \left\{ -2a^2 \kappa \int \frac{d^2p}{4\pi^2} e^{ipx} D(p) p^2 \tau(p) \right\} >_\sigma = \exp \left\{ a^4 g^2 \int \frac{d^2p}{4\pi^2} \kappa D^2(p) p^4 + \frac{n\xi}{2\pi} \int d^2x \left[ \cos I(x) - 1 - \frac{1}{2} I^2(x) \right] \right\}
\]

and

\[
< \exp \left\{ -\kappa \pi \int \frac{d^2p}{4\pi^2} e^{ipx} \tau(p) p^2 D(p) \omega(p) \right\} >_\sigma = \exp \left\{ 2\pi a^2 n \int \frac{d^2p}{4\pi^2} e^{ipx} \frac{p^2 D(p)}{\kappa p^2 D(p) + \xi} \right\}
\]

with \( I(x) \) and \( J(x) \) defined by

\[
I(x) = \frac{4\pi a^2 \kappa}{n} \int \frac{d^2p}{4\pi^2} e^{ipx} \kappa p^2 D(p) \frac{p^2 D(p)}{\kappa p^2 D(p) + \xi}
\]

\[
J(x) = \frac{2n}{4\pi^2} \int \frac{d^2p}{4\pi^2} e^{ipx} \kappa p^2 D(p) \omega(p) \frac{p^2 D(p)}{\kappa p^2 D(p) + \xi}
\]

We will now use these expressions to minimize the average energy. We will consider the cases \( n > 8 \) and \( n < 8 \) separately.

**4.2 \( n > 8 \)**

As was discussed in the previous section, for \( n > 8 \) the magnetic vortex gas is in the dipole phase. The vortex fugacity \( z \) vanishes, and in the dilute gas approximation they do not contribute to the energy.

The electric vortex gas will also be treated here in the dilute gas approximation. As explained in the previous section, the minimization here is performed on the set of variational functions \( D(p) \) which have the constant value \( D(p) = 1/2\kappa \) for all momenta.
above some scale $\mu << a^{-1}$. For these variational functions, the electric vortex gas is indeed dilute, and for $n > 8$ is in the molecular phase. The electric fugacity $\zeta$ also vanishes, and therefore the electric vortices also do not contribute to the energy.

The calculation is further simplified, since as can be checked explicitly with these variational functions $D(p)$ the expectation values in eqs.(4.78) and (4.81) for large $n$ are close to one. Therefore the relevant exponentials can be expanded up to first order in Taylor series.

In this approximation, therefore the expectation value of energy is the same as for the noncompact theory and the minimization with respect to $D(p)$ gives

$$D^{-1}(p) = \sqrt{p^2 + 4\kappa^2}, \quad p^2 < \mu^2$$

(4.90)

We therefore find that for $n > 8$ both magnetic and electric vortices are irrelevant, and the compact theory at small momenta is indistinguishable from the noncompact one.

One assumption that was made in the previous discussion is that the value of the propagator $D$ at large momenta is exactly $1/2\kappa$. The general argument given in the previous section pretty much establishes that this value should be of order $1/\kappa$, but does not establish the coefficient. This magic number $1/2$ enabled us to conclude that the critical value of $n$ for the electric vortex gas is the same as for magnetic vortex gas, $n = 8$. To verify this picture further, we would like to demonstrate explicitly that above the scale $\mu$ this is indeed the correct behaviour of the propagator. To this end we will minimize the energy calculated in the “$\tau$-approximation”. This calculation is valid as long as the fugacity $\xi$ is small. To make sure this condition is satisfied, we will first consider only such functions $D(p)$ that behave $D(p) \propto 1/\sqrt{p^2}$ for momenta $\lambda^2 < p^2 < a^{-2}$ with $\lambda < a^{-1}$. Note, that although $\lambda$ must be smaller than the ultraviolet cutoff, it does not have to be parametrically smaller, so that for example $\lambda a$ may remain a finite small constant in the
limit $a \rightarrow 0$.

The terms depending on $I$ and $J$ (see (4.89), (4.85)) in the expression for the expectation value of the energy can be neglected at large $n$, since they give corrections suppressed by powers of $1/n^2$. Now, substituting the formulae of the previous section into equations (2.32) and (2.33) we obtain for the expectation value of the energy $H$

$$2V^{-1} < H > = \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \left[ D^{-1}(p) + (p^2 + 4\kappa^2)D(p)\frac{\xi}{\kappa p^2 D(p) + \xi} \right]$$

Minimizing this with respect to $D(p)$ we obtain

$$D^{-2}(p) = Z(p^2 + M^2)\frac{\xi^2}{(\kappa p^2 D(p) + \xi)^2}$$

where the constants $Z$ and $M$ are

$$Z = 1 + \frac{\kappa}{\xi} \frac{\partial \ln \xi}{\partial \ln \xi D} \frac{(\kappa p^2 D + \xi)^2}{\kappa p^2 D} \int \frac{d^2p}{4\pi^2} (p^2 + 4\kappa^2)\frac{\kappa p^2 D^2(p)}{(\kappa p^2 D(p) + \xi)^2}$$

$$M^2 = 4\kappa^2 Z^{-1}$$

The solution of eq.(4.92) is

$$D^{-1}(p) = Z^{1/2}(p^2 + M^2)^{1/2} - \frac{\kappa}{\xi} p^2$$

Since $D^{-1}$ is bounded from below, this solution is valid for momenta for which $D^{-1} > 2\kappa$, that is for

$$p^2 < \mu^2 = \frac{4\xi^2}{M^2} (1 - \frac{M^2}{\xi})$$

For momenta greater than $\mu$ and smaller than $\lambda$

$$D^{-1}(p^2 > \mu) = 2\kappa$$

We see that the physics corresponding to this solution is precisely the same as we were describing earlier. Even though we have not determined the value of the scale $\mu$ yet, let
us for the moment assume that it satisfies the relation $M \ll \mu \ll a^{-1}$. Then at low momenta $p^2 \ll \mu^2$ the variational propagator is the same as in the noncompact case. Therefore the physics it describes at all physical scales is the same as in the noncompact theory. The only difference is that there appears a nontrivial mass and wave function renormalization given by the same factor $Z$ in eqs.(4.93,4.94), which is small as long as $\xi$ is small enough. At scales $p^2 > \mu$ the propagator is frozen, and therefore at these momenta the theory does not contain propagating degrees of freedom. Thus effectively, the ultraviolet cutoff has been also changed from $a^{-1}$ to a lower value $\mu$. Note also that for $\xi \ll \lambda^2$, which is where the approximation is valid, the scale $\mu$ also satisfies $\mu \ll \lambda$.

The emergence of the scale $\mu$ and the change in the behaviour of the propagator at large momenta is therefore seen already in the regime where the $\tau$-approximation is valid. It is again difficult to establish the precise value of this scale $\mu$. The point is that it grows very fast when the auxiliary scale $\lambda$ approaches the ultraviolet cutoff. It is clear from eqs.(4.95,4.96) that the dynamics favours precisely this situation. However, when $\lambda$ is close to $a^{-1}$, the $\tau$-approximation ceases to be valid.

A rough estimate of $\mu$ can be obtained by pursuing the $\tau$-approximation to the end. Calculating the integral in (3.75) using an explicit expression for $D(p)$ (4.95) one has after some algebra

$$\xi = \frac{4\pi}{n} a^{-2} \exp \left( -\frac{1}{3n} \frac{\xi}{M^2} - \frac{1}{n} \ln \frac{M}{2\xi a} \right)$$

(4.98)

Using the fact that $\ln(1/Ma) \gg 1$ the expression for fugacity is as follows

$$\xi = 3nM^2 \ln \frac{4\pi}{na^2 M^2} [1 + O(\ln\ln(1/Ma)/\ln(1/Ma))]$$

(4.99)

From this we get

$$\mu = 6nM \ln \frac{4\pi}{na^2 M^2}$$

(4.100)
We see indeed, that $\mu / M \rightarrow a \rightarrow 0 \propto$.

We stress again, that the exact value of $\mu$ as given in eq.(4.100) is not to be trusted literally. It is obtained in the region of variational functions $D(p)$ where the nonlinearities in the effective $\tau$ theory are not at all small. For example, there is a large (logarithmic in cutoff) renormalization of the fugacity $\xi$ due to the cosine interaction already at the two loop level. On the other hand we believe that this renormalization will be the leading effect. If it is taken into account properly, the expressions (4.95, 4.99, 4.100) will not be changed drastically if $\xi$ appearing in them is understood as the complete renormalized fugacity. We have checked for example, that the $I$ and $J$ dependent terms in the energy expectation value, which appear due to the nonlinearity in the $\tau$ action, are suppressed relative to the terms we have kept in eq.(4.91) even for fugacities $\xi$ of eq.(4.99). We believe therefore that the estimate for the crossover scale $\mu$ eq.(4.100) is qualitatively correct.

To summarize this part, we find that for $n > 8$ the compactness of the theory is not important for infrared physics. The best variational propagator at physical momenta is identical to the propagator in the noncompact theory. The only effect of the finite radius of compactness, $1 / g$ is that it “freezes” the propagation of the modes with high momentum, thereby effectively just changing the ultraviolet cutoff.

4.3 $n < 8$.

Let us now turn to a more interesting case $n < 8$. At these values of $n$ both, magnetic and electric vortices are in the plasma phase. The fugacities $z$ and $\zeta$ do not vanish, and in the dilute gas approximation we obtain for the expectation value of the energy
\[2V^{-1} < H > \quad = \quad \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \left( \int \frac{D^{-1}(p)}{2\pi n^2 p^2} \right) \left[ 1 - \frac{2z}{\frac{2z}{\pi n^2} p^2 D(p) + 2z} \right] \]

\[+ \quad (p^2 + 4\kappa^2) D(p) \left[ 1 - \frac{2\zeta}{2\pi n^2 p^2 D^{-1}(p) + 2\zeta} \right] \]

\[- \quad \frac{n}{2\pi \kappa} a^{-4} \zeta \int d^2x (\cosh X - 1 - \frac{1}{2} X^2) \]

\[- \quad \frac{4\kappa n}{\pi} a^{-2} \zeta \int d^2x (\cosh Y - 1 - \frac{1}{2} Y^2) \]

\[- \quad \frac{4\kappa n}{\pi} a^{-2} z \int d^2x (\cosh L - 1 - \frac{1}{2} L^2) \]

The last three terms appear due to nonlinearities in the \(\psi\) and \(\chi\) Lagrangians. They are negligible as long as the appropriate gases are dilute, that is \(\zeta\) and \(z\) are smaller than the ultraviolet cutoff. We present these terms here for completeness, but will neglect them in the following derivations. It can be checked that on the solution we obtain, these terms are indeed small.

Minimizing the energy with respect to \(D(p)\) we obtain after some algebra the following equation

\[
\left( \frac{2\kappa}{\pi n} p^2 D(p) + 2z \right)^2 \left[ 1 - \frac{1}{2\pi n^2 \kappa^2} p^2 (p^2 + 4\kappa^2) + A\zeta \right] = \]

\[
\left( \frac{1}{2\pi n^2 \kappa^2} p^2 + 2\zeta D(p) \right)^2 4\kappa^2 \left[ \frac{2\kappa}{\pi n} p^2 + Bz \right] \]

with constants

\[
A = \frac{1}{16\pi^4 \kappa} \left[ 1 - \zeta \int d^2q \frac{1}{4\pi^2 \left( \frac{1}{2\pi \kappa} q^2 D(q) + 2\zeta \right)^2} \right]^{-1} \int d^2q \frac{q^2 (q^2 + 4\kappa^2)}{4\pi^2 \left( \frac{1}{2\pi \kappa} q^2 D^{-1}(q) + 2\zeta \right)^2} \]

(4.103)

and

\[
B = \frac{4\kappa}{8\pi^3 \kappa} \left[ 1 - z \int d^2q \frac{1}{4\pi^2 \left( \frac{2\kappa}{\pi n} q^2 D(q) + 2z \right)^2} \right]^{-1} \int d^2q \frac{q^2}{4\pi^2 \left( \frac{2\kappa}{\pi n} q^2 D(q) + 2z \right)^2} \]

(4.104)
The solution of this quadratic equation is

\[
D(p) = \left\{ \frac{p^2}{4\pi n\kappa} (\zeta F(p) - 4\kappa^2 z) - (z\zeta - \frac{p^4}{4\pi^2 n^2})\sqrt{F(p)} \right\} \left[ \zeta^2 F(p) - \frac{\kappa^2 p^4}{\pi^2 n^2} \right]^{-1}
\] (4.105)

where we have defined

\[
F(p) = \frac{4\kappa^2 [\frac{2\kappa}{\pi n} p^2 + B z]}{\frac{1}{2\pi n} p^2 (p^2 + 4\kappa^2) + A z}
\] (4.106)

Let us analyze these expressions. First, note that both values of fugacities are of the order of the fugacity of the Coulomb gas. This is true, since as we already know from the previous analysis for \( p^2 > \mu \), the variational propagator is a constant \( 1/2\kappa \). Then the integrals in the definitions of the chemical potentials \( \mu_\rho \) and \( \mu_\sigma \) in (3.65) and (3.71) get contributions only from momenta lower than \( \mu << a^{-1} \).

The fugacity of the Coulomb gas behaves like

\[
a^{-2} e^{-\frac{8\kappa}{\pi n}}
\] (4.107)

Therefore for \( n \neq 8 \) it is smaller, but of the order of the ultraviolet cutoff \( a^{-2} \). We can then estimate the constants \( A \) and \( B \) as

\[
A \propto \frac{a^{-2}}{\kappa}, \quad B \propto \kappa
\] (4.108)

This gives

\[
F(0) \propto \frac{\kappa^4 z}{a^{-2}\zeta}
\] (4.109)

Substituting this into the expression in eq.(4.105) we find

\[
D(0) = \frac{z}{\zeta \sqrt{F(0)}} \propto \sqrt{\frac{z}{\zeta}} \frac{a^{-1}}{\kappa} \frac{1}{\kappa} >> \frac{1}{2\kappa}
\] (4.110)

It is also easy to see that the function \( D(p) \) as given in eq. (4.105) is a growing function at small momenta, and is of the same order of magnitude up to momenta of order \( p^2 \sim \kappa a^{-1} \).
On the other hand the variational function $D(p)$ by definition, can not exceed the value $1/2\kappa$. It follows therefore that the solution eq. (4.105) is unphysical and we have to choose for $D(p)$ the end point value

$$D(p) = \frac{1}{2\kappa}$$

(4.111)

at all momenta.

We therefore discover that for $n < 8$ the propagator is constant for all momenta, or is completely local in the coordinate space. This should remind one of the behaviour of correlation functions in topological theories: there correlation functions of any local operator are completely local. The situation therefore is such that for $n < 8$ the Maxwell term in our theory becomes totally irrelevant, and the theory degenerates into a pure Chern - Simons theory. This conclusion is further confirmed by inspection of the expectation value of the energy (4.102). Note, that the structure of this equation is such that for momenta which are smaller than $z$ and $\zeta$ the contributions of the electric and magnetic vortices cancell completely the “noncompact” contributions. Since on our solution $\zeta$ and $z$ are both of the order of the ultraviolet cutoff\footnote{Note that with this function $D(p)$ $z$ and $\zeta$ are both equal to the fugacity of the Coulomb gas.}, physical momenta do not contribute at all to the energy. This is again consistent with a topological theory, in which the Hamiltonian vanishes.

### 4.4 Spontaneous breaking of magnetic flux at $n < 8$.

Let us discuss in more detail the physical properties of the theory for $n < 8$. As we have just argued, it has many similarities with a pure topological theory. It is clear, however, that it can not be equivalent to a simple noncompact pure Chern - Simons theory. First, as is obvious from the previous calculation, compactness of the gauge group is crucial
in this phase. It is responsible for the appearance of the magnetic and electric vortex gases, which are in the plasma phase and thereby determine the infrared properties of the model. Second, our derivation in the previous subsection is in large measure independent of the assumption \( \kappa \ll a^{-1} \). For large \( \kappa = O(a^{-1}) \), the “photon mass” is of the order of the ultraviolet cutoff, and this case corresponds to the pure Chern-Simons limit of the TMGT. It is clear therefore, that the compact pure Chern-Simons theory also undergoes phase transition at \( n = 8 \), the small \( n \) phase being qualitatively different from the large \( n \) phase.

We conclude that the compact TMGT at \( n < 8 \) is equivalent to compact pure Chern-Simons model, but they both must be different from the noncompact Chern-Simons theory. What distinguishes the two phases of the pure Chern-Simons theory?

The fundamental property of a noncompact Chern-Simons theory is the Bohm-Aharonov interaction between charged particles. It is interesting to check, therefore whether this interaction is still there in the compact theory for \( n < 8 \). A straightforward way to study this question is to calculate the expectation value of the Wilson loop in a state which contains a unit external charged. The Bohm-Aharonov interaction should show up as the Bohm-Aharonov phase in this expectation value. Introduction of a unit external charge at the point \( x = 0 \) leads to the following modification of the Gauss’ constraint equation

\[
\partial_i E_i(x) - 2\kappa B(x) = g\delta^2(x)
\]

The ground state wave functional in this sector should be well approximated by a projected Gaussian with a nonzero shift in the vector potential. For simplicity, we will take the width of the Gaussian to be the same as in the zero charge sector \( G^{-1}(p) = \kappa \).

\[
|1\rangle = \int Ds_i D\phi \exp \left\{ i\kappa \left[ (\epsilon_{ij}(s_i - \partial_i \phi)A_j + \epsilon_{ij}\partial_i \phi s_j) - ig\phi(0) - \frac{\kappa}{2} (A_i^{\phi,s} - a_i)^2 \right] \right\}
\]
The function $a_i$ should be treated as a variational function. Since we are dealing with the pure Chern-Simons theory, the function $a_i$ should be holomorphic

$$a_1 + ia_2 = 0 \quad (4.114)$$

Additional constraint on $a_i$ follows from the requirement that the state $|1>$ be normalizable. This forces $a_i$ to satisfy at zero momentum

$$\text{Re} \, \epsilon_{ij} \partial_i a_j + \text{Im} \, \partial_i a_i = -\frac{g}{\kappa} \delta^2(x) \quad (4.115)$$

We will see that the result does not depend on the detailed form of $a_i$. The expectation value of the Wilson loop in this state can be calculated in a straightforward manner.

$$\langle \exp(i g \int_S B dS) \rangle = \exp(i g \int_S \epsilon_{ij} \partial_i a_j dS) \langle \exp\left(\frac{g}{2} \int_S \partial^2 \phi dS\right) \rangle_{\phi} \langle \exp\left(-\pi \int_S \sigma dS\right) \rangle_{\sigma} \quad (4.116)$$

where for holomorphic functions $a_i$, the weight for the $\sigma$ averaging is the same as in the vacuum state, while the weight for the $\phi$ averaging is given by

$$\exp\{-S[\phi]\} = \exp\left\{-\frac{\kappa}{2}(a_i + \partial_i \phi)^2 + \frac{\kappa}{4}(a_i + a_i^* + \partial_i \phi + i \epsilon_{ij} \partial_j \phi)^2 + ig\phi(0)\right\} \quad (4.117)$$

Finally we obtain

$$\langle \exp\left(i g \int_S B dS\right) \rangle = R(S) \exp\left(-i \frac{2\pi}{n}\right) \quad (4.118)$$

where $R(S)$ is a real number. The value of $R(S)$ depends on the phase of the theory, but the value of the phase does not. Also, $R(S)$ is nonzero in both phases.

We conclude therefore that the Bohm-Aharonov interaction of external charges is the same in the compact theory for $n > 8$ and $n < 8$.

Nevertheless, the two phases are distinguishable. The operator that distinguishes them is not the Wilson loop, but is closely related to its dual. It is the 2+1 dimensional analog.
of the t’Hooft loop [9] - the operator that creates magnetic vortices. The significance of this operator is, that it is an order parameter for spontaneous breaking of the magnetic flux symmetry [10].

In fact, the fate of the magnetic flux symmetry in the compact TMGT is an interesting question. Recall, that in noncompact electrodynamics in 2+1 dimensions (both with and without the Chern - Simons term) the homogeneous Maxwell equation

\[ \partial_\mu \tilde{F}_\mu = 0 \]  

ensures the existence of the conserved current. The global charge associated with this current is the magnetic flux through the plain

\[ \Phi = \int d^2 x B(x) \]  

In QED without Chern - Simons term, this charge is spontaneously broken in the Coulomb phase. This breaking is accompanied by the appearance of the massless Goldstone boson - the photon. In noncompact TMGT the photon is massive. Accordingly the magnetic flux is not broken, but annihilates the vacuum state. In a compact theory the magnetic flux is not conserved anymore [10]. The magnetic “monopole - instantons” change the magnetic flux through the plain by an integer multiple of \( 2\pi/g \). Consequently, only the following subgroup of the flux group remains the symmetry of the theory

\[ U_N = e^{iN\Phi} \]  

for integer \( N \). In QED without Chern - Simons term, the magnetic flux is not quantized, and therefore the operators \( U_N \) constitute the group of integers \( \mathbb{Z} \). This symmetry again is spontaneously broken. Since the group is discreet, its spontaneous breakdown does not require an existence of a massless particle, and the photon in compact QED is massive. In
TMGT the situation is slightly different. Recall, that the compactification of the gauge group in TMGT requires the magnetic flux $\Phi$ to be quantized in units of $2\pi/ng$. On these states only the operators $U_N$ for $N = 1, ..., n - 1$ are represented nontrivially. The magnetic flux symmetry in compact TMGT is therefore just $Z_n$. Let us note, that it is this reduction of the magnetic flux group to $Z_n$ that is responsible for the mixing of the states with $n/2$ positively and negatively charged particles in the spontaneously broken SU(2) TMGT discussed in paper [11].

The natural question is, what is the realization of this $Z_n$ flux symmetry in compact TMGT. To answer this question we may calculate the vacuum expectation value of the appropriate order parameter. This order parameter should be an operator which is gauge invariant under both, compact and noncompact gauge groups, and should transform nontrivially under the flux symmetry. The suitable operator is

$$v_m(x, C) = \exp\left\{i\frac{2\pi m}{gn} \int_C \epsilon_{ij} dx_i E_j\right\}$$

The contour $C$ is a semiinfinite line with an endpoint at $x$. For integer $m < n$ this operator commutes with the elements of the compact gauge group. The calculation of the expectation value of $v$ is straightforward and parallels exactly the calculation of the electric part of the expectation value of the energy. Without giving the details here, we just describe the result.

For $n > 8$, there are no vortex contributions and the result is basically the same as in the noncompact theory. It has the form

$$\langle v_m \rangle = \exp\{-Ka^{-1}L\}$$

where $K$ is a numerical constant, and $L$ is the length of the curve $C$. In the infinite volume, $L \rightarrow \infty$, and we find

$$\langle v \rangle = 0, \quad n > 8$$
For $n < 8$ the contributions of the magnetic and electric vortices cancel exactly the noncompact contributions, precisely in the same way as they did in the calculation of the energy expectation value. There is therefore no linear divergence in the infinite volume limit, and we obtain

$$<v> \neq 0, \quad n < 8$$

Equation (4.125)

We conclude therefore, that for $n > 8$, the $Z_n$ magnetic flux group in compact TMGT is unbroken, and for $n < 8$ it is broken spontaneously. These different ways the symmetry is realized in the vacuum distinguish between the two phases of the model.

5 Discussion.

To summarize, the result of our variational calculation is the following. For $n > 8$ the magnetic and electric vortices in the wave function are bound in pairs. Their only effect is to introduce an intermediate scale $\mu$, below which the physics is the same as in the noncompact theory. This scale $\mu$ becomes infinite in the limit of infinite ultraviolet cutoff, and therefore in the continuum limit the effects of compactness disappear. This result is in qualitative agreement with the results of previous studies [3]. Here we want to make the following remark. The basic picture of ref. [3] is that the monopoles in the compact Chern-Simons theory are bound in pairs by a linear potential. On the other hand the magnetic (and electric) vortices in our ground state wave functional have interaction which is logarithmic at large distances. It is important to realize that these two claims are not inconsistent with each other. The point is, that the object which in path integral formalism is represented by a monopole-antimonopole pair has a segment of a Dirac string, stretched between them. This Dirac string becomes “observable” in the Chern-Simons theory and carries a finite action density, which is the origin of the linear potential.
between the monopoles. In our Hamiltonian description, a widely separated monopole - antimonopole pair is represented by a configuration in the wave function which has a nonvanishing magnetic vortex density $\rho$ at two points, but also a string of electric vortex dipoles $\sigma$ along the line that connects these points. This line of electric vortex dipoles corresponds precisely to the “observable” segment of the Dirac string. Configurations of this type are indeed suppressed in our wave functional by an exponential of the length of the dipole string segment. In this sense it is indeed true, that monopoles in our wave functional interact with a linear potential.

The new result we find, is that for $n < 8$ the nature of the ground state of the theory is very different. Due to the liberation of magnetic and electric vortices, the correlation functions of local observables (such as $B$ or $E$) become completely local. The theory therefore does not describe any propagating degrees of freedom. The liberation of the vortices also leads to spontaneous breaking of the magnetic flux symmetry in this phase. In this respect this phase is similar to the confining compact QED without the Chern-Simons term, whose vacuum also spontaneously breaks magnetic flux.

The dichotomy between the “monopoles” on one hand and the Chern-Simons term on the other hand, is resolved therefore in this nontrivial way.

Finally, let us remark, that in recent years TMGT have seen many applications in condensed matter physics, both in relation to Quantum Hall Effect and high temperature superconductivity. In the latter case a prominent role is played by the so-called semions. Those are charged particles coupled to Chern-Simons theory with $n = 4$. It is clear from our results, that the question of compactness of the Chern-Simons theory must be crucial for the “semion physics”. The physics usually discussed, corresponds to the noncompact theory. In the compact case the semions must have a very different behaviour. We hope
to return to this question in future work.

**Acknowledgements.** One of us (I.K.) is grateful to A.I. Vainshtein, M.A. Shifman and all members of TPI for many interesting discussions and hospitality during the visits in March-April and Summer of 1995. A.K. would like to thank members of Theoretical Physics Department at Oxford for hospitality during his visit in the Fall 1994, when this project has been started. We are grateful to A. Bochkarev for many useful discussions. This work was supported in part by DOE under grant number DE-FG02-94ER40823, PPARC grant GR/J 21354 and by Balliol College, University of Oxford.
References


   R. D. Pisarski, *Phys. Rev.* D 34, 3851 (1986);


    and B. Rosenstein, *Int. J. Mod. Phys.* A7, 7419 (1992);