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IS THE COULOMB GAUGE ALWAYS REALIZABLE?

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Abstract:

A class of field configurations, those that satisfy $(\vec{D} \cdot \vec{\partial} \mp \vec{\partial} \cdot \vec{A}) = 0$ with $\vec{\partial} \cdot \vec{A} = 0$, and satisfying spherical symmetry is shown to be not transformable to the Coulomb gauge in the context of L^2 fields. This is shown by considering all kinds of gauge transformations - infinitesimal, finite, and large gauge transformations.

1.0 Introduction

It is generally assumed that the Coulomb gauge condition is always realizable, i.e., any field configuration that does not satisfy $\bar{\partial} \cdot \bar{A} \neq 0$ can always be gauge transformed to the Coulomb gauge surface. This conjecture was further bolstered when Zwanziger and Dell'Antonio⁽¹⁾ apparently proved that all orbits pass inside the Gribov horizon. Their proof rests on the assumption that a minimizing sequence always exists for

$$\|\bar{A}\Omega\|^2 = \int d^3x \left[\Omega \bar{A} \Omega^{-1} - (\partial\Omega) \Omega^{-1} \right]^2, \quad (1)$$

and the sequence is shown to always achieve an absolute minimum in the fundamental modular region. Relative minima of (1) on the other hand represent the copies inside the Gribov horizon.

In a recent paper⁽²⁾, it was conjectured that the Coulomb gauge is an incomplete gauge-fixing condition in the sense that there are field configurations whose orbit will not intersect the Coulomb surface. These are the field configurations that satisfy

$$(\bar{D} \cdot \bar{\partial})(\bar{\partial} \cdot \bar{A}) = 0, \quad (2a)$$

$$\bar{\partial} \cdot \bar{A} = 0, \quad (2b)$$

$$\det(\bar{D} \cdot \bar{\partial}) = 0. \quad (2c)$$

Equations (2, a, b, c) state that the field configuration $\bar{A}$ is not on the Coulomb surface, the operator $\bar{D} \cdot \bar{\partial}$ is singular and its zero mode is precisely $\bar{\partial} \cdot \bar{A}$ itself.

2.0 An Explicit Example

In this paper, we will construct a class of field configurations that explicitly realize equations (2; a, b, c) and then explicitly show that these configurations cannot be gauge transformed to the Coulomb gauge. Consider the following ansatz (take the case of SU(2))

$$A_i^a = \delta_i^a f(r) + \frac{x^a x^i}{r^2} g(r) \quad (3)$$

Equation (3) assumes spherically symmetric fields specified by the two functions f and g . Taking the divergence yields

$$\partial_i A_i^a = \frac{x^a}{r} \left(\frac{df}{dr} + \frac{dg}{dr} + \frac{2}{r} g \right) \quad (4)$$

and for this not to be zero, we define

$$\frac{df}{dr} + \frac{dg}{dr} + \frac{2}{r} g = s(r) = 0. \quad (5)$$

The non-linear gauge condition given by equation (2a) on the other hand yields the condition

$$\frac{d^3 f}{dr^3} - \frac{2}{r} \frac{d^2 f}{dr^2} - \frac{2}{r^2} \frac{df}{dr} - \frac{d^3 g}{dr^3} - \frac{4}{r} \frac{d^2 g}{dr^2} - \frac{2}{r^2} \frac{dg}{dr} - \frac{4g}{r^3} = 0. \quad (6)$$

Using equation (5), we find that $\frac{x^a}{r} s(r)$ is a zero mode of $(\bar{D} \cdot \bar{\partial})$ if $s(r)$ satisfies

$$\frac{d^2 s}{dr^2} + \frac{2}{r} \frac{ds}{dr} - \frac{2}{r^2} s = 0. \quad (7)$$

The solution to this equation is

$$s(r) = \frac{a}{r^2} + br. \quad (8)$$

Equations (4) to (8) state that the field configuration A_i^a given by equation (3) is not on the Coulomb surface (unless $a = b = 0$) and that $\bar{\partial} \cdot \bar{A}^a = \frac{x^a}{r} s(r)$ is a zero mode of the operator $\bar{D}(\bar{A}) \cdot \bar{\partial}$. Varying the constants a and b is equivalent to generating the parallel surfaces $\bar{\partial} \cdot \bar{A}^a = k^a(\bar{x})$.

3.0 The Infinitesimal Case

We will be considering the problem of a field configuration that is infinitesimally close to the Coulomb surface, satisfying equations (2) to (8). Since we will be considering fields that should be transformable to the Coulomb gauge, we will be imposing the square integrability condition (L^2). The reason for this is the observation of Maskawa and Nakajima⁽³⁾ that the Coulomb gauge is geometrically and intrinsically related to L^2 fields.

Under an infinitesimal gauge transformation given by $\Omega = 1 + i \frac{x^a}{r} T^a \lambda(r)$, the A_i^a given by equation (3) transforms into

$$A_i'^a = \delta_i^a h(r) + \frac{x^a x^i}{r^2} m(r) + \epsilon^{aij} \frac{x^j}{r} n(r) \quad (9)$$

where

$$h(r) = f(r) + \frac{\lambda(r)}{r}, \quad (10a)$$

$$m(r) = g(r) - \frac{\lambda(r)}{r} - \frac{d\lambda}{dr}, \quad (10b)$$

$$n(r) = \lambda(r) f(r) \quad (10c)$$

Imposing the Coulomb gauge on $A_i'^a$ constrains h and m to satisfy

$$\frac{dh}{dr} - \frac{dm}{dr} - \frac{2}{r} m = 0. \quad (11)$$

Using equations (10; a, b, c), (5) and (8) in equation (11) yields

$$\frac{d^2 \lambda}{dr^2} + \frac{2}{r} \frac{d\lambda}{dr} - \frac{2\lambda}{r^2} = - \left(\frac{a}{r^2} + br \right). \quad (12)$$

Notice that the operator acting on λ has a zero mode λ_0 given by

$$\lambda_o = \frac{a}{r^2} + br, \quad (13)$$

which is precisely the source. This λ_o will yield a square-integrable zero mode for

$(\bar{D} \cdot \bar{\partial})$ (which is $\frac{x^a}{r} \lambda_o(r)$) if the constants a and b behave like $a(R) \sim \frac{1}{R^{1/2}}$ and

$b(R) \sim \frac{1}{R^{5/2}}$ (we put the system in a sphere of radius R and regularize the

ultraviolet limit by $\frac{1}{R}$). Since the L^2 zero mode is the source itself then equation

(12) does not have a well-defined solution. Thus there is no gauge transformation that will take the field configuration A_i^a given by equations (2) to (8) to the Coulomb surface.

An observant reader may point out that the behaviour of $b(R)$ will not lead to an L^2 gauge field because of equation (5). This is not really a problem as we will argue below. Even with $b(R)$ behaving like $R^{-5/2}$ the norm of A_i^a which is

$$\|\vec{A}\|^2 = \int_0^R r^2 (3f^2 - 2fg - g^2) dr, \quad (14)$$

can still be made finite because the solution of (5) involves both particular solutions f_p and g_p and homogeneous solutions f_o and g_o . Thus, even if for example $g_p \sim br^2$ (which leads to $\|g_p\|^2 = O(R^2)$), there is enough leeway in the choices of f_p, g_o, f_o to make $\|\vec{A}\|^2 = O(R^0)$, i.e., finite.

Another important observation ~~is~~ that the reader may point out that equation (12) has a particular solution,

$$\lambda_p(r) = \frac{1}{2}a - \frac{b}{10}r^3. \quad (15)$$

However, this solution leads to a non L^2 gauge transformation because

$$\int_0^R r^2 \left(\frac{1}{2}a - \frac{b}{10}r^3 \right)^2 dr = O(R^4).$$

Furthermore, even the resulting gauge transformed A_3^{tr} whose norm is

$$\|A_3^{\text{tr}}\|^2 = \int r^2 (3h^2 + 2m(r)h(r) + m^2 + 2n^2) dr \quad (16)$$

will not be L^2 also. There are too many terms that are not $O(R^0)$ that even if $\|A_3^{\text{tr}}\|^2$ is finite, the resulting $\|A_3^{\text{tr}}\|^2$ cannot be finite. An example is the last term of equation (16) which is different from the rest because it is proportional to $\int \lambda_p^2 f^2 r^2 dr$. This term can never be finite regardless whether $\|f\|^2$ is finite or not because of the r^6 term that is contributed by λ_p^2 .

To summarize this section, we have explicitly shown that within the context of L^2 condition, the field configuration given by equations (3) to (8) is not transformable to the Coulomb gauge.

4.0 Finite and Large Gauge Transformations

We now complete the proof that the field configuration defined by equations (3) to (8) is not transformable to the Coulomb gauge by considering finite and large (topology changing) gauge transformations. Consider an Ω given by

$$\Omega = \alpha(r) 1 + i \frac{\vec{x} \cdot \vec{T}}{r} \beta(r) \quad (17)$$

This form of Ω takes into account finite gauge transformations if $\alpha = \cos \theta(r)$ and

$\beta = \sin \theta(r)$ so that Ω is exponentiable to $\exp\left(i \frac{\vec{x} \cdot \vec{T}}{r} \theta(r)\right)$. This form will also

take into account large gauge transformations if the Pontryagin index

corresponding to Ω is non-zero (for example $\alpha = \frac{r^2 - \lambda^2}{r^2 + \lambda^2}$; $\beta = \frac{4\lambda r}{r^2 + \lambda^2}$ yields a

Pontryagin index equal to one).

The gauge transformed field using Ω given by equation (17) is

$$A'_i = T_k \left\{ \frac{x_i x_k}{r^2} \left(\frac{d\alpha}{dr} \beta - \frac{\alpha d\beta}{dr} \right) - \left(\delta_{ik} - \frac{x_i x_k}{r^2} \right) \left(\frac{1}{2} f \beta^2 - \frac{1}{r} \alpha \beta \right) \right. \\ \left. - \delta_{ik} f - \frac{x_i x_k}{r^2} g - \epsilon^{ijk} \frac{x_j}{r} \left(\alpha f \beta - \frac{1}{2} \frac{\beta^2}{r} \right) \right\} \quad (18)$$

Imposing the Coulomb gauge condition on (18), we get a second-order differential equation for α and β

$$\beta \frac{d^2 \alpha}{dr^2} - \alpha \frac{d^2 \beta}{dr^2} + \frac{2}{r} \left(\frac{d\alpha}{dr} \beta - \alpha \frac{d\beta}{dr} \right) + \frac{2}{r^2} \alpha \beta - \frac{1}{r} f \beta^2 = - \left(\frac{a}{r^2} + br \right). \quad (19)$$

Defining

$$\frac{du}{dr} = \frac{d\alpha}{dr} \beta - \frac{\alpha d\beta}{dr} \quad (20)$$

and using $\alpha^2 + \frac{\beta^2}{4} = 1$ leads to a simple relation between u' and α ($du = 2(1 - \alpha^2)^{-1/2} d\alpha$) which can easily be integrated to yield $u = -2 \cos^{-1} \alpha + c_0$, where c_0 an integration constant. Using this result, equation (19) becomes

$$\frac{d^2 u'}{dr^2} + \frac{2}{r} \frac{du'}{dr} - \frac{2}{r^2} \sin u' + \frac{4}{r} f(r) \sin^2 \frac{u'}{2} = - \left(\frac{a}{r^2} + br \right), \quad (21)$$

where $u' = u - c_0$.

Equation (21) is an inhomogeneous non-linear equation and it is the principal equation that must be solved to get a finite or even large transformation that will take the field configuration A_i (given by equations (3) to (8)) to the Coulomb surface. We need not solve this equation to infer the nature of the

solution. We will just rewrite this equation to get a problem whose result is known. Considering only the first three terms in equation (21), it can be rewritten (by defining $t = \ell n r$) as a pendulum problem with frictional force⁽⁴⁾. This equation is known not to have an L^2 solution and the resulting gauge transformed field is also not L^2 . If we now also consider the source given by the RHS of (21), the problem becomes a pendulum problem with friction and driven by a constant force and an exponentially increasing force ($a + b e^{3t}$). This problem can only have a runaway (exponentially increasing) solution and the resulting gauge transformation will not be L^2 . Adding the fourth term ($e^t f(e^t) \sin^2 \frac{u'}{2}$) will not change the conclusion about the nature of the solution because the L^2 nature of $\bar{A}$ says that f must behave at best like $e^{-\left(\frac{3}{2} - \epsilon\right)t}$ (or $\frac{1}{r^{3/2+\epsilon}}$) as $t \rightarrow \infty$, with ϵ an infinitesimal constant.

In short, equation (21) will not yield a gauge transformation that will take the field A_i (given by equations (3) to (8)) to the Coulomb surface. This completes the proof that there are field configurations that cannot be gauge transformed to the Coulomb gauge. And this then hints that the Coulomb gauge needs to be generalized to be able to include field configurations, such as those given in this paper, in the path-integral.

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