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***A SOLUBLE GAUGE MODEL
WITH GRIBOV-TYPE COPIES***

R. FRIEDBERG, T. D. LEE

COLUMBIA UNIVERSITY, NEW YORK, N.Y. 10027

Y. PANG

COLUMBIA UNIVERSITY, NEW YORK, N.Y. 10027

AND

BROOKHAVEN NATIONAL LABORATORY, UPTON, N.Y. 11973

AND

H. C. REN

THE ROCKEFELLER UNIVERSITY, NEW YORK, N.Y. 10021

See 9534

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ABSTRACT

A soluble gauge model is presented which can exhibit the typical characteristics of Gribov's gauge-equivalent copies that exist in the Coulomb gauge of QCD. From the explicit solutions of the model, we show that correct results can be obtained by including all such copies, both in the Hamiltonian approach using Schrödinger wave functions and in the path-integration formalism using Feynman rules. This leads to an alternative proposal on how to resolve the complications of gauge-equivalent copies different from the one suggested by Gribov.

1. INTRODUCTION

In an important paper [1], Gribov correctly pointed out that, e.g., in the Coulomb gauge of quantum chromodynamics (QCD) the divergence-free condition on the gluon potential field does not fix its gauge; i.e., there exist “gauge-equivalent” copies, each of which satisfies the same divergence-free condition, and yet is related to the others through gauge transformations. Gribov proposed to restrict the field configurations only to those with positive Faddeev-Popov determinants. (In order to avoid double counting, a suitable normalization factor has to be introduced which depends on the multiplicity of such copies with positive determinants.) These limitations on the field configuration were then suggested as a way to understand color confinement [1, 2].

In this paper we present a soluble gauge model that can exhibit the characteristics of Gribov-type copies. As we shall see, from the explicit solution, there are some serious questions concerning the implementation of Gribov’s suggestion, especially with regard to the Feynman diagram approach. Instead, we propose an alternative resolution to remove these complications. However, the question of color confinement is not discussed.

The model consists of a single nonrelativistic particle of unit mass whose three Cartesian coordinates are X, Y and Z . In addition, there is a fourth coordinate ξ . The Lagrangian of the system is given by

$$\mathcal{L} = \frac{1}{2} \left[(\dot{X} + gY\xi)^2 + (\dot{Y} - gX\xi)^2 + (\dot{Z} - \xi)^2 \right] - U \quad (1.1)$$

where $g > 0$ is the coupling constant, the dot denotes the time-derivative d/dt , and U can be *any* function of $X^2 + Y^2$. Because $\dot{\xi}$ is absent in $\mathcal{L}$, the gauge transformation with an arbitrary time-dependent function α ,

$$X(t) \rightarrow X(t) \cos \alpha(t) - Y(t) \sin \alpha(t),$$

$$Y(t) \rightarrow X(t) \sin \alpha(t) + Y(t) \cos \alpha(t),$$

$$Z(t) \rightarrow Z(t) + \frac{\alpha(t)}{g}$$

and

$$\xi(t) \rightarrow \xi(t) + \frac{\dot{\alpha}(t)}{g},$$

(1.2)

leaves the Lagrangian invariant. The X, Y and Z coordinates are the analog of the spatial-components of a gauge field A_μ and ξ its fourth-component A_0 .

The conjugate momenta are

$$P_X = \dot{X} + gY\xi, \quad P_Y = \dot{Y} - gX\xi \quad \text{and} \quad P_Z = \dot{Z} - \xi. \quad (1.3)$$

The classical Lagrangian equation for ξ is given by $\partial\mathcal{L}/\partial\xi = 0$; i.e.,

$$P_Z + g(XP_Y - YP_X) = 0, \quad (1.4)$$

or alternatively

$$\xi = \frac{\dot{Z} + g(X\dot{Y} - Y\dot{X})}{1 + g(X^2 + Y^2)}. \quad (1.5)$$

In a classical theory, (1.5) can be substituted into the Lagrangian $\mathcal{L}$, which then becomes a new function of $\dot{X}, \dot{Y}$ and $\dot{Z}$, without ξ . This procedure fails in a quantum theory, since (1.4) is in conflict with the commutation relation ($\hbar = 1$)

$$[P_X, X] = [P_Y, Y] = [P_Z, Z] = -i. \quad (1.6)$$

Thus, for quantization, one has to make a choice of the gauge.

In a gauge theory of the continuum field A_μ , in order to use the canonical Hamiltonian formalism there are, among others, three commonly used gauges:

(i) the time-axial gauge $A_0 = 0$, (ii) the space-axial gauge in which one of the space-components, say $A_z = 0$ and (iii) the Coulomb gauge $\nabla_i A_i = 0$ (where A_i refers to the space components of A_μ). In this simple model, the analog of the time-axial gauge is

$$\xi = 0 \quad (1.7)$$

and the space-axial gauge corresponds to

$$Z = 0; \quad (1.8)$$

the analog of the Coulomb gauge (or its generalization, the Coulomb-like gauge $\nabla_i A_i = 0$) would be to choose a linear homogeneous function of (X, Y, Z) to be zero. Because of the rotational invariance in the (X, Y) plane, we may choose this linear function to be independent of Y . To be specific, set

$$Z - \lambda X = 0 \quad \text{with} \quad \lambda > 0, \quad (1.9)$$

which will be referred to as the λ -gauge.

We shall begin with the quantum theory in the time-axial gauge ($\xi = 0$), and then transform it into the λ (or Coulomb-like) gauge and also to the space-axial gauge ($Z = 0$). As we shall see, the transformation from the time-axial gauge to the space-axial gauge is unique but, in general, that to the λ -gauge is not, except in the weak-coupling limit. Thus, there is a realization of Gribov's "gauge-equivalent copies", as in the Coulomb gauge for QCD, even though the gauge transformation (1.2) in the model is Abelian.

In the following, for notational clarity we reserve (X, Y, Z) for the coordinates in the time-axial gauge, (x, y) with $z = \lambda x$ for those in the λ -gauge and (x_0, y_0) with $z_0 = 0$ for the space-axial gauge. The exact quantum states of the model in these gauges are given in Section 2. As we shall see, the Jacobian (or the Faddeev-Popov determinant) in the λ -gauge is

$$\mathcal{J} = \lambda y + g^{-1}, \quad (1.10)$$

which is negative if $y < -(\lambda g)^{-1}$. The mapping between the time-axial gauge and the space-axial gauge is one-to-one, but the mapping from either one to the λ -gauge can be 1 to μ . Thus, e.g., the entire (x_0, y_0) plane in the space-axial gauge can be divided into domains of different multiplicity μ , in which each point (x_0, y_0) is mapped into μ gauge-equivalent copies

$$(x_1, y_1), (x_2, y_2), \dots, (x_\mu, y_\mu) \quad (1.11)$$

in the (x, y) plane of the λ -gauge, with $\mu = 1, 3, 5, \dots$. As we shall see in Section 3, the boundary of these domains of different multiplicities is described by the involute of a circle in (x_0, y_0) , illustrated in Figure 4; correspondingly, the entire (x, y) plane is also divided into different areas, with each mapped image (x_i, y_i) located at one of those areas, as shown in Figures 6 and 7. The simplicity of the model allows us to exhibit explicit solutions of these locations.

As will be shown, all the eigenstates of the Hamiltonian are single-valued functions of (x, y) in the λ -gauge; they are well-behaved at the "horizon", defined as the zero Jacobian boundary, $y = -(\lambda g)^{-1}$, as well as across the multiplicity regions of different μ . In Section 4.1, the exact propagators in the λ -gauge are calculated

following the Dyson-Wick approach, using the Hamiltonian. Again, they are regular in the entire (x, y) plane, including the horizon and across all multiplicity domains.

In passing from the Hamiltonian approach to the path-integration formulation in the λ -gauge, special attention must be given to paths that can travel backward in time. Their appearance can be understood as follows: Consider the mapping of a continuous forward-moving path in (x_0, y_0) of the space-axial gauge into the λ (or Coulomb-like) gauge. Whenever the multiplicity μ is > 1 in (1.11), the mapped path, though still continuous, has to switch its time-direction $\mu - 1$ times. This is analyzed in Section 4.2. In Section 5 we give the path-integration formulation in the λ -gauge, which extends over the entire (x, y) plane; i.e., includes all gauge-copies. In addition, we derive an extra-interaction term

$$V = -\frac{g^2(2 + 3\lambda^2) + \lambda^3 g^3 y}{8(1 + \lambda g y)^3} + \frac{\lambda^2 g^4 x^2(1 + \lambda^2)}{8(1 + \lambda g y)^4} \quad (1.12)$$

due to the ordering of quantum operators. The presence of such an additional interaction term is a well-known characteristic [3-11] of path integrations in curvilinear coordinates. The new feature is that for different gauge copies in (1.11), V is different; therefore, this V function is manifestly non-gauge invariant.

Rules for Feynman diagrams are derived in Section 6. Again, they are applied to the entire (x, y) space in the λ -gauge, including all gauge-copies. These rules are used to evaluate all the one-loop contributions to the propagators, which agree with the exact results derived in Section 4.1; they are also used to calculate all the two-loop diagrams for the confirmation of V , the additional manifest non-gauge-invariant interaction due to operator ordering.

The solutions of this soluble model give an explicit proof that by including all gauge copies we can derive correct results in either the Hamiltonian approach, or the path-integration formalism, or the Feynman diagram analysis. The suggestion made by Gribov, while valid in principle, is difficult to implement. In particular, it is not clear how to derive Feynman rules for perturbative calculations if only one of these gauge copies is included.

As we shall show, the proposal that we made for resolving the complications of gauge-equivalent copies can be applied to a much wider class of problems, in which there are mappings from Cartesian to curvilinear coordinates that can be *one-to-several*. Thus, the general methodology that we develop in this paper is relevant to gauge theories as well as to theories without gauge invariance.

2. HAMILTONIAN FORMALISM IN DIFFERENT GAUGES

2.1. “Time-Axial” Gauge

In accordance with (1.7), analogous to the time-axial gauge $A_0 = 0$ in a gauge-field theory, we choose

$$\xi = 0.$$

The corresponding quantum-mechanical Hamiltonian is

$$\mathcal{H} = \frac{1}{2} (P_X^2 + P_Y^2 + P_Z^2) + U \quad (2.1)$$

with the classical equation (1.4) replaced by the supplementary condition

$$[P_Z + g(XP_Y - YP_X)] |> = 0 \quad (2.2)$$

on the state vector $|>$, where P_X, P_Y and P_Z satisfy (1.6). Introduce

$$X = \rho \cos \Phi \quad \text{and} \quad Y = \rho \sin \Phi. \quad (2.3)$$

The general solution of (2.2) in the coordinate space is

$$\langle X, Y, Z |> = e^{im(\Phi - gZ)} R(\rho) \quad (2.4)$$

in which m can be any integer, $0, \pm 1, \pm 2, \dots$. In terms of $R(\rho)$, the eigenvalue equation $\mathcal{H}|> = E|>$ becomes

$$-\frac{1}{2\rho} \frac{d}{d\rho} \rho \frac{dR}{d\rho} + \left[U + \frac{m^2}{2} \left(\frac{1}{\rho^2} + g^2 \right) \right] R = ER. \quad (2.5)$$

As mentioned before,

$$U = U(\rho). \quad (2.6)$$

In the particular case

$$U = \frac{1}{2} \omega^2 (X^2 + Y^2), \quad (2.7)$$

we have

$$E = \omega(n_+ + n_- + 1) + \frac{g^2}{2} (n_+ - n_-)^2 \quad (2.8)$$

where n_+ and n_- are two independent positive integers (including zero), related to m by

$$m = n_+ - n_- . \quad (2.9)$$

2.2. The λ (or “Coulomb-Like”) Gauge

Starting from the time-axial gauge X, Y, Z and $\xi = 0$, we may transform to a new set of coordinates x, y, z and $\xi \neq 0$ through (1.2):

$$x = X \cos \alpha - Y \sin \alpha , \quad (2.10)$$

$$y = X \sin \alpha + Y \cos \alpha , \quad (2.11)$$

$$z = Z + (\alpha / g) \quad (2.12)$$

and

$$\xi = \dot{\alpha} / g . \quad (2.13)$$

In this model, as mentioned in the Introduction, the analog of the usual Coulomb gauge $\nabla_i A_i = 0$ (or its generalization [3] $\Gamma_i A_i = 0$) is the λ -gauge given by (1.9); i.e., in terms of the transformed coordinates x and z , we set

$$z - \lambda x = 0 \quad \text{with} \quad \lambda > 0 , \quad (2.14)$$

Substituting the gauge-condition into (2.12) we have

$$\lambda x = Z + (\alpha / g) . \quad (2.15)$$

Equations (2.10)-(2.11) and (2.15) define the coordinate transformation from (X, Y, Z) to (x, y, α) . The Jacobian is

$$\mathcal{J} = -\frac{\partial(X, Y, Z)}{\partial(x, y, \alpha)} = (1 + \lambda g y) / g . \quad (2.16)$$

In terms of

$$p \equiv \frac{1}{i} \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ -\partial/\partial \alpha \end{pmatrix} \equiv \begin{pmatrix} p_x \\ p_y \\ -p_\alpha \end{pmatrix} \quad (2.17)$$

the Hamiltonian (2.1) becomes

$$\mathcal{H} = \mathcal{K} + U(\rho) \quad (2.18)$$

where $\rho^2 = x^2 + y^2$ and

$$\mathcal{K} = \frac{1}{2\mathcal{J}} \tilde{p}(\mathcal{J} \mathcal{M}^{-1} p) + U(\rho) \quad (2.19)$$

with $\sim$ denoting the transpose, $\mathcal{M}^{-1}$ the inverse of the “mass-matrix”

$$\mathcal{M} = \begin{pmatrix} \lambda^2 + 1 & 0 & -y + g^{-1}\lambda \\ 0 & 1 & x \\ -y + g^{-1}\lambda & x & x^2 + y^2 + g^{-2} \end{pmatrix} \quad (2.20)$$

and

$$\mathcal{J} = (\det \mathcal{M})^{\frac{1}{2}} \quad (2.21)$$

being the analog of the Faddeev-Popov determinant [12, 13]. Since

$$-g p_\alpha = P_Z + g(X P_Y - Y P_X), \quad (2.22)$$

the supplementary condition (2.2) means that in the coordinate representation $\psi = \langle x, y, \alpha |$ is independent of α ; i.e.,

$$\psi = \psi(x, y) \quad (2.23)$$

in the λ gauge. Correspondingly, the p_α -dependence of the kinetic energy operator $\mathcal{K}$ can be eliminated, and we may replace $\mathcal{K}$ by

$$K = \frac{1}{2\mathcal{J}} (p_x \ p_y) \mathcal{J} \begin{pmatrix} \mathcal{M}_{xx}^{-1} & \mathcal{M}_{xy}^{-1} \\ \mathcal{M}_{yx}^{-1} & \mathcal{M}_{yy}^{-1} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} \quad (2.24)$$

and the Hamiltonian $\mathcal{H}$ by

$$H = K + U(\rho) \quad (2.25)$$

where, on account of (2.20),

$$\begin{aligned} (\mathcal{M}^{-1})_{xx} &= \mathcal{J}^{-2}(y^2 + g^{-2}), \\ (\mathcal{M}^{-1})_{xy} &= (\mathcal{M}^{-1})_{yx} = \mathcal{J}^{-2}x \left(\frac{\lambda}{g} - y \right), \\ (\mathcal{M}^{-1})_{yy} &= \mathcal{J}^{-2} \left[\left(\lambda y + \frac{1}{g} \right)^2 + x^2(\lambda^2 + 1) \right]. \end{aligned} \quad (2.26)$$

In the λ -gauge, the time-independent Schrödinger equation becomes

$$H\psi = [K + U(\rho)]\psi = E\psi. \quad (2.27)$$

It is useful to define the angular momentum operator

$$J \equiv (1 + \lambda g y)^{-1} \frac{1}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right). \quad (2.28)$$

One can directly verify that the commutators

$$[J, K] = 0 \quad \text{and} \quad [J, H] = 0. \quad (2.29)$$

In terms of the polar coordinates

$$x = \rho \cos \phi \quad \text{and} \quad y = \rho \sin \phi, \quad (2.30)$$

the eigenvalue equation

$$J\psi = m\psi \quad (2.31)$$

can be readily solved, with $\psi(x, y)$ given by

$$\psi = R(\rho) e^{im(\phi - \lambda g \rho \cos \phi)}. \quad (2.32)$$

Substituting ψ into (2.27) yields

$$-\frac{1}{2\rho} \frac{d}{d\rho} \rho \frac{dR}{d\rho} + \left[U + \frac{m^2}{2} \left(\frac{1}{\rho^2} + g^2 \right) \right] R = ER \quad (2.33)$$

which is identical to (2.5). In order for ψ to be a *single-valued* function, the eigenvalue m of J must be an integer, the same as (2.9).

In terms of ρ and ϕ , the volume-differential is

$$(1 + \lambda \rho g \sin \phi) \rho d\rho d\phi, \quad (2.34)$$

the angular momentum operator (2.28) is

$$J = (1 + \lambda \rho g \sin \phi)^{-1} \frac{1}{i} \frac{\partial}{\partial \phi} \quad (2.35)$$

and the kinetic energy operator (2.24) becomes

$$\begin{aligned}
K = & -\frac{1}{2\rho} (1 + \lambda\rho g \sin \phi)^{-1} \left[\frac{\partial}{\partial \rho} \rho(1 + \lambda\rho g \sin \phi) \frac{\partial}{\partial \rho} \right. \\
& + \frac{\partial}{\partial \rho} \lambda\rho g \cos \phi \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi} \lambda\rho g \cos \phi \frac{\partial}{\partial \rho} \\
& \left. + \frac{\partial}{\partial \phi} (1 + \lambda\rho g \sin \phi)^{-1} (\rho g^2 + \rho^{-1} + \lambda^2 \rho g^2 \cos^2 \phi) \frac{\partial}{\partial \phi} \right].
\end{aligned} \tag{2.36}$$

The operators J, K and the Hamiltonian H in the λ -gauge are all Hermitian with the measure (2.34) and the domain of integration:

$$\begin{aligned}
\rho & \text{ from } 0 \text{ to } \infty, \\
\phi & \text{ from } 0 \text{ to } 2\pi.
\end{aligned} \tag{2.37}$$

In the Hamiltonian approach, the Hilbert space Ω spans all the eigenstates $|\nu\rangle$ of the Hamiltonian

$$\Omega = \{|\nu\rangle\}. \tag{2.38}$$

Regarding these states $|\nu\rangle$ as abstract ket-vectors, the Hilbert space is a *gauge-invariant* concept (as will be further discussed in Section 4.2 below). In contrast, the coordinate-representation of these ket-vectors

$$\psi(x, y) = \langle x, y | \nu \rangle$$

is, of course, gauge dependent, and so is the form of the Hamiltonian H in terms of these coordinates. From (2.32), one sees that in the λ -gauge, $\psi(x, y)$ is well-behaved at the “horizon” defined by the zero Jacobian, $1 + \lambda\rho g \sin \phi = 0$. In Sections 4 and 5, we shall convert the operator e^{-iHt} into the path-integration formulation. As we shall see, the “Feynman rule” thus derived is to integrate over all the gauge-equivalent copies within the entire physical region defined by (2.37) with the proper Jacobian.

2.3. “Space-Axial” Gauge

It is useful to compare the formulation in the λ -gauge with another special gauge by setting $\lambda = 0$ (i.e., using $z = 0$ as the gauge-fixing condition); as mentioned in the Introduction, this gauge will be referred to as the *space-axial*

gauge, analogous to the “space-axial gauge $A_z = 0$ ” in the usual gauge-field theory. Let x_0, y_0, α_0 and ξ_0 be the variables in the space-axial gauge. From (2.10)-(2.15) and by setting $\lambda = 0$, we have the following relations between these variables and those in the time-axial gauge:

$$x_0 = X \cos \alpha_0 - Y \sin \alpha_0, \quad (2.39)$$

$$y_0 = X \sin \alpha_0 + Y \cos \alpha_0, \quad (2.40)$$

$$\alpha_0 = -g Z \quad (2.41)$$

and

$$\xi_0 = \dot{\alpha}_0 / g. \quad (2.42)$$

The Jacobian $-\partial(X, Y, Z) / \partial(x_0, y_0, \alpha_0)$ of this transformation is g^{-1} ; thus, the mapping from (X, Y, Z) to (x_0, y_0, α_0) is one to one.

Substituting (2.41) into (2.15), we have

$$\lambda g x = \alpha - \alpha_0. \quad (2.43)$$

Therefore, the coordinates x_0, y_0 in the space-axial gauge and x, y in the λ -gauge are related through

$$x_0 = x \cos \lambda g x + y \sin \lambda g x$$

$$\text{and} \quad (2.44)$$

$$y_0 = -x \sin \lambda g x + y \cos \lambda g x.$$

Introducing

$$x_0 = \rho \cos \phi_0 \quad \text{and} \quad y_0 = \rho \sin \phi_0, \quad (2.45)$$

we find that the phase angles ϕ_0 and ϕ in these two gauges satisfy

$$\phi_0 = \phi - \lambda \rho g \cos \phi. \quad (2.46)$$

Correspondingly, the angular momentum operator (2.35), the kinetic energy operator (2.36), and the wave function (2.32) in the space-axial gauge become

$$J = \frac{1}{i} \frac{\partial}{\partial \phi_0}, \quad (2.47)$$

$$K = -\frac{1}{2\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \left(\frac{1}{2\rho^2} + \frac{g^2}{2} \right) \frac{\partial^2}{\partial \phi_0^2} \quad (2.48)$$

and

$$\psi_0(x_0, y_0) = R(\rho) e^{im\phi_0}, \quad (2.49)$$

with $R(\rho)$ the solution of (2.33), as before. The volume element is simply $dx_0 dy_0 = \rho d\rho d\phi_0$.

In this soluble model, the space-axial gauge is particularly useful for the derivation of matrix elements. In the case when

$$U = \frac{1}{2} \omega^2 \rho^2 = \frac{1}{2} \omega^2 (x_0^2 + y_0^2), \quad (2.50)$$

we define the annihilation and creation operators a, b and their Hermitian conjugates $a^\dagger, b^\dagger$ by

$$\begin{aligned} x_0 &= \frac{1}{2\sqrt{\omega}} (a + b + a^\dagger + b^\dagger), \\ y_0 &= \frac{i}{2\sqrt{\omega}} (a - b - a^\dagger + b^\dagger), \\ \frac{1}{i} \frac{\partial}{\partial x_0} &= -i \frac{\sqrt{\omega}}{2} (a + b - a^\dagger - b^\dagger) \end{aligned} \quad (2.51)$$

and

$$\frac{1}{i} \frac{\partial}{\partial y_0} = \frac{\sqrt{\omega}}{2} (a - b + a^\dagger - b^\dagger)$$

with the commutators

$$[a, a^\dagger] = [b, b^\dagger] = 1 \quad (2.52)$$

and $[a, b] = [a, b^\dagger] = 0$. The eigenvalue equation can be written as

$$H |n_+, n_- \rangle = E_{n_+ n_-} |n_+, n_- \rangle \quad (2.53)$$

where, as in (2.8),

$$E_{n_+ n_-} = \omega(n_+ + n_- + 1) + \frac{g^2}{2} (n_+ - n_-)^2, \quad (2.54)$$

with the corresponding eigenvector

$$|n_+, n_- \rangle = \frac{1}{\sqrt{n_+! n_-!}} (a^\dagger)^{n_+} (b^\dagger)^{n_-} |0 \rangle \quad (2.55)$$

in which the ground state $|0\rangle = |0,0\rangle$ is determined by

$$a|0\rangle = b|0\rangle = 0. \quad (2.56)$$

The Heisenberg equation of motion gives the time dependence of $x_0(t)$ and $y_0(t)$. From that we find the propagator in the space-axial gauge,

$$\mathcal{D}_0(t-t') \equiv \langle 0|T(x_0(t)x_0(t'))|0\rangle = \langle 0|T(y_0(t)y_0(t'))|0\rangle, \quad (2.57)$$

to be

$$\mathcal{D}_0(t-t') = \begin{cases} \frac{1}{2\omega} e^{-i(\omega+\frac{g^2}{2})(t-t')} & \text{if } t \geq t' \\ \frac{1}{2\omega} e^{-i(\omega+\frac{g^2}{2})(t'-t)} & \text{if } t' \leq t \end{cases} \quad (2.58)$$

with T denoting the time-ordered product. In terms of the Fourier transform, we may write

$$\mathcal{D}_0(t) = \left(1 + \frac{g^2}{2\omega}\right) \int \frac{i}{k_0^2 - (\omega + \frac{g^2}{2})^2 + i0+} e^{-ik_0 t} dk_0. \quad (2.59)$$

Before ending this section, we discuss the wave function of an eigenstate, $|n_+, n_-\rangle$ in the coordinate space. In terms of the polar coordinates introduced in (2.45), the operators $a, a^\dagger, b, b^\dagger$ can be written as

$$\begin{aligned} a &= \frac{1}{2\sqrt{\omega}} e^{-i\phi_0} \left(\omega\rho + \frac{\partial}{\partial\rho} - \frac{i}{\rho} \frac{\partial}{\partial\phi_0} \right), \\ b &= \frac{1}{2\sqrt{\omega}} e^{i\phi_0} \left(\omega\rho + \frac{\partial}{\partial\rho} + \frac{i}{\rho} \frac{\partial}{\partial\phi_0} \right), \\ a^\dagger &= \frac{1}{2\sqrt{\omega}} e^{i\phi_0} \left(\omega\rho - \frac{\partial}{\partial\rho} - \frac{i}{\rho} \frac{\partial}{\partial\phi_0} \right), \\ b^\dagger &= \frac{1}{2\sqrt{\omega}} e^{-i\phi_0} \left(\omega\rho - \frac{\partial}{\partial\rho} + \frac{i}{\rho} \frac{\partial}{\partial\phi_0} \right). \end{aligned} \quad (2.60)$$

The wave function of the ground state, $\langle \rho, \phi_0 | 0, 0 \rangle$, is determined by the condition:

$$\left(\omega\rho + \frac{\partial}{\partial\rho} - \frac{i}{\rho} \frac{\partial}{\partial\phi_0} \right) \langle \rho, \phi_0 | 0, 0 \rangle = 0,$$

$$\left(\omega \rho + \frac{\partial}{\partial \rho} + \frac{i}{\rho} \frac{\partial}{\partial \phi_0} \right) \langle \rho, \phi_0 | 0, 0 \rangle = 0. \quad (2.61)$$

We find

$$\langle \rho, \phi_0 | 0, 0 \rangle = \sqrt{\frac{\omega}{\pi}} e^{-\frac{1}{2} \omega \rho^2} \quad (2.62)$$

which is normalized, i.e.,

$$\int_0^\infty \rho d\rho \int_0^{2\pi} d\phi_0 |\langle \rho, \phi_0 | 0, 0 \rangle|^2 = 1. \quad (2.63)$$

The wave function of an arbitrary excited state is computed in Appendix A, through repeated operations of the differential operators $a^\dagger, b^\dagger$ in (2.60) on $\langle \rho, \phi_0 | 0, 0 \rangle$, i.e.,

$$\langle \rho, \phi_0 | n_+, n_- \rangle = \frac{1}{\sqrt{n_+! n_-!}} (a^\dagger)^{n_+} (b^\dagger)^{n_-} \langle \rho, \phi_0 | 0, 0 \rangle. \quad (2.64)$$

The result is that

$$\langle \rho, \phi_0 | n_+, n_- \rangle = R_{n_+, n_-}(\rho) e^{i(n_+ - n_-)\phi_0} \quad (2.65)$$

where

$$R_{n_+, n_-}(\rho) = (-)^n \frac{\omega^{\frac{1}{2}(1+|m|)}}{\sqrt{\pi}} \left(\frac{n!}{(n+|m|)!} \right)^{\frac{1}{2}} \rho^{|m|} e^{-\frac{1}{2} \omega \rho^2} L_n^{|m|}(\omega \rho^2) \quad (2.66)$$

with $L_n^{|m|}(\omega \rho^2)$ the Laguerre polynomial defined by

$$L_n^\alpha(z) = e^z \frac{z^{-\alpha}}{n!} \frac{d^n}{dz^n} (e^{-z} z^{n+\alpha}), \quad (2.67)$$

$m = n_+ - n_-$ the azimuthal quantum number, as before, and $n = \frac{n_+ + n_- - |m|}{2}$ the radial quantum number, equal to the number of nodes of the radial wave function, $R_{n_+, n_-}(\rho)$. The corresponding energy (2.54) in terms of m, n reads

$$E_{n_+, n_-} = \omega (2n + |m| + 1) + \frac{g^2}{2} m^2. \quad (2.68)$$

3. GAUGE-EQUIVALENT COPIES

3.1. From Time-Axial Gauge to λ -Gauge

Consider the mapping from a *given* X, Y, Z in the time-axial ($\xi = 0$) gauge into x, y, α in the λ -gauge through the gauge transformation (2.10)-(2.13) and with the gauge-fixing condition $z = \lambda x$. In this section, we examine the question of multiplicity in this mapping.

Combining (2.3) with (2.10), one has

$$x = \rho \cos(\Phi + \alpha). \quad (3.1)$$

In the (x, α) plane, the intersection of the cosine curve (3.1) and the straight line (2.15),

$$x = \frac{1}{\lambda} \left(Z + \frac{\alpha}{g} \right), \quad (3.2)$$

gives the configuration in the λ -gauge. One can readily verify that for

$$\rho < (\lambda g)^{-1} \quad (3.3)$$

there is only one such intersection (for each given ρ, Φ and Z), as illustrated in Figure 1(a). Complications arise when

$$\rho > (\lambda g)^{-1}. \quad (3.4)$$

The slope $\partial x / \partial \alpha$ of the straight line (3.2) is $(\lambda g)^{-1}$, while that of the cosine curve is $-y$, whose maximum value is ρ . For $\rho > (\lambda g)^{-1}$, by varying either Z or Φ , one can have three or more intersections, as illustrated in Figures 1(b) and 1(c). Each of these intersections gives a different configuration (x, y, α) in the same λ -gauge. Furthermore, all these configurations are connected by the gauge transformation, giving rise to the gauge-equivalent copies (or simply gauge copies) referred to by Gribov [1].

The value of the Jacobian $\mathcal{J} = g^{-1} + \lambda y$ is λ times the difference of the slopes of the straight line (3.2) and the cosine curve (3.1) at their intersection. For $\rho < (\lambda g)^{-1}$, the gauge-fixing condition $z = \lambda x$ uniquely determines a configuration with $\mathcal{J} > 0$. For $\rho > (\lambda g)^{-1}$, depending on the given values of X, Y and Z , there can be μ odd number of gauge copies in the λ -gauge. From Figure 1(b) or

1(c), one sees that the Jacobian of these successive intersections switches its sign from one intersection to the next.

It is instructive to view these gauge copies in the three-dimensional (x, y, α) space, in which the plane

$$\mathcal{J} = \frac{1}{g} + \lambda y = 0 \quad (3.5)$$

denotes the Gribov horizon in this model. For a given ρ (and also Φ and Z), the corresponding λ -gauge configurations must lie on the helix, determined by (3.1) and

$$y = \rho \sin(\Phi + \alpha). \quad (3.6)$$

Equation (3.2) now denotes a plane parallel to the y -axis, whose intersections with the helix give all the gauge copies in the λ -gauge. As can be seen from Figures 2(a) and 2(b), for $\rho > (\lambda g)^{-1}$, when one moves continuously along the helix, these discrete intersections appear alternatively on different sides of the Gribov-horizon plane (3.5), resulting in the sign-switching of the Jacobian.

The transformation from the time-axial gauge to the λ -gauge can also be formulated in terms of a minimum principle (analogous to that from the time-axial gauge to the Coulomb gauge in the continuum gauge field theory). Consider again a fixed point (X, Y, Z) in the time-axial gauge. Through (2.10)-(2.12), regard x, y and z as functions of α ; hence,

$$\begin{aligned} \frac{\partial x}{\partial \alpha} &= -y, & \frac{\partial y}{\partial \alpha} &= x, \\ \frac{\partial z}{\partial \alpha} &= \frac{1}{g} & \text{and} & \quad \frac{\partial \rho}{\partial \alpha} = 0 \end{aligned} \quad (3.7)$$

where $\rho = (x^2 + y^2)^{\frac{1}{2}}$, as before. Define

$$I(\alpha) \equiv \frac{1}{2} z^2 - \frac{1}{g} \lambda y. \quad (3.8)$$

Therefore $I(\alpha) \geq -(\lambda \rho / g)$ is bounded from below. The extremum of $I(\alpha)$ is determined by

$$\frac{\partial I}{\partial \alpha} = \frac{1}{g} (z - \lambda x) = 0, \quad (3.9)$$

which satisfies the λ -gauge; i.e., all the gauge copies in the λ -gauge are the extrema of $I(\alpha)$. Furthermore, since the second derivative is

$$\frac{\partial^2 I}{\partial \alpha^2} = \frac{\mathcal{J}}{g} = \frac{1}{g^2} (\lambda g y + 1),$$

all minima of I must be at $y > -(\lambda g)^{-1}$. Combining (3.9) with (3.8), we find the value of I at the extremum to be

$$\frac{1}{2} (\lambda x)^2 - g^{-1} \lambda y = \frac{1}{2} [(\lambda \rho)^2 + g^{-2} - (\lambda y + g^{-1})^2]; \quad (3.10)$$

hence, the absolute minimum of $I(\alpha)$ corresponds to the gauge copy that has the largest Jacobian value; which is also the one that is furthest above the $\mathcal{J} = 0$ line on the (x, y) plane in the λ -gauge. For each X, Y, Z in the time-axial gauge, there is at least one gauge copy in the λ -gauge which is the absolute minimum of $I(\alpha)$. Designate

$$\Lambda \quad (3.11)$$

to be the set of all such absolute minima (analogous to a similar set studied by Zwanziger [2] for QCD). Its location in (x, y) will be examined below.

3.2. From Space-Axial Gauge to λ -Gauge

As shown in Section 2.3, the transformation connecting the time-axial and space-axial gauges is always one-to-one. On the other hand, starting from the space-axial gauge the mapping onto the λ -gauge simplifies, because it can be reduced to a one-dimensional transformation $\phi_0 \rightarrow \phi$, defined by (2.46):

$$\phi_0 = \phi - \lambda \rho g \cos \phi. \quad (3.12)$$

Note that as $\phi_0 \rightarrow \phi_0 + 2\pi$, the corresponding $\phi \rightarrow \phi + 2\pi$. Since the Schrödinger wave functions (2.32) and (2.49) are both periodic in their angular variables, we need only examine a period from 0 to 2π in either ϕ_0 or ϕ . From Figures 3(a) and 3(b), we see that, within a 2π period in ϕ_0 , when ρ is $> (\lambda g)^{-1}$, the mapping $\phi_0 \rightarrow \phi$ can be either one to one, or one to three, or one to five, etc. On the other hand, since

$$\int_0^{2\pi} f(\phi_0) d\phi_0 = \int_0^{2\pi} (1 + \lambda \rho g \sin \phi) f(\phi - \lambda \rho g \cos \phi) d\phi$$

for any periodic function $f(\phi_0)$, the question of “uniqueness” or “non-uniqueness” does not require any special attention in such an integration, so long as the integral in ϕ covers all the relevant “copies” in a continuous way.

The mapping details in this model are quite interesting. For a given ϕ , the transformation (3.12) determines a unique ϕ_0 ; the converse is not true. If we limit ϕ to the period between 0 and 2π , the mapping $\phi_0 \rightarrow \phi$ is either one to one, or one to three, with ϕ_0 , $\phi_0 \pm 2\pi$, $\phi_0 \pm 4\pi$, etc. being identified as the same point (since $\mathcal{J} = (1 + \lambda \rho g \sin \phi) / g$ can at most have two zeroes within $0 \leq \phi < 2\pi$). On the other hand, if we restrict ϕ_0 to lying between 0 and 2π , the multiplicity pattern in $\phi_0 \rightarrow \phi$ becomes more complex. For example, in Figure 3(b), the curve (3.12) is broken into seven disconnected pieces within the interval $0 \leq \phi_0 < 2\pi$ (i.e., between the solid and dashed vertical lines). However, as can be seen from the figure, these seven pieces can be re-assembled into a continuous curve, satisfying the same (3.12), either from point a' (with $\phi_0 = 0$) to point b' (with $\phi_0 = 2\pi$), or from point a (with $\phi = 0$) to point b (with $\phi = 2\pi$). In Appendix B, we give an analytic expression for $\phi = \phi(\phi_0)$, the inverse function of (3.12).

Next, consider the two-dimensional point-transformation (2.44) from $(x_0, y_0) \rightarrow (x, y)$. Let μ be the multiplicity of the mapping. The boundary of the same multiplicity μ region must satisfy $\mathcal{J} = g^{-1} + \lambda y = 0$. Thus, replacing y by $-(\lambda g)^{-1}$ in (2.44) and writing

$$r \equiv (\lambda g)^{-1} \quad \text{and} \quad \theta \equiv \lambda g x,$$

we find these boundaries to be described by

$$\begin{aligned} x_0 &= r(\theta \cos \theta - \sin \theta) \\ y_0 &= -r(\theta \sin \theta + \cos \theta), \end{aligned} \tag{3.13}$$

which is plotted in Figure 4, with alternate white and black areas denoting regions of the same multiplicity $\mu = 1, 3, 5, \dots$. Equation (3.13) is the *involute* of a circle. There is a simple construct: First, coil tightly a long string (with P as its endpoint) around a circle (centered at 0) of radius r . At the beginning, P is on the circle with $\overline{P0} = r$ and along the y_0 axis, as shown in Figure 4. Then, unwind the coiled string, by moving P away from the circle but maintaining at all times a taut configuration, with the uncoiled portion $QP = r\theta$ of the string kept as a straight line, tangent to the circle at point Q . The trajectories of P and its mirror reflection (with respect to the y_0 -axis) give (3.13).

Figure 4 gives the multiplicity domain structure of the mapping $(x_0, y_0) \rightarrow (x, y)$ on the (x_0, y_0) plane. We shall now examine the locations of these gauge copies on the (x, y) plane. Consider a straight line trajectory

$$y_0 = x_0 \tan \phi_0 \quad (3.14)$$

in which

$$\begin{cases} x_0 \text{ varies from } -\infty \text{ to } \infty \\ \text{at a fixed } \phi_0 \text{ between } -\frac{\pi}{2} \text{ and } \frac{\pi}{2} \end{cases} \quad (3.15)$$

as shown in Figure 5(a). This straight line in the space-axial gauge is mapped into an infinite number of branches of

$$y = x \tan(\phi_0 + \lambda g x) \quad (3.16)$$

in the λ -gauge, as shown in Figure 5(b) for $\phi_0 = 0$, 5(c) for $\phi_0 = \frac{\pi}{4}$ and 5(d) for $\phi_0 = \pm \frac{\pi}{2}$. We see that of these branches, there is only one that is entirely convex down, and it will be referred to as the *convex branch* (in contrast, all other branches of (3.16) extend from $y = -\infty$ to $y = \infty$ and therefore are partly concave down). Since only the convex branch passes through the origin $x = y = 0$ (which corresponds uniquely to the origin $x_0 = y_0 = 0$ in (x_0, y_0)), it is also the only branch that covers the entire x_0 -range from $-\infty$ to ∞ in the mapping from (3.15) to (3.16).

The same straight line (3.14) can also be viewed as the *union* of two radial lines in which

$$\begin{cases} \text{the radial variable } \rho \text{ varies from } 0 \text{ to } \infty \\ \text{but there are two azimuthal angles: } \phi_0 \text{ and } \pi + \phi_0. \end{cases} \quad (3.17)$$

For example, in Figure 5(a) the polar coordinates of the points q_0 and p_0 are (ρ, ϕ_0) and $(\rho, \pi + \phi_0)$. In the mapping $(x_0, y_0) \rightarrow (x, y)$, each point q_0 or p_0 gives μ or μ' gauge copies; together these two points yield $\mu + \mu'$ gauge copies. Their locations in the (x, y) space of the λ -gauge are given by the intersections between (3.16) and the circle

$$x^2 + y^2 = \rho^2. \quad (3.18)$$

For ρ sufficiently small there is only one copy ($\mu = \mu' = 1$) for each (x_0, y_0) ; but for larger ρ , μ can be $3, 5, \dots$ of which $\frac{1}{2}(\mu + 1)$ are of positive Jacobian $\mathcal{J} = g^{-1}(1 + \lambda g y)$ and $\frac{1}{2}(\mu - 1)$ of negative Jacobian, and likewise for μ' . In

Figures 5(b) and 5(c), we have $\mu = \mu' = 3$, with the mapping $q_0 \rightarrow q_1, q_2, q_3$ and $p_0 \rightarrow p_1, p_2, p_3$. There is a reflection symmetry with respect to the y -axis between the locations of gauge copies for ϕ_0 and $-\phi_0$, since (3.16) and (3.18) are both invariant under

$$\phi_0 \rightarrow -\phi_0, \quad x \rightarrow -x \quad \text{and} \quad y \rightarrow y, \quad (3.19)$$

and so are their intersections.

In general, take any point q_0 in (x_0, y_0) . In the mapping $(x_0, y_0) \rightarrow (x, y)$, we have

$$q_0 \rightarrow q_1, q_2, \dots, q_\mu \quad (3.20)$$

with $\mu = 1, 3, 5, \dots$. Let $q_1 \equiv q_1(x_0, y_0)$ be the copy in (x, y) that has the *largest* Jacobian; i.e., farthest above the zero Jacobian line,

$$y = -(\lambda g)^{-1}. \quad (3.21)$$

As can be seen from Figures 5(b) and 5(c), q_1 stays on the convex branch of (3.16). In accordance with (3.11), we have

$$\Lambda = \{q_1(x_0, y_0)\} \quad (3.22)$$

to be the domain occupied by $q_1(x_0, y_0)$ in (x, y) for all (x_0, y_0) . Thus, the boundary of Λ is simply the envelope of the convex branches of (3.16) with $\phi_0 = \frac{\pi}{2}$, and with $\phi_0 = -\frac{\pi}{2}$; i.e.,

$$y = -x \cot \lambda g x, \quad (3.23)$$

for $|\lambda g x| \leq \pi$.

Every point in Λ is mapped uniquely to a point in (x_0, y_0) through (2.44), and vice versa (except its boundary, which will be discussed below). In Figure 5(d), the shaded region is Λ ; its boundary is tangent to the dotted line (3.21); i.e., $\mathcal{J} = 0$. Substituting (3.23) into (2.44), we find that the boundary of Λ in (x, y) is mapped from a straight line in (x_0, y_0) described by

$$x_0 = 0 \quad (3.24)$$

and

$$y_0 = -x \csc \lambda g x, \quad \text{varying from } -(\lambda g)^{-1} \text{ to } -\infty,$$

which can be visualized as a *cut* along the negative y_0 -axis in Figure 4, from the point P to $-\infty$.

The domain Λ in (x, y) can, in turn, be decomposed into a sum:

$$\Lambda = \Lambda_1 + \Lambda_3 + \Lambda_5 + \dots \quad (3.25)$$

where each point p_1 in Λ_μ has $(\mu - 1)$ gauge copies $p_2, p_3, \dots, p_\mu$ all outside Λ . This is illustrated in Figures 6 and 7. We see that, unlike $\Lambda_1, \Lambda_5, \Lambda_9, \dots$, each of $\Lambda_3, \Lambda_7, \Lambda_{11}, \dots$ is separated into two regions. This can be readily understood: In the mapping $(x_0, y_0) \rightarrow \Lambda$, the straight line (3.24) is mapped onto the boundary of Λ . Since this line cuts each of the black areas (of multiplicity $\mu = 3, 7, 11, \dots$) into two, the same separation is preserved in the mapping.

Consider now the mapping of a continuous curve in (x_0, y_0) . If the curve does not cross the cut (3.24), we may regard the mapping, instead of $(x_0, y_0) \rightarrow$ the entire (x, y) plane, as simply

$$(x_0, y_0) \rightarrow \Lambda \text{ in } (x, y); \quad (3.26)$$

the mapped curve remains continuous. Otherwise it does not. In order to preserve continuity in a natural way, a convenient approach (and the one that we adopt in this paper) is to regard the mapping as

$$(x_0, y_0) \rightarrow \text{entire } (x, y) \text{ plane}, \quad (3.27)$$

instead of (3.26). In the path-integration formulation, each path is also time-dependent. It is easy to see that, whenever the mapping (3.27) has a multiplicity $\mu > 1$, a forward-moving continuous path in (x_0, y_0) is mapped into a continuous one in (x, y) , but having both forward- and backward-moving segments in time. This interesting feature will be carefully examined in the next section.

There is an alternative approach in which one alters the topology of Λ by regarding the two opposite points (i.e., having the same y but different signs of x) on the boundary of Λ as the *same* point, since they are originally mapped from the same point on the straight line (3.24). However, as we shall discuss in subsequent sections, it is difficult to incorporate such an alteration into the usual Feynman diagram formulation.

REMARKS.

The region Λ is by no means the only single-domain in (x, y) that has the one-to-one mapping relationship with the (x_0, y_0) plane. Similar to the separation of Riemann sheets, there is a great deal of arbitrariness in the location of the cut. For example, in the above description of the boundary (3.13) between the multiplicity domains, the involute of the circle consists of two smooth continuous trajectories, called $\mathcal{A}$ and $\mathcal{B}$: At the initial point P in Figure 4, one can uncoil the string either by swinging to the left (i.e., towards the negative x_0 -axis) which gives $\mathcal{A}$, or to the right which gives its mirror image $\mathcal{B}$. If we take $\mathcal{A}$ to be the cut in the (x_0, y_0) plane (instead of the straight line (3.24)), then excluding $\mathcal{A}$ itself, every point in (x_0, y_0) is uniquely mapped into a point in an area $\Lambda(\mathcal{A})$ in (x, y) , and *vice versa*. To derive the boundary of $\Lambda(\mathcal{A})$, we start from the point A in Figure 7, located at $x = \infty$ and $y = -(\lambda g)^{-1}$, move along the zero-Jacobian line $y = -(\lambda g)^{-1}$ to $x = 0$, then trace the entire left side boundary of the tear-drop shaped area labeled 1, and continue on smoothly without any sharp turn towards A' at ∞ . Likewise, its mirror image gives $\Lambda(\mathcal{B})$ in (x, y) which also has a one-to-one mapping relationship with the entire (x_0, y_0) plane. Furthermore, every point in $\Lambda(\mathcal{A})$, or $\Lambda(\mathcal{B})$, is of positive Jacobian.

4. PROPAGATORS AND HILBERT SPACE IN THE λ -GAUGE

4.1. Propagators

Since the original question of Gribov's gauge-equivalent copies concerns duplications in the path-integration approach, we must examine the soluble model in the same context. We shall first derive the exact expressions for propagators by following the Dyson-Wick approach [14, 15] which is based on the Hamiltonian. Next, in transforming the operator e^{-iHt} in the λ -gauge into the path-integration form [16] in terms of the Lagrangian, special attention will be paid to the formation of the backward-traveling path in time, as will be analyzed in Section 4.2. Only then can we address in full the appropriate role of "Feynman rules" with regard to gauge copies.

Let $x(t)$ and $y(t)$ be the operators that satisfy the Heisenberg equations of motion in the λ -gauge. Their propagators are defined by

$$\mathcal{D}_{ab}(t-t') = \langle 0 | T(x_a(t) x_b(t')) | 0 \rangle_C \quad (4.1)$$

with the subscripts a (or b) and the corresponding x_a (or x_b) = x or y , T denoting the time-ordered product, $|0\rangle$ the ground state vector of the Hamiltonian H and the subscript C referring to the connected part of the expectation value; i.e.

$$\begin{aligned} \langle 0 | T(x_a(t) x_b(t')) | 0 \rangle_C &= \langle 0 | T(x_a(t) x_b(t')) | 0 \rangle \\ &\quad - \langle 0 | x_a(t) | 0 \rangle \langle 0 | x_b(t') | 0 \rangle . \end{aligned}$$

As we shall show, $\langle 0 | x(t) | 0 \rangle = 0$, but $\langle 0 | y(t) | 0 \rangle \neq 0$. (See (4.22) below.) Because the λ -gauge variables $x(t)$ and $y(t)$ are related to the space-axial gauge variables $x_0(t)$ and $y_0(t)$ through (2.44), the symmetry of H under

$$x_0 \rightarrow -x_0, \quad y_0 \rightarrow y_0,$$

implies also the symmetry under

$$x \rightarrow -x \quad \text{and} \quad y \rightarrow y. \quad (4.2)$$

Thus, the propagator

$$\mathcal{D}_{xy}(t-t') = 0. \quad (4.3)$$

To derive the explicit expressions for $\mathcal{D}_{xx}$ and $\mathcal{D}_{yy}$, we consider the special case

$$U(\rho) = \frac{1}{2} \omega^2 (x^2 + y^2). \quad (4.4)$$

As we shall prove, for $t > 0$

$$\mathcal{D}_{xx}(t) = 2(\lambda g)^{-2} \sum_{m=1}^{\infty} m^{-2} \left[\exp \left(-\frac{1}{2} g^2 m^2 \left(\frac{\lambda^2}{\omega} + it \right) \right) \right] I_m(z_m), \quad (4.5)$$

$$\begin{aligned} \mathcal{D}_{yy}(t) &= (4\omega^2)^{-1} \lambda^2 g^2 e^{-2i\omega t} \\ &+ 2(\lambda g)^{-2} \sum_{m=1}^{\infty} m^{-4} \left[\exp \left(-\frac{1}{2} g^2 m^2 \left(\frac{\lambda^2}{\omega} + it \right) \right) \right] \left(z_m \frac{d}{dz_m} - \frac{\lambda^2 g^2 m^2}{2\omega} \right)^2 I_m(z_m), \end{aligned} \quad (4.6)$$

where

$$z_m = \frac{\lambda^2 g^2 m^2}{2\omega} e^{-i\omega t} \quad (4.7)$$

and $I_m(z)$ is the hyperbolic Bessel function, related to the usual Bessel function $J_m(z)$ by

$$I_m(z) = \left(\frac{1}{i} \right)^m J_m(iz) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+m}}{n!(n+m)!} \quad (4.8)$$

(hence, as $z \rightarrow \infty$, $I_m(z) \sim (2\pi z)^{\frac{1}{2}} e^z$). For $t < 0$,

$$\mathcal{D}_{xx}(t) = \mathcal{D}_{xx}(-t) = \mathcal{D}_{xx}(|t|)$$

and

$$\mathcal{D}_{yy}(t) = \mathcal{D}_{yy}(-t) = \mathcal{D}_{yy}(|t|).$$

To establish (4.5) and (4.6), we start with the definition of the propagator (4.1). For $t > 0$, we have

$$\langle 0, 0 | x(t) x(0) | 0, 0 \rangle = \sum'_{n_+ n_-} |\langle 0, 0 | x(0) | n_+, n_- \rangle|^2 e^{-i(E_{n_+ n_-} - E_0)t} \quad (4.10)$$

and

$$\langle 0, 0 | y(t) y(0) | 0, 0 \rangle = \sum'_{n_+ n_-} |\langle 0, 0 | y(0) | n_+, n_- \rangle|^2 e^{-i(E_{n_+ n_-} - E_0)t} \quad (4.11)$$

where the sum $\sum'_{n_+n_-}$ excludes the groundstate $n_+ = n_- = 0$, $E_{n_+n_-}$ is given by (2.54) and $E_0 \equiv E_{00} = \omega$. The matrix elements, $\langle 0, 0 | x | n_+, n_- \rangle$ and $\langle 0, 0 | y | n_+, n_- \rangle$ can be evaluated readily. According to (2.32), the wave function of a state $|n_+, n_- \rangle$ in the λ -gauge reads

$$\langle \rho, \phi | n_+, n_- \rangle = R_{n_+n_-}(\rho) e^{i(n_+-n_-)(\phi-\lambda g\rho\cos\phi)} \quad (4.12)$$

where ρ, ϕ are the polar coordinates of x, y introduced in (2.30), and $R_{n_+n_-}(\rho)$ is the same function in (2.65). The volume-differential for the integration over ρ, ϕ is given by (2.34) with a Jacobian $(1 + \lambda g\rho \sin \phi)$. Therefore,

$$\begin{aligned} \langle 0, 0 | x | n_+, n_- \rangle &= \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi (1 + \lambda g\rho \sin \phi) \\ &\quad R_{00}(\rho) R_{n_+n_-}(\rho) e^{i(n_+-n_-)(\phi-\lambda g\rho\cos\phi)} \rho \cos \phi \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} \langle 0, 0 | y | n_+, n_- \rangle &= \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi (1 + \lambda g\rho \sin \phi) \\ &\quad R_{00}(\rho) R_{n_+n_-}(\rho) e^{i(n_+-n_-)(\phi-\lambda g\rho\cos\phi)} \rho \sin \phi. \end{aligned} \quad (4.14)$$

For $n_+ \neq n_-$, a partial integration over ϕ brings (4.13) and (4.14) into

$$\langle 0, 0 | x | n_+, n_- \rangle = \frac{i}{(n_+ - n_-) f} u(f) \quad (4.15)$$

and

$$\langle 0, 0 | y | n_+, n_- \rangle = -\frac{1}{(n_+ - n_-)^2} \frac{d}{df} u(f) \quad (4.16)$$

where $f = \lambda g$ and

$$u(f) = \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi R_{00}(\rho) R_{n_+n_-}(\rho) e^{i(n_+-n_-)(\phi-f\rho\cos\phi)}. \quad (4.17)$$

We observe that the integration (4.17), after re-designating ϕ as ϕ_0 , is identical to the matrix element of $e^{-i(n_+-n_-)fx_0}$ with x_0 given by (2.51). It follows from (2.51)-(2.53) that

$$u(f) = \frac{(-i)^{n_++n_-}}{\sqrt{n_+!n_-!}} \left[\frac{(n_+ - n_-)f}{2\sqrt{\omega}} \right]^{n_++n_-} e^{-(n_+-n_-)^2 f^2 / 4\omega} \quad (4.18)$$

and consequently

$$\langle 0,0|x|n_+,n_- \rangle = -\frac{(-i)^{n_++n_-+1}}{(n_+-n_-)\sqrt{n_+!n_-!}} f \left[\frac{(n_+-n_-)f}{2\sqrt{\omega}} \right]^{n_++n_-} e^{-(n_+-n_-)^2 f^2/4\omega} \quad (4.19)$$

and

$$\begin{aligned} \langle 0,0|y|n_+,n_- \rangle = & -\frac{(-i)^{n_++n_-}}{(n_+-n_-)^2\sqrt{n_+!n_-!}} f \left[n_++n_- - \frac{(n_+-n_-)^2 f^2}{2\omega} \right] \\ & \cdot \left[\frac{(n_+-n_-)f}{2\sqrt{\omega}} \right]^{n_++n_-} e^{-(n_+-n_-)^2 f^2/4\omega}. \end{aligned} \quad (4.20)$$

For $n_+ = n_- = n$, we have

$$\langle 0,0|x|n,n \rangle = \int_0^\infty \rho d\rho R_{00}(\rho) R_{nn}(\rho) \int_0^{2\pi} d\phi (1+f\rho \sin \phi) \rho \cos \phi = 0, \quad (4.21)$$

$$\begin{aligned} \langle 0,0|y|n,n \rangle &= \int_0^\infty \rho d\rho R_{00}(\rho) R_{nn}(\rho) \int_0^{2\pi} d\phi (1+f\rho \sin \phi) \rho \sin \phi \\ &= f \langle 0,0|y_0^2|n,n \rangle = \frac{f}{2\omega} (\delta_{n0} + \delta_{n1}). \end{aligned} \quad (4.22)$$

A direct evaluation of the integral (4.17) using the expression (2.66) for $R_{n_+n_-}(\rho)$ is given in Appendix A. In the same appendix, we also tabulate the matrix elements of x and y between two arbitrary eigenstates, $\langle n'_+, n'_- | x | n_+, n_- \rangle$ and $\langle n'_+, n'_- | y | n_+, n_- \rangle$.

Substituting (4.17) into (4.10), we end up with an expression for $\mathcal{D}_{xx}(t)$ for $t > 0$, which contains a double sum over n_+ and n_- . Convert the sum over n_+, n_- to the sum over the azimuthal quantum number, $m = n_+ - n_-$, and the radial quantum number $n = \frac{1}{2}(n_+ + n_- - |m|)$. The sum over n gives rise to the hyperbolic Bessel function (4.8) and we are left with a single sum over m . The result gives (4.5) for $\mathcal{D}_{xx}(t)$. Similarly, we can establish (4.6) for $\mathcal{D}_{yy}(t)$.

4.2. Hilbert Space

As mentioned in (2.38), the Hilbert space Ω is to be constructed out of the set of all the orthonormal eigenstates of the Hamiltonian. It is important to observe that each of these eigenstates can appear in two distinct forms:

(i) as an abstract ket-vector $|\nu\rangle$, with

$$\Omega = \{|\nu\rangle\}, \quad (4.23)$$

or (ii) as a function of coordinates by taking its coordinate representation, which has to be different in different gauges. For example, in the λ -gauge, the coordinate-representation of $|\nu\rangle$,

$$\psi(q) = \langle q | \nu \rangle, \quad (4.24)$$

is given by (2.32), with

$$q = (x, y) \quad \text{or} \quad (\rho, \phi), \quad (4.25)$$

whereas in the space-axial gauge, the coordinate-representation of $|\nu\rangle$ changes to

$$\psi_0(q_0) = \langle q_0 | \nu \rangle \quad (4.26)$$

given by (2.49), with

$$q_0 = (x_0, y_0) \quad \text{or} \quad (\rho, \phi_0). \quad (4.27)$$

Since $\langle q | \nu \rangle$ is equal to $\langle q_0 | \nu \rangle$ when q_0 and q are related by a simple coordinate transformation $\phi_0 \rightarrow \phi$ through (2.46) with

$$\phi_0 = \phi - \lambda \rho g \cos \phi,$$

we are in fact dealing with the same ket-vector $|\nu\rangle$, independent of the gauge. Thus, in (i) by regarding the eigenstate as an abstract ket-vector $|\nu\rangle$ denoting the physical state, we have achieved a gauge-invariant representation; consequently, the Hilbert space $\Omega = \{|\nu\rangle\}$ is also *gauge invariant*. (The selection of the Hamiltonian as the relevant operator for $|\nu\rangle$ is only for convenience, as will be discussed below in Remark 2.)

Next, we examine the consequences of (ii) by taking the coordinate representation of $|\nu\rangle$. In the space-axial gauge, any function $f(q_0)$ in the q_0 -space can be expanded in terms of the set $\langle q_0 | \nu \rangle$; i.e., the basis vectors $\langle q_0 | \nu \rangle$ of the Hilbert space form a *complete* set of functions in q_0 . In terms of Dirac's δ -function, we write

$$\delta^2(q_0 - q'_0) = \langle q_0 | q'_0 \rangle \equiv \sum_{\nu} \langle q_0 | \nu \rangle \langle \nu | q'_0 \rangle \quad (4.28)$$

where, as usual, $\langle \nu | q'_0 \rangle$ is the complex conjugate of $\langle q'_0 | \nu \rangle$ and

$$\delta^2(q_0 - q'_0) = \delta(x_0 - x'_0) \delta(y_0 - y'_0) = \rho^{-1} \delta(\rho - \rho') \delta(\phi_0 - \phi'_0). \quad (4.29)$$

The situation is quite different in the λ -gauge. Because the measure (2.34),

$$\mathcal{J}(q) d^2q \equiv (1 + \lambda \rho g \sin \phi) \rho d\rho d\phi, \quad (4.30)$$

is no longer always positive, the set of the basis vectors of the Hilbert space Ω in the coordinate representation

$$\{\langle q | \nu \rangle\} \quad (4.31)$$

do *not* form a complete set of functions in the q -space. In other words, $\{\langle q | \nu \rangle\}$ forms a complete set of basis vectors for the Hilbert space Ω of the physical problem, but *not* for the space of arbitrary functions of q .

Define the identity operator I in Ω to be

$$I = \sum_{\nu} |\nu\rangle \langle \nu|;$$

hence

$$\langle q | I | q' \rangle = \sum_{\nu} \langle q | \nu \rangle \langle \nu | q' \rangle \quad (4.32)$$

where, as in (4.28), $\langle \nu | q' \rangle = \langle q' | \nu \rangle^*$. Take any two state vectors $|\alpha\rangle$ and $|\beta\rangle$ in Ω :

$$\begin{aligned} |\alpha\rangle &= \sum_{\nu} \alpha_{\nu} |\nu\rangle, \\ |\beta\rangle &= \sum_{\nu} \beta_{\nu} |\nu\rangle. \end{aligned} \quad (4.33)$$

The scalar product $\langle \alpha | \beta \rangle$, defined by $\sum_{\nu} \alpha_{\nu}^* \beta_{\nu}$, can be written as

$$\langle \alpha | \beta \rangle \equiv \int \mathcal{J}(q) d^2q \mathcal{J}(q') d^2q' \langle \alpha | q \rangle \langle q | I | q' \rangle \langle q' | \beta \rangle. \quad (4.34)$$

As we shall see, although $\langle q | I | q' \rangle \neq \delta^2(q - q')$, $\langle \alpha | \beta \rangle$ satisfies the following simplified expression:

Theorem.

$$\langle \alpha | \beta \rangle = \int \mathcal{J}(q) d^2q \langle \alpha | q \rangle \langle q | \beta \rangle. \quad (4.35)$$

(In polar coordinates it is convenient to normalize $d^2q = \rho d\rho d\phi$ implying

$$\mathcal{J}(q) = 1 + \lambda \rho g \sin \phi \quad (4.36)$$

which differs from the Cartesian Jacobian (2.16) by a factor g .)

Proof. At a given ρ , take any two points $q_0 = (\rho, \phi_0)$ and $q'_0 = (\rho, \phi'_0)$ in the space-axial gauge. On account of (3.18), the mapping (3.20) into the λ -gauge carries

$$\begin{aligned} q_0 &\rightarrow q_1, q_2, \dots, q_\mu, \\ q'_0 &\rightarrow q'_1, q'_2, \dots, q'_{\mu'}, \end{aligned} \quad (4.37)$$

all with the same ρ . In terms of the azimuthal angles of $q_i = (\rho, \phi_i)$ and $q'_i = (\rho, \phi'_i)$, the above mapping becomes

$$\begin{aligned} \phi_0 &\rightarrow \phi_1, \phi_2, \dots, \phi_\mu, \\ \phi'_0 &\rightarrow \phi'_1, \phi'_2, \dots, \phi'_{\mu'}. \end{aligned} \quad (4.38)$$

Select ϕ to be *any* member in the set $\{\phi_i\}$, with $i = 1, 2, \dots, \mu$; i.e., ϕ can be any solution that satisfies (2.46),

$$\phi_0 = \phi - \lambda \rho g \cos \phi. \quad (4.39)$$

Now, a δ -function in ϕ_0 can be written in terms of ϕ as

$$\delta(\phi_0 - \phi'_0) = \sum_{i=1}^{\mu} \left| \frac{\partial \phi_0}{\partial \phi} \right|_{\phi'_i}^{-1} \delta(\phi - \phi'_i) \quad (4.40)$$

in which $|\partial \phi_0 / \partial \phi|_{\phi'_i}$ is the absolute value of

$$\begin{cases} \frac{\partial \phi_0}{\partial \phi} = 1 + \lambda \rho g \sin \phi \\ \text{at } \phi = \phi'_i. \end{cases} \quad (4.41)$$

To verify (4.40), we note that when ϕ differs from any ϕ'_i (where $i = 1, 2, \dots, \mu'$), ϕ_0 must be different from ϕ'_0 and therefore both sides of (4.40) are zero. When ϕ is near one of the ϕ'_i , say within a small interval between $\phi'_i - \epsilon$ and $\phi'_i + \epsilon$, (4.40) can be established by integrating both sides over this interval; furthermore, we may let the sum over i vary from 1 to μ .

Integrating (4.40) over the entire range of ϕ from 0 to 2π , we have

$$\sum_{i=1}^{\mu} \left(\frac{\partial \phi_0}{\partial \phi} \right)_{\phi_i} \left| \frac{\partial \phi_0}{\partial \phi} \right|_{\phi_i}^{-1} = 1, \quad (4.42)$$

or in terms of q_i in (4.37):

$$\sum_{i=1}^{\mu} \mathcal{J}(q_i) \left| \mathcal{J}(q_i) \right|^{-1} = 1. \quad (4.43)$$

Likewise, select ϕ' to be any one of the ϕ'_i in (4.38). Writing

$$\begin{aligned} \phi_0 &= \phi_0(\phi), \\ \phi'_0 &= \phi_0(\phi') \end{aligned} \quad (4.44)$$

with $\phi_0(\phi)$ denoting the function of ϕ on the right-hand side of (4.39), we can express $\delta(\phi_0 - \phi'_0)$ as the following function of ϕ and ϕ' , or of $q = (\rho, \phi)$ and $q' = (\rho, \phi')$:

$$\Delta_\rho(\phi, \phi') \equiv \delta(\phi_0(\phi) - \phi_0(\phi')).$$

It follows from (4.40) that

$$\Delta_\rho(\phi, \phi') = \Delta_\rho(\phi', \phi) = \sum_{i=1}^{\mu} \left| \mathcal{J}(q'_i) \right|^{-1} \delta(\phi - \phi'_i). \quad (4.45)$$

Since $\langle q | \nu \rangle$ and $\langle q_0 | \nu \rangle$ are given by (2.32) and (2.49), we write

$$\begin{aligned} \langle q | \nu \rangle &= R(\rho) e^{im(\phi - \lambda \rho g \cos \phi)}, \\ \langle q_0 | \nu \rangle &= R(\rho) e^{im \phi_0} \end{aligned} \quad (4.46)$$

and therefore, by using the definitions (4.28) and (4.32),

$$\langle q | I | q' \rangle = \rho^{-1} \delta(\rho - \rho') \Delta_\rho(\phi, \phi') = \sum_{i=1}^{\mu} \left| \mathcal{J}(q'_i) \right|^{-1} \delta^2(q - q'_i) \quad (4.47)$$

where $\delta^2(q - q') = \delta(x - x') \delta(y - y')$. Because of the symmetry in q and q' , we have

$$\langle q | I | q' \rangle = \langle q' | I | q \rangle = \sum_{i=1}^{\mu} \left| \mathcal{J}(q_i) \right|^{-1} \delta^2(q' - q_i). \quad (4.48)$$

Substituting (4.48) into (4.34) and integrating over $d^2 q'$, we derive

$$\langle \alpha | \beta \rangle = \int \mathcal{J}(q) d^2 q \langle \alpha | q \rangle \sum_{i=1}^{\mu} \mathcal{J}(q_i) \left| \mathcal{J}(q_i) \right|^{-1} \langle q_i | \beta \rangle.$$

Since $|\beta\rangle$ is within the Hilbert space Ω , from (4.33) and (4.46), we see that

$$\langle q_i | \beta \rangle = \langle q | \beta \rangle \quad (4.49)$$

is independent of i . The Theorem follows on account of (4.43).

REMARKS.

1. Equation (4.47) holds for any (similar) mapping from the Cartesian coordinates $q_0 \rightarrow q_1, q_2, \dots, q_\mu$ in curvilinear coordinates with $\mu > 1$; its validity is not restricted to the gauge theory. When $\mu = 1$, (4.47) reduces to the more familiar expression

$$\delta^2(q_0 - q'_0) = \langle q | I | q' \rangle = |\mathcal{J}(q)|^{-1} \delta^2(q - q')$$

in the curvilinear coordinates.

2. As mentioned before, in our definition (4.23) of the Hilbert space $\Omega = \{ |\nu\rangle \}$, it is only for convenience that we single out the eigenvectors of the Hamiltonian H . In the space-axial gauge, we can choose the basis of Ω to be any complete set of functions of q_0 ; the basis vectors $|\nu\rangle$ do not have to be the eigenstates of the Hamiltonian. Through the kinematic transformation (4.37) (or more specifically (2.46)), all the above relations such as (4.34)-(4.35) and (4.48) can be derived without any reference to the Hamiltonian. Since the abstract Hilbert space Ω is gauge invariant, the Hilbert space in the λ -gauge is also well-defined; its basis vectors can be derived from those in the space-axial gauge through the same kinematic transformation ((4.37), or (2.46)). Likewise, any operators $A, B, \dots$ that are well-defined in the time-axial or space-axial gauge are also well-defined in the λ -gauge.

In a general gauge theory, we would begin by using the Cartesian time-axial gauge as the standard, and proceed as follows: (a) Define the Hilbert space $\Omega = \{|\nu\rangle\}$ in terms of all physical states $|\nu\rangle$ in the time-axial gauge (with the constraint of Gauss' Law). (b) Regard this abstract Hilbert space Ω as gauge-invariant, and (c) Its coordinate representation $\{ \langle q | \nu \rangle \}$ in any gauge is to be derived purely based on the coordinate-transformation from the time-axial gauge to the particular gauge that one is interested in. (In the soluble model, the space-axial gauge happens to be an even more convenient standard than the time-axial gauge.)

3. In the λ (or Coulomb-like) gauge, the Hilbert space in its coordinate representation $\{ \langle q | \nu \rangle \}$, is embedded in a *much larger* functional space

$$\{ \langle q | \rangle \} \quad (4.50)$$

whose basis vectors $\langle q |$ form a complete set of functions in q . Viewed from the larger functional space, once the coordinate representation, say $\langle q_0 | A | q'_0 \rangle$, is known in the space-axial gauge, then through the kinematic transformation (4.37) or (2.46), $\langle q | A | q' \rangle$ is also defined in the λ -gauge (hence, A is well-defined both in Ω and in the larger functional space (4.50)).

As an example, take the angular momentum operator in the space-axial gauge: In accordance with (2.47), we have

$$J = \frac{1}{i} \frac{\partial}{\partial \phi_0};$$

through the coordinate transformation $\phi_0 = \phi - \lambda \rho g \cos \phi$, J becomes

$$J = (1 + \lambda \rho g \sin \phi)^{-1} \frac{1}{i} \frac{\partial}{\partial \phi}$$

given by (2.35) in the λ -gauge, which is well-defined in both the Hilbert space and the functional space (4.50). Next, consider the function

$$e^{im\phi}$$

which belongs to the larger functional space $\{ \langle q | \rangle \}$, but not to the Hilbert space $\Omega = \{ \langle q | \nu \rangle \}$. Likewise,

$$J e^{im\phi} = (1 + \lambda \rho g \sin \phi)^{-1} m e^{im\phi}$$

belongs to $\{ \langle q | \rangle \}$, but not to Ω .

4. In the λ -gauge, in taking products of two operators A and B in the Hilbert space Ω , we must follow the usual rule in Ω ; i.e.,

$$AB = \sum_{\nu} A|\nu\rangle\langle\nu|B; \quad (4.51)$$

or, in terms of the coordinates,

$$AB = \int A|q\rangle\mathcal{J}(q)d^2q\langle q|I|q'\rangle\mathcal{J}(q')d^2q'\langle q'|B. \quad (4.52)$$

This expression simplifies, when AB acts on any ket-vector $|\alpha\rangle$ belonging to Ω ; we have, as in (4.35)

$$AB|\alpha\rangle = \int A|q\rangle\mathcal{J}(q)d^2q\langle q|B|\alpha\rangle. \quad (4.53)$$

5. In the path-integration formulation, we decompose

$$e^{-iH(t-t_1)} = (e^{-iH\epsilon})^{\mathcal{N}}$$

with $\mathcal{N}\epsilon = t-t_1$. In accordance with (4.51)-(4.52), we see that the matrix elements of $e^{-iH(t-t_1)}$ can be written as

$$\begin{aligned} \langle q|e^{-iH(t-t_1)}|q_1\rangle &= \int \langle q|e^{-iH\epsilon}|q_{\mathcal{N}}\rangle\mathcal{J}(q_{\mathcal{N}})d^2q_{\mathcal{N}}\langle q_{\mathcal{N}}|I|q'_{\mathcal{N}}\rangle\mathcal{J}(q'_{\mathcal{N}})d^2q'_{\mathcal{N}} \\ &\cdot \langle q'_{\mathcal{N}}|e^{-iH\epsilon}|q_{\mathcal{N}-1}\rangle\mathcal{J}(q_{\mathcal{N}-1})d^2q_{\mathcal{N}-1}\langle q_{\mathcal{N}-1}|I|q'_{\mathcal{N}-1}\rangle\mathcal{J}(q'_{\mathcal{N}-1})d^2q'_{\mathcal{N}-1} \cdots \\ &\cdot \langle q'_3|e^{-iH\epsilon}|q_2\rangle\mathcal{J}(q_2)d^2q_2\langle q_2|I|q'_2\rangle\mathcal{J}(q'_2)d^2q'_2\langle q'_2|e^{-iH\epsilon}|q_1\rangle. \end{aligned} \quad (4.54)$$

According to (4.47), $\langle q_n|I|q'_n\rangle$ is non-local, whenever q_n or q'_n is located at one of the multiplicity domains with $\mu > 1$ (shown in Figure 7); when we identify both q_n and q'_n as the positions of q at $t_n = (n-1)\epsilon + t_1$, the continuous path $q(t)$ can move forward as well as backward in time. On account of (4.53), when we multiply (4.54) by $\mathcal{J}(q_1)\langle q_1|\alpha\rangle$ and integrate over d^2q_1 , the expression simplifies into

$$\begin{aligned} &\langle q|e^{-iH(t-t_1)}|\alpha\rangle \\ &= \int \langle q|e^{-iH\epsilon}|q_{\mathcal{N}}\rangle\mathcal{J}(q_{\mathcal{N}})d^2q_{\mathcal{N}}\langle q_{\mathcal{N}}|e^{-iH\epsilon}|q_{\mathcal{N}-1}\rangle\mathcal{J}(q_{\mathcal{N}-1})d^2q_{\mathcal{N}-1} \\ &\quad \cdots \langle q_2|e^{-iH\epsilon}|q_1\rangle\mathcal{J}(q_1)dq_1\langle q_1|\alpha\rangle, \end{aligned} \quad (4.55)$$

provided that the ket-vector $|\alpha\rangle$ belongs to Ω .

6. So far we have written only $|\nu\rangle, |\rangle, \dots$ as ket-vectors, and $\langle q_0|\nu\rangle, \langle q|\nu\rangle, \langle q|\rangle, \dots$ for their coordinate representations; we have avoided the separate notations of $|q_0\rangle$ and $|q\rangle$ whose meanings are not yet specified. However, one may derive some insight into equations such as (4.34) and (4.35) by allowing $|q_0\rangle$ and $|q\rangle$ as individual entities and setting

$$I|q\rangle = |q_0\rangle \quad (4.56)$$

where q_0 is derived from q by the usual coordinate transformation and I is the identity operator in Ω , as before. Thus if two different coordinates q_i and q_j are mapped from the same q_0 (as in (4.37) with i and $j = 1, 2, \dots, \mu$), we have $I|q_i\rangle = I|q_j\rangle = |q_0\rangle$, but $|q_i\rangle \neq |q_j\rangle$. The kets $|q_0\rangle$ can be regarded as belonging to Ω , and are gauge invariant, but $|q\rangle$ belong to the larger space $\{|\rangle\}$ whose coordinate representation is (4.50). The set of all $|q_0\rangle$ is Ω , while the set of all $|q\rangle$ is the larger space $\{|\rangle\}$, with I acting as a projection operator through (4.56).

5. PATH INTEGRATION (λ -GAUGE)

To cast the operator $e^{-iH(t-t_1)}$ in the λ -gauge into a path integration formalism, in terms of the Lagrangian, there are several steps:

(i) Assume at $t = t_1$, the initial state vector $|t_1\rangle$ belongs to the Hilbert space Ω , given by (4.23); i.e., similar to (4.33)

$$|t_1\rangle = \sum_{\nu} c_{\nu} |\nu\rangle. \quad (5.1)$$

Hence (4.55) holds, with $|\alpha\rangle$ replaced by $|t_1\rangle$.

(ii) Rewrite H in a Weyl-ordered form H_W which, as we shall see, contains an additional interaction function V (analog to V_1 and V_2 in Ref. 3 for QCD). According to a general theorem, $e^{-iH_W(t-t_1)}$ can then be formulated in terms of path integration in the phase-space variables ($dx dy dp_x dp_y$ at different time t in our case).

(iii) Replacing unity by a product of Gaussian integrals over $d\xi(t)$, and carrying out the $dp_x dp_y$ integrals, we can derive the final path integration form over the $dx dy d\xi$ (i.e., $\delta(z - \lambda x) dx dy dz d\xi$) space at different t .

(iv) Alternatively, after (ii) we may carry out the integration over $dp_x dp_y$ directly and derive an equivalent path integration formulation over a reduced space, with only $dx dy$ at each t .

5.1. Weyl-Ordered Hamiltonian

To proceed with step (ii), we start from the wave function ψ and the kinetic energy operator K in the λ -gauge, in accordance with (2.23) and (2.24). Define χ by

$$\psi = \frac{\chi}{\sqrt{\mathcal{J}}} \quad (5.2)$$

where, as before, $\mathcal{J} = \lambda y + g^{-1}$ is the Jacobian. By direct differentiation, we find

$$\sqrt{\mathcal{J}} K \frac{\chi}{\sqrt{\mathcal{J}}} = (K_W + V) \chi \quad (5.3)$$

with

$$V = -\frac{g^2(2 + 3\lambda^2) + \lambda^3 g^3 y}{8(1 + \lambda g y)^3} + \frac{\lambda^2 g^4 x^2 (1 + \lambda^2)}{8(1 + \lambda g y)^4} \quad (5.4)$$

and K_W , the Weyl-ordered form of K , given by

$$K_W = \frac{1}{8} (p_a p_b \mathcal{M}_{ab}^{-1} + 2p_a \mathcal{M}_{ab}^{-1} p_b + \mathcal{M}_{ab}^{-1} p_a p_b) \quad (5.5)$$

in which the repeated indices a and b are summed over x and y with $p_x = -i \partial / \partial x$, $p_y = -i \partial / \partial y$ and $\mathcal{M}_{ab}^{-1}$ given by (2.26). In terms of χ , the time-dependent Schrödinger equation

$$H \psi = i \dot{\psi}$$

becomes

$$H_W \chi = i \dot{\chi} \quad (5.6)$$

where, on account of $H = K + U$,

$$H_W = K_W + U + V. \quad (5.7)$$

Next, we consider the time development of $\chi(x, y, t)$ from an initial time t_1 to a later time t . By using (5.6), we have

$$\chi(x, y, t) = \int \langle x, y | e^{-iH_W(t-t_1)} | x_1 y_1 \rangle \chi(x_1, y_1, t_1) dx_1 dy_1. \quad (5.8)$$

where $\chi(x_1, y_1, t_1)$ is related to $|t_1\rangle$ of (5.1) by

$$\langle x_1, y_1 | t_1 \rangle = \frac{1}{\sqrt{\mathcal{J}}} \chi(x_1, y_1, t_1). \quad (5.9)$$

Take any $\psi = \chi / \sqrt{\mathcal{J}}$ and $\psi' = \chi' / \sqrt{\mathcal{J}}$,

$$\int \psi'^* \psi \mathcal{J} dx dy = \int \chi'^* \chi s(\mathcal{J}) dx dy \quad (5.10)$$

where

$$s(\mathcal{J}) = \mathcal{J}(x, y) / |\mathcal{J}(x, y)| \quad (5.11)$$

is the sign of the Jacobian.

By using (4.55), we can rewrite the right-hand side of (5.8) in the form of a phase-space path integral [3, 17]:

$$\chi(x, y, t) = \lim_{\epsilon \rightarrow 0} \int \prod_{n=1}^{\mathcal{N}} (2\pi)^{-2} dx_n dy_n dk_{x_n} dk_{y_n} s(n) e^{i\epsilon L(n)} \chi(x_1, y_1, t_1) \quad (5.12)$$

where

$$L(n) \equiv k_{x_n} \dot{x}_n + k_{y_n} \dot{y}_n - H_{\text{cl}}(\bar{x}_n, \bar{y}_n, k_{x_n}, k_{y_n}), \quad (5.13)$$

$$s(n) = \mathcal{J}(\bar{x}_n, \bar{y}_n) / |\mathcal{J}(\bar{x}_n, \bar{y}_n)|, \quad (5.14)$$

$$\begin{aligned} \bar{x}_n &= \frac{1}{2} (x_{n+1} + x_n), \quad \bar{y}_n = \frac{1}{2} (y_{n+1} + y_n), \\ \dot{x}_n &= \frac{1}{\epsilon} (x_{n+1} - x_n), \quad \dot{y}_n = \frac{1}{\epsilon} (y_{n+1} - y_n) \end{aligned} \quad (5.15)$$

and $H_{\text{cl}}(x, y, k_x, k_y)$ is the classical Hamiltonian function of c no. variables x, y, k_x and k_y , given by

$$H_{\text{cl}} = \frac{1}{2} k_a \mathcal{M}_{ab}^{-1} k_b + U + V. \quad (5.16)$$

In the integral (5.12), we divide $t - t_1$ into $\mathcal{N}$ intervals of ϵ each and the integration is over x_n, y_n, k_{x_n} and k_{y_n} which may be regarded as the values of x, y, k_x and k_y at $t_n = t_1 + (n - 1)\epsilon$. After the integration, the right-hand side of (5.12) depends only on the final coordinate variables $x_{\mathcal{N}+1} = x$ and $y_{\mathcal{N}+1} = y$ at $t_{\mathcal{N}+1} = t$. To complete the final path integration form, we must carry out step (iii) by returning to the original Lagrangian $\mathcal{L}$ (1.1), but in the λ -gauge.

5.2. Path Integration in the (x, y, ξ) -Space

Substituting x, y and λx for X, Y and Z in (1.1), we write

$$\mathcal{L}_{\text{cl}} = \mathcal{K}_{\text{cl}} - U \quad (5.17)$$

where

$$\mathcal{K}_{\text{cl}} = \frac{1}{2} \tilde{q} \mathcal{M} \dot{q} \quad (5.18)$$

with

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ -g\xi \end{pmatrix} \quad (5.19)$$

and $\mathcal{M}$ given by (2.20). It is convenient to express $\mathcal{M}$ in terms of a 2×2 matrix M , a 2×1 column matrix u and a number c :

$$\mathcal{M} = \begin{pmatrix} M & u \\ \tilde{u} & c \end{pmatrix} \quad (5.20)$$

where

$$M = \begin{pmatrix} 1 + \lambda^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} -y + \frac{\lambda}{g} \\ x \end{pmatrix} \quad (5.21)$$

and c is given by

$$c = x^2 + y^2 + g^{-2}. \quad (5.22)$$

Likewise, the inverse $\mathcal{M}^{-1}$ can be written as

$$\mathcal{M}^{-1} = \begin{pmatrix} N^{-1} & v \\ \tilde{v} & c' \end{pmatrix} \quad (5.23)$$

where N is a 2×2 matrix, v a 2×1 column matrix and c' is 1×1 :

$$\begin{aligned} N &= M - \frac{1}{c} u \tilde{u}, \\ v &= -\mathcal{J}^{-2} (1 + \lambda^2) M^{-1} u \end{aligned} \quad (5.24)$$

and

$$c' = \mathcal{J}^{-2} (1 + \lambda^2)$$

with the Jacobian $\mathcal{J} = \lambda y + g^{-1}$ and the matrix elements of N^{-1} identical to $(\mathcal{M}^{-1})_{ab}$ of (2.26).

We observe that the classical functions

$$K'_{\text{cl}} \equiv k_x \dot{x} + k_y \dot{y} - \frac{1}{2} k_a \mathcal{M}_{ab}^{-1} k_b \quad (5.25)$$

and $\mathcal{K}_{\text{cl}}$ of (5.18) are related by the following algebraic identity:

$$K'_{\text{cl}} + Q_1 = \mathcal{K}_{\text{cl}} + Q_2 \quad (5.26)$$

where Q_1 and Q_2 are two quadratic functions:

$$Q_1 = (2c')^{-1} (g\xi + \tilde{v} k)^2 \quad (5.27)$$

with

$$k = \begin{pmatrix} k_x \\ k_y \end{pmatrix}, \quad (5.28)$$

and

$$Q_2 = -\frac{1}{2} \tilde{k}' M^{-1} k' \quad (5.29)$$

with

$$k' = k + g\xi u - M \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}. \quad (5.30)$$

Insert

$$1 = g |\mathcal{J}| \sqrt{\frac{\epsilon}{2\pi i(1+\lambda^2)}} \int_{-\infty}^{\infty} e^{i\epsilon Q_1} d\xi \quad (5.31)$$

into the integrand of

$$\int e^{i\epsilon K'_{cl}} (2\pi)^{-2} s(\mathcal{J}) dx dy dk_x dk_y. \quad (5.32)$$

Using the identities (5.26) and

$$\mathcal{J} = |\mathcal{J}| s(\mathcal{J}), \quad (5.33)$$

and carrying out the k_x, k_y integrations, we find the integral (5.32) to be

$$\left(\frac{1}{2\pi i}\right)^{3/2} \frac{g}{\sqrt{\epsilon}} \int \mathcal{J} dx dy d\xi e^{i\epsilon K_{cl}}. \quad (5.34)$$

Thus, apart from a constant multiplicative factor, (5.12) becomes

$$\begin{aligned} \chi(x, y, t) &\equiv \langle x, y | e^{-iH_W(t-t_1)} | t_1 \rangle \\ &= \lim_{\epsilon \rightarrow 0} \int \prod_{n=1}^{\mathcal{N}} dx_n dy_n d\xi_n e^{i\epsilon \mathcal{L}(n)} \chi(x_1, y_1, t_1) \end{aligned} \quad (5.35)$$

where $|t_1\rangle$ denotes the ket-vector at time t_1 ,

$$\chi(x_1, y_1, t_1) = \langle x_1 y_1 | t_1 \rangle,$$

$$\mathcal{L}(n) = \mathcal{K}_{cl}(\bar{x}_n, \bar{y}_n, \dot{x}_n, \dot{y}_n, \xi_n) - U(\bar{x}_n, \bar{y}_n) - V(\bar{x}_n, \bar{y}_n) - i\epsilon^{-1} \ln(g\mathcal{J}_n) \quad (5.36)$$

with

$$\mathcal{J}_n = \lambda \bar{y}_n + g^{-1}$$

and $\bar{x}_n, \bar{y}_n, \dot{x}_n, \dot{y}_n$ given by (5.15). For $\mathcal{J}_n > 0$, $\ln \mathcal{J}_n$ is real; however,

$$\ln \mathcal{J}_n = i\pi + \text{real} \quad \text{for } \mathcal{J}_n < 0.$$

While $\ln \mathcal{J}_n$ has a cut from $\bar{y}_n = -(\lambda g)^{-1}$ to $-\infty$, the integrand $\exp(i\epsilon \mathcal{L}(n))$, being proportional to $\exp(\ln \mathcal{J}_n)$, does not.

In this derivation, the right-hand side of (5.35) gives a well-defined path integration; the question of “gauge-equivalent” copies plays *no special role* in the

formulation, provided that the “ghost term” [12, 13], $-i\epsilon^{-1} \ln \mathcal{J}_n$, and the additional vertex term [3], $-V(\bar{x}_n, \bar{y}_n)$, are taken into account.

REMARKS.

1. The additional interaction $V(x, y)$, given by (5.4), is due to the ordering of quantum operators; its correctness will be confirmed by a two-loop calculation in Section 5.3. In Figures 8(a) and 8(b), the function $V(x, y)$ is plotted in the three-dimensional (V, x, y) space. Comparing that against the locations of gauge copies in Figure 7, we see that $V(x, y)$ can be quite different at different sites of gauge copies. The origin of V is connected with the mapping (3.20)

$$q_0 \rightarrow q_1, q_2, \dots, q_\mu$$

with $q_0 = (x_0, y_0)$ in the space-axial gauge and $q_i = (x_i, y_i)$ in the λ -(or Coulomb-like) gauge. A *rectilinear* path in the space-axial gauge

$$(x_0, y_0) \text{ at } t \rightarrow (x_0 + \delta x_0, y_0 + \delta y_0) \text{ at } t + \epsilon, \quad (5.37)$$

is mapped into *different* curves (no longer rectilinear) at different sites (x_i, y_i) of gauge copies, which are manifestly non-gauge-invariant (as is evident from the fact that different $\mathcal{J}(q_i)$ implies different slopes, and different derivations of $\mathcal{J}(q_i)$ give different curvatures, etc.). The presence of this manifest non-gauge-invariant interaction V is to compensate for the corresponding manifest non-gauge-invariant aspects of different curvatures at different gauge copy-sites, just as the Jacobian $\mathcal{J}$ compensates for different “slopes” at different sites. (The Jacobian correction for a general Feynman diagram formulation in the curvilinear coordinates was proved in Ref. [12], and also by Faddeev and Popov [13] using a different approach. For QCD in the Coulomb gauge, the additional V interaction derived here is the analog of the $V_1 + V_2$ interaction obtained in Ref. 3, of which the V_1 term was found earlier by Schwinger [4].)

2. One may wonder, since the classical action integrals over different curvilinear paths of gauge-equivalent copies (as the result of the mapping from (5.37)) should be identical, why would there be any compensating factors (like $\mathcal{J}$ and $\exp(-i\epsilon V)$)? This can be understood as follows:

In the path integration formulation (5.35)-(5.36), the action within t_n and $t_{n+1} = t_n + \epsilon$ is written as a sum:

$$\epsilon \mathcal{L}(n) = \epsilon \mathcal{L}_{\text{cl}}(n) - \epsilon V_n - i \ln \mathcal{J}_n$$

where $V_n = V(\bar{x}_n, \bar{y}_n)$ and $\mathcal{L}_{\text{cl}}(n)$ is the classical Lagrangian (5.17),

$$\mathcal{L}_{\text{cl}}(n) = \mathcal{K}_{\text{cl}}(q_n, \dot{q}_n) - U(q_n),$$

with q_n denoting $\bar{x}_n, \bar{y}_n$ and ξ_n , and $\dot{q}_n$ for $\dot{x}_n$ and $\dot{y}_n$, given by (5.15). The actual path from t_1 to t , though continuous, consists of $\mathcal{N}$ piece-wise rectilinear segments of ϵ time-interval each. It is important to note that except for Cartesian coordinates the integral

$$\int \prod_{i=1}^{\mathcal{N}} [dq_n] e^{i\epsilon \mathcal{L}_{\text{cl}}(n)} \quad (5.38)$$

with $[dq_n] = dx_n dy_n d\xi_n$ differs from the superposition of exponentials of the classical action along various smooth continuous paths, even in the limit $\epsilon \rightarrow 0$. Thus, for different gauge copies, the integrand in the path integral (5.38) is actually *not* gauge equivalent, which necessitates the compensating factors $\mathcal{J}_n$ and $\exp(-i\epsilon V_n)$, as mentioned above.

The difference between a ‘‘Gaussian-type’’ path integral (5.38) and that of a classical action integral (along a smooth path) is a common feature to any problem employing curvilinear coordinates. In deriving Feynman diagrams, one may either (i) start with the Lagrangian [16], or (ii) use the Hamiltonian following the Dyson-Wick approach [14, 15]. For the mapping of (5.37) from Cartesian to curvilinear coordinates, in (i) one is dealing with transformations of time-dependent paths which are more complicated, whereas in (ii) the problem can be reduced to time-independent transformations between Hamiltonian representations in different coordinates; the latter method [3, 17] is simpler, and is the one that we follow in this paper.

3. In (5.35), the ξ_n integration can be readily carried out, since it is a Gaussian integral (as is evident from the way that the ξ_n -dependence is introduced). If one is interested in a perturbative expansion of (5.35) in powers of g , then one needs the limiting form as $g \rightarrow 0$,

$$\mathcal{L}(n) \rightarrow \mathcal{L}_0(n) = L_0(n) + \frac{1}{2} (\xi_n - \lambda \dot{x}_n)^2 \quad (5.39)$$

where

$$L_0(n) = \frac{1}{2} (\dot{x}_n^2 + \dot{y}_n^2) - U(\bar{x}_n, \bar{y}_n). \quad (5.40)$$

Let the average of the m^{th} power of ξ_n be

$$\langle \xi_n^m \rangle \equiv \int_{-\infty}^{\infty} e^{i\epsilon L_0(n)} \xi_n^m d\xi_n / \int_{-\infty}^{\infty} e^{i\epsilon L_0(n)} d\xi_n. \quad (5.41)$$

We find

$$\begin{aligned} \langle \xi_n \rangle &= \lambda \dot{x}_n, & \langle \xi_n^2 \rangle &= (\lambda \dot{x}_n)^2 + (i/\epsilon), \\ \langle \xi_n^3 \rangle &= (\lambda \dot{x}_n)^3 + 3\lambda \dot{x}_n (i/\epsilon), \\ \langle \xi_n^4 \rangle &= (\lambda \dot{x}_n)^4 + 6(\lambda \dot{x}_n)^2 (i/\epsilon) + 3(i/\epsilon)^2 \end{aligned} \quad (5.42)$$

and in general

$$\begin{aligned} \langle \xi_n^m \rangle &= (\lambda \dot{x}_n)^m + \frac{1}{2} m(m-1) (\lambda \dot{x}_n)^{m-2} (i/\epsilon) \\ &\quad + \frac{m!}{(m-4)! 4!!} (\lambda \dot{x}_n)^{m-4} (i/\epsilon)^2 + \dots \\ &\quad + \frac{m!}{(m-2\ell)! (2\ell)!!} (\lambda \dot{x}_n)^{m-2\ell} (i/\epsilon)^\ell + \dots \end{aligned} \quad (5.43)$$

These expressions and the integrations over x_n and y_n lead to perturbative Feynman diagrams in the λ -gauge, as will be discussed in Section 6.

5.3. Path Integration in the Reduced Space

The path integration formula (5.12) is over the phase space (x, y, k_x, k_y) . The integrations over k_{x_n} and k_{y_n} are all Gaussian, and can be readily carried out:

$$\begin{aligned} \int (2\pi)^{-2} dk_x dk_y \exp \left[i\epsilon (k_x \dot{x} + k_y \dot{y} - \frac{1}{2} k_a \mathcal{M}_{ab}^{-1} k_b) \right] \\ = \left(\frac{1}{2\pi i\epsilon} \right) |\mathcal{J}'| \exp \left(\frac{1}{2} i\epsilon \dot{x}_a N_{ab} \dot{x}_b \right) \end{aligned} \quad (5.44)$$

where, as before, the repeated indices a and b are summed over x and y (or, 1 and 2 with $x_1 = x$ and $x_2 = y$), $\mathcal{J}'$ is the *new* Jacobian

$$\mathcal{J}'(x, y) = (1 + \lambda g y) / (1 + g^2 \rho^2)^{\frac{1}{2}} \quad (5.45)$$

with $\rho^2 = x^2 + y^2$ and N is the 2×2 matrix,

$$N(x, y) = (1 + g^2 \rho^2)^{-1} \begin{pmatrix} g^2(1 + \lambda^2)x^2 + (1 + \lambda gy)^2 & -gx(\lambda - gy) \\ -gx(\lambda - gy) & 1 + g^2 y^2 \end{pmatrix}, \quad (5.46)$$

whose inverse is the same as the upper left 2×2 block matrix of the inverse $\mathcal{M}^{-1}$ of the original mass matrix (2.20); i.e., $N_{ab}^{-1} = \mathcal{M}_{ab}^{-1}$ given by (2.26). Since the sign of $\mathcal{J}'$ is the same as that of $\mathcal{J}$, we have

$$\mathcal{J}' = |\mathcal{J}'| s(\mathcal{J}) \quad (5.47)$$

Equation (5.12) becomes, apart from a constant multiplicative factor,

$$\chi(x, y, t) = \lim_{\epsilon \rightarrow 0} \int \prod_{n=1}^{\mathcal{N}} dx_n dy_n e^{i\epsilon L'(n)} \chi(x_1, y_1, t_1) \quad (5.48)$$

with

$$L'(n) = \frac{1}{2} \tilde{r}_n N(\bar{x}_n, \bar{y}_n) \dot{r}_n - U(\bar{x}_n, \bar{y}_n) - V(\bar{x}_n, \bar{y}_n) - i\epsilon^{-1} \ln \mathcal{J}'_n, \quad (5.49)$$

$$\dot{r}_n = \begin{pmatrix} \dot{x}_n \\ \dot{y}_n \end{pmatrix}, \quad \mathcal{J}'_n = (1 + \lambda g \bar{y}_n) / [1 + g^2(\bar{x}_n^2 + \bar{y}_n^2)]^{\frac{1}{2}} \quad (5.50)$$

and $\bar{x}_n, \bar{y}_n, \dot{x}_n, \dot{y}_n$ given by (5.15). We note that the first term in $L'(n)$ can also be derived by substituting (1.5) into the original Lagrangian (1.1) and replacing X, Y, Z by x, y and $z = \lambda x$; the resulting classical kinetic energy at time t_n is $\frac{1}{2} \tilde{r}_n N(\bar{x}_n, \bar{y}_n) \dot{r}_n$. For $\mathcal{J}'_n > 0$, $\ln \mathcal{J}'_n$ is real; for $\mathcal{J}'_n < 0$, $\ln \mathcal{J}'_n = i\pi + \text{real}$. Again, as in (5.35), the integrand $\exp(i\epsilon L'(n))$ is analytic at $\mathcal{J}'_n = 0$; i.e., $\bar{y}_n = -(\lambda g)^{-1}$.

Equation (5.35) is the path-integration formulation in terms of paths in the (x, y, ξ) space, whereas (5.48) gives an equivalent form in terms of paths in the (x, y) -space. (Both formulations closely mirror the corresponding formulations in the Coulomb gauge for QCD.) The Feynman diagrams for these two formulations are different, as will be analyzed in the next section. All physical results are, of course, the same.

6. FEYNMAN DIAGRAMS (λ -GAUGE)

6.1. Feynman Rules in Terms of x, y, ξ

Decompose $\mathcal{L}(n)$ of (5.36) into a sum of two terms:

$$\mathcal{L}(n) = \mathcal{L}_0(n) + \mathcal{L}_1(n) \quad (6.1)$$

and treat $\mathcal{L}_1(n)$ as the perturbation with the expansion

$$e^{i\epsilon \mathcal{L}(n)} = e^{i\epsilon \mathcal{L}_0(n)} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} [i\epsilon \mathcal{L}_1(n)]^{\ell} \quad (6.2)$$

where $\mathcal{L}_0(n)$ is given by (5.39). In this section, we consider only the special case

$$U(x, y) = \frac{1}{2} \omega^2 (x^2 + y^2), \quad (6.3)$$

so that the unperturbed ground state $|0\rangle$ is simply that of a two-dimensional harmonic oscillator of frequency ω . It is convenient to introduce $x_{\mu}(t)$ with

$$x_0 = \xi, \quad x_1 = x, \quad x_2 = y \quad (6.4)$$

and the subscript $\mu = 0, 1$ and 2 . The zeroth-order propagator $D_{\mu\nu}(t-t')$ is given by the average of $x_{\mu}(t) x_{\nu}(t')$ over the path integration

$$\prod_{n=1}^{\mathcal{N}} e^{i\epsilon \mathcal{L}_0(n)} dx_n dy_n d\xi_n \quad (6.5)$$

with $|0\rangle$ as both the initial and final states and $t_{\mathcal{N}+1} - t_1 = \mathcal{N}\epsilon$, in the limit $\epsilon \rightarrow 0$, $t_1 \rightarrow -\infty$ and $t_{\mathcal{N}+1} \rightarrow \infty$. We find

$$D_{11}(t-t') = D_{22}(t-t') = \int \frac{ie^{-ik_0(t-t')}}{k_0^2 - (\omega - i0+)^2} \frac{dk_0}{2\pi}, \quad (6.6)$$

$$\begin{aligned} D_{00}(t-t') &= \int \frac{i[(1+\lambda^2)k_0^2 - \omega^2]}{k_0^2 - (\omega - i0+)^2} e^{-ik_0(t-t')} \frac{dk_0}{2\pi} \\ &= \lambda^2 \frac{\partial^2}{\partial t \partial t'} D_{11}(t-t') + i\delta(t-t'), \end{aligned} \quad (6.7)$$

$$D_{01}(t-t') = \int \frac{k_0 e^{-ik_0(t-t')}}{k_0^2 - (\omega - i0+)^2} \frac{dk_0}{2\pi} = \lambda \frac{\partial}{\partial t} D_{11}(t-t'), \quad (6.8)$$

$$D_{10}(t-t') = -D_{01}(t-t') = \lambda \frac{\partial}{\partial t'} D_{11}(t-t') \quad (6.9)$$

and other $D_{\mu\nu}(t-t')$ are all zero. Equation (6.7) follows from the average $\langle \xi_n^2 \rangle$ of (5.42), with $(\lambda \dot{x}_n)^2$ related to $\lambda^2 \partial^2 D_{11} / \partial t \partial t'$ and i/ϵ replaced by $i \delta(t-t')$. Likewise, the average $\langle \xi_n \rangle = \lambda \dot{x}_n$ gives (6.8)-(6.9). The systematics of other remaining averages in (5.42) insures that products of D_{00} and D_{01} can be used in contractions involving higher powers of ξ .

From (5.18)-(5.19), (5.36) and (6.1), we have (suppressing the index n and denoting $\bar{x}_n, \bar{y}_n$ by x, y and $\dot{x}_n, \dot{y}_n$ by $\dot{x}, \dot{y}$)

$$i\mathcal{L}_1 = ig\xi(\dot{x}y - \dot{y}x) + \frac{ig^2}{2} \xi^2(x^2 + y^2) - iV + \epsilon^{-1} \ln g\mathcal{J} \quad (6.10)$$

where V is given by (5.4) and the last term can be written as

$$\epsilon^{-1} \ln g\mathcal{J} = \epsilon^{-1} \ln(1 + \lambda gy) = \sum_{\ell=1}^{\infty} \epsilon^{-1} (-\ell^{-1}) (-\lambda gy)^\ell; \quad (6.11)$$

each $-\lambda gy = i$ times $(i\lambda gy)$, with the second factor $i\lambda gy$ assigned to be the “ghost- y -ghost” vertex and the first factor i to be part of the ghost propagator

$$i \delta(t-t') = i \int e^{-ik_0(t-t')} \frac{dk_0}{2\pi}. \quad (6.12)$$

Thus, the summand $\epsilon^{-1}(-\ell^{-1})(-\lambda gy)^\ell$ in (6.11) can be represented by a loop with ℓ ghost- y -ghost vertices, all connected by internal ghost-lines with the factor $(-\ell^{-1})$ denoting an overall minus sign divided by ℓ , the symmetry number of the “ghost-loop”. The integration over the $\delta(t-t')$ functions in the ghost propagators gives another overall $\delta(0)$ which is the ϵ^{-1} factor in (6.11). The “ghost-loop” can be either the whole diagram or a part of a diagram.

To summarize: the λ -gauge Feynman rules in the $x_\mu = (x, y, \xi)$ space are

$D_{\mu\nu}(t-t')$	for the propagator $x_\nu(t')$ to $x_\mu(t)$,	
$i \delta(t-t')$	for the ghost propagator,	
$-ig\xi(x\dot{y} - y\dot{x})$	for the three-point vertex,	
$\frac{ig^2}{2} \xi^2(x^2 + y^2)$	for the four-point vertex,	(6.13)
$i\lambda gy$	for the ghost- y -ghost vertex,	

$-i V$

for the additional interaction vertices
due to quantum operator ordering,

and an overall minus sign for each ghost loop. The V function and the propagator $D_{\mu\nu}$ are given by (5.4) and (6.6)-(6.9). On account of the definitions of $\bar{x}_n, \bar{y}_n$ and $\dot{x}_n, \dot{y}_n$ given by (5.15), the contractions of time derivatives of $x_\mu(t) x_\nu(t')$ are the same as time derivatives of the contractions.

In Appendix C, we will evaluate the one-loop corrections to $D_{\mu\nu}(t - t')$, and compare the results with the exact expressions derived in Section 4.1.

6.2. Feynman Rules in the Reduced Space

We start with the path integration formulation (5.48) in the (x, y) space. For $U = \frac{1}{2} \omega^2(x^2 + y^2)$, the zeroth-order propagators connecting $x(t')$ or $y(t')$ to $x(t)$ or $y(t)$ are

$$D_{xx}(t - t') = D_{yy}(t - t') = \int \frac{ie^{-ik_0(t-t')}}{k_0^2 - (\omega - i0+)^2} \frac{dk_0}{2\pi} \quad (6.14)$$

and $D_{xy}(t - t') = 0$. Write

$$\mathcal{J}' = 1 - G, \quad (6.15)$$

$$G(x, y) = 1 - \frac{1 + \lambda gy}{(1 + g^2 \rho^2)^{\frac{1}{2}}} \quad (6.16)$$

and

$$\epsilon^{-1} \ln \mathcal{J}' = \sum_{\ell=1}^{\infty} \epsilon^{-1} (-\ell^{-1}) G^\ell. \quad (6.17)$$

In this alternative formalism, all paths are in the reduced space $(x(t), y(t))$. The corresponding λ -gauge Feynman rules are given by (6.14) and, in place of (6.13),

$$i \delta(t - t') \quad \text{for the ghost propagator,} \quad (6.18)$$

$$i \frac{1}{2} \tilde{r} [N(x, y) - 1] \dot{r} \quad \text{for the 3-point, 4-point, } \dots \text{ vertices,} \quad (6.19)$$

$$-iG(x, y) \quad \text{for the ghost-}(x, y)\text{-ghost vertices,} \quad (6.20)$$

$$-i V \quad \text{for the additional interaction vertices} \\ \text{due to quantum operator ordering,} \quad (6.21)$$

As in the previous section, the summand $\epsilon^{-1}(-\ell^{-1}) G^\ell$ in (6.17) is to be represented by a loop with ℓ ghost- G -ghost vertices, all connected by internal ghost-lines with

the factor $(-\ell^{-1})$ denoting an overall minus sign divided by the symmetry number ℓ of the “ghost-loop”. The integration over the $\delta(t - t')$ functions in the ghost propagators gives an overall $\delta(0)$ to each ghost-loop, which is the ϵ^{-1} factor in (6.17). Expand in powers of g :

$$\begin{aligned} i \frac{1}{2} \tilde{r}(N-1) \dot{r} &= i\lambda g(y\dot{x}^2 - x\dot{x}\dot{y}) \\ &+ i \frac{1}{2} g^2 \{ [\lambda^2 x^2 + (\lambda^2 - 1)y^2] \dot{x}^2 + 2xy\dot{x}\dot{y} - x^2\dot{y}^2 \} \\ &+ O(g^3) \end{aligned} \quad (6.22)$$

$$-iG = i\lambda g y - \frac{i}{2} g^2 (x^2 + y^2) + O(g^3) \quad (6.23)$$

and, from (5.4),

$$\begin{aligned} -iV &= \frac{i}{8} g^2 (2 + 3\lambda^2) - \frac{i}{4} \lambda (3 + 4\lambda^2) g^3 y \\ &- \frac{i}{8} \lambda^2 g^4 [(1 + \lambda^2)x^2 - 3(4 + 5\lambda^2)y^2] + \dots \end{aligned} \quad (6.24)$$

where $\dots$ denotes cubic and higher-power functions of x and y . As before, the above $x, y, \dot{x}$ and $\dot{y}$ stand for $\bar{x}_n, \bar{y}_n, \dot{x}_n$ and $\dot{y}_n$ given by (5.15).

The Feynman diagrams derived in this section can also be arrived at by using the Feynman rules (6.13) of the previous section, but summing over all diagrams with ξ -propagators and ξ -vertices.

6.3. Confirming V

As mentioned before, the additional interaction V , given by (5.4), is due to the ordering of quantum operators. In a perturbative series expansion, V is relevant to two- and higher-loop Feynman diagrams. In this section, we shall confirm the g^2 term in V by computing the groundstate energy using diagrams.

In Figure 9, we list all the relevant two-loop diagrams in which the vertices are given by (6.19) and (6.20). Each solid line represents the propagator $\mathcal{D}_{xx}$ or $\mathcal{D}_{yy}$ given by (6.14) and each dashed line represents the ghost propagator (6.18). Let $a, b, \dots, g$ denote i times the contributions due to the diagrams (a), (b), $\dots$, (g) in Figure 9. Including V , the shift of groundstate energy to $O(g^2)$ is

$$E_{\text{gd}}(g^2) = a + b + c + d + e + f + g - \frac{1}{8} g^2 (2 + 3\lambda^2). \quad (6.25)$$

From the exact expression (2.8), the groundstate carries the quantum numbers $n_+ = n_- = 0$: its energy is ω , independent of g^2 . Thus,

$$E_{\text{gd}}(g^2) = 0. \quad (6.26)$$

As will be shown in Appendix D,

$$a + b + c + d = g^2 \left(\frac{1}{4} + \frac{\lambda^2}{2} \right) \quad (6.27)$$

and

$$e + f + g = -g^2 \frac{\lambda^2}{8}. \quad (6.28)$$

Their sum gives

$$\frac{1}{8} g^2 (2 + 3\lambda^2) \quad (6.29)$$

which cancels the g^2 term in V and yields the correct groundstate energy, with the energy shift $E_{\text{gd}}(g^2) = 0$.

7. CONCLUDING REMARKS

The question raised by Gribov concerning gauge-equivalent copies in the Coulomb gauge of quantum chromodynamics is central to our understanding of the theory in that gauge. Yet the complexity of QCD makes it difficult to penetrate the various complications implied by the existence of such copies. The soluble gauge model presented in this paper shares many analogous features of Gribov-type copies; by construction the model can be solved exactly and thereby we may provide some insight into the problem.

Gribov's proposal is to select only one copy with a positive Jacobian. This necessitates the imposition of an artificial boundary that is both gauge-dependent and *coupling-constant*-dependent. As a consequence, it requires a new boundary condition at the artificial boundary, which changes the topology of the problem. The fact that the boundary itself depends on the coupling constant makes it especially difficult to develop Feynman rules for perturbative calculations. While Gribov's suggestion [1], especially with Zwanziger's improvement [2], might work in some cases, it is unclear how to develop general Feynman diagrammatic formulation, that has been so very useful in our analysis of the theory.

In this paper we have proposed an alternative method which allows the inclusion of all gauge copies. From the explicit solutions of the model, we see that correct results can be derived by including *all* the gauge copies in the λ (or Coulomb-like-) gauge. In that gauge, all state vectors are well-behaved in the entire (x, y) space, including the zero Jacobian boundary as well as the boundary of different multiplicity regions. The same holds for the path-integration formulation, and for a systematic development of the Feynman rules for diagrams.

As we have shown, many of the complications of gauge-equivalent copies are in common with any problem in which there is a coordinate transformation from a set of Cartesian coordinates q_0 to a set of curvilinear coordinates q with

$$q_0 \rightarrow q = q_1, q_2, \dots, q_\mu$$

and μ can be > 1 , as in (3.20). Thus, the method that we develop for resolving these complications can be applied to a general class of such cases, for both gauge theories as well as non-gauge theories.

APPENDIX A

A.1. The Wave Function of an Arbitrary Eigenstate $|n_+, n_- \rangle$ in the "Space-Axial" Gauge

Let ρ, ϕ_0 be the polar coordinates of x_0 and y_0 . For an arbitrary function $F(\rho)$ of ρ , it follows from (2.60) that

$$\begin{aligned} a^\dagger F(\rho) e^{i\nu\phi_0} &= \frac{1}{2\sqrt{\omega}} e^{i(\nu+1)\phi_0} \left(\omega\rho - \frac{d}{d\rho} + \frac{\nu}{\rho} \right) F(\rho) \\ &= -\frac{1}{\sqrt{\omega}} e^{i(\nu+1)\phi_0} e^{\frac{1}{2}\omega\rho^2} \rho^{\nu+1} \frac{d}{d\rho^2} \left[\rho^{-\nu} e^{-\frac{1}{2}\omega\rho^2} F(\rho) \right]. \end{aligned} \quad (\text{A.1})$$

Likewise, we have

$$b^\dagger F(\rho) e^{i\nu\phi_0} = -\frac{1}{\sqrt{\omega}} e^{i(\nu-1)\phi_0} e^{\frac{1}{2}\omega\rho^2} \rho^{-\nu+1} \frac{d}{d\rho^2} \left[\rho^\nu e^{-\frac{1}{2}\omega\rho^2} F(\rho) \right]. \quad (\text{A.2})$$

To build up the wave function for the state $|n_+, n_- \rangle$ with $n_+ \geq n_-$, we may start with the groundstate wave function (2.62)

$$F(\rho) = \sqrt{\frac{\omega}{\pi}} e^{-\frac{1}{2}\omega\rho^2} \quad (\text{A.3})$$

and operate on it n_+ times by $a^\dagger$

$$\frac{1}{\sqrt{n_+!}} (a^\dagger)^{n_+} \langle \rho, \phi_0 | 0, 0 \rangle = \sqrt{\frac{\omega}{\pi}} \omega^{\frac{n_+}{2}} \frac{1}{\sqrt{n_+!}} \rho^{n_+} e^{-\frac{1}{2}\omega\rho^2} e^{in_+\phi_0}. \quad (\text{A.4})$$

Then we let $b^\dagger$ act on (A.4) n_- times and obtain

$$\begin{aligned} \langle \rho, \phi_0 | n_+, n_- \rangle &= \frac{1}{\sqrt{n_+! n_-!}} (b^\dagger)^{n_-} (a^\dagger)^{n_+} \langle \rho, \phi_0 | 0, 0 \rangle \\ &= (-)^{n_-} \frac{\omega^{\frac{1+n_+-n_-}{2}}}{\sqrt{\pi}} \sqrt{\frac{n_-!}{n_+!}} \rho^{n_+-n_-} e^{-\frac{1}{2}\omega\rho^2} L_{n_-}^{n_+-n_-}(\omega\rho^2) e^{i(n_+-n_-)\phi_0} \end{aligned} \quad (\text{A.5})$$

with $L_{n_-}^{n_+-n_-}(\omega\rho^2)$, the Laguerre polynomial, given by (2.65). In the case $n_+ < n_-$ we can apply $b^\dagger$ n_- times and then $a^\dagger$ n_+ times on $\langle \rho, \phi_0 | 0, 0 \rangle$. The result is

$$\langle \rho, \phi_0 | n_+, n_- \rangle = (-)^{n_+} \frac{\omega^{\frac{1+n_- - n_+}{2}}}{\sqrt{\pi}} \sqrt{\frac{n_+!}{n_-!}} \rho^{n_- - n_+} e^{-\frac{1}{2}\omega\rho^2} L_{n_+}^{n_- - n_+}(\omega\rho^2) e^{i(n_+ - n_-)\phi_0}. \quad (\text{A.6})$$

Introducing the azimuthal quantum number $m = n_+ - n_-$, as before, and

$$n = \frac{1}{2} (n_+ + n_- - |m|) = \begin{cases} n_- & n_+ \geq n_- \\ n_+ & n_+ \leq n_- \end{cases}, \quad (\text{A.7})$$

we find both (A.5) and (A.6) can be unified as

$$\langle \rho, \phi_0 | n_+ - n_- \rangle = (-)^n \frac{\omega^{\frac{1+|m|}{2}}}{\sqrt{\pi}} \sqrt{\frac{n!}{(n+|m|)!}} \rho^{|m|} e^{-\frac{1}{2}\omega\rho^2} L_n^{|m|}(\omega\rho^2) e^{im\phi_0}. \quad (\text{A.8})$$

which corresponds to (2.64) and (2.65).

A.2. The Integral (4.17)

Using (2.65), the integral (4.17) becomes

$$\begin{aligned} & (-)^n \frac{\omega^{1+\frac{|m|}{2}}}{\pi} \sqrt{\frac{n!}{(n+|m|)!}} \int_0^\infty \rho d\rho e^{-\omega\rho^2} \rho^{|m|} L_n^{|m|}(\omega\rho^2) \int_0^{2\pi} d\phi e^{im(\phi - f\rho \cos \phi)} \\ & = 2(-)^n i^{-m} \omega^{1+\frac{|m|}{2}} \sqrt{\frac{n!}{(n+|m|)!}} \int_0^\infty d\rho \rho^{|m|+1} e^{-\omega\rho^2} L_n^{|m|}(\omega\rho^2) J_{|m|}(|m|f\rho) \end{aligned} \quad (\text{A.9})$$

where $J_{|m|}(|m|f\rho)$ is the Bessel function. The integration over ρ can be carried out with the aid of the formulae

$$\int_0^\infty dx x^{\frac{\alpha}{2}} e^{-x} L_n^\alpha(x) J_\alpha(2\sqrt{ax}) = n! a^{n+\frac{\alpha}{2}} e^{-a}. \quad (\text{A.10})$$

We obtain

$$(4.17) = \frac{(-i)^{2n+|m|}}{\sqrt{n!(n+|m|)!}} e^{-\frac{m^2 f^2}{4\omega}} \left(\frac{m^2 f^2}{4\omega} \right)^{n+\frac{|m|}{2}} \quad (\text{A.11})$$

which is identical to (4.18) through the relation (A.7).

A.3. The Matrix Elements $\langle n'_+, n'_- | x | n_+, n_- \rangle$ and $\langle n'_+, n'_- | y | n_+, n_- \rangle$

Introduce

$$\Delta \equiv n_+ - n_- - n'_+ + n'_- \quad (\text{A.12})$$

we have

$$\begin{aligned} \langle n'_+, n'_- | x | n_+, n_- \rangle &= \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi (1 + f\rho \sin \phi) \\ &\quad \cdot R_{n'_+ n'_-}(\rho) R_{n_+ n_-}(\rho) e^{i(\phi - f\rho \cos \phi)\Delta} \rho \cos \phi \end{aligned} \quad (\text{A.13})$$

and

$$\begin{aligned} \langle n'_+, n'_- | y | n_+, n_- \rangle &= \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi (1 + f\rho \sin \phi) \\ &\quad \cdot R_{n'_+ n'_-}(\rho) R_{n_+ n_-}(\rho) e^{i(\phi - f\rho \cos \phi)\Delta} \rho \sin \phi. \end{aligned} \quad (\text{A.14})$$

For $\Delta \neq 0$, a partial integration over ϕ gives rise to

$$\langle n'_+, n'_- | x | n_+, n_- \rangle = \frac{i}{f\Delta} U(f), \quad (\text{A.15})$$

$$\langle n'_+, n'_- | y | n_+, n_- \rangle = \frac{-1}{\Delta^2} \frac{d}{df} U(f), \quad (\text{A.16})$$

where

$$\begin{aligned} U(f) &= \int_0^\infty \rho d\rho \int_0^{2\pi} d\phi R_{n'_+ n'_-}(\rho) R_{n_+ n_-}(\rho) e^{i(\phi - f\rho \cos \phi)\Delta} \\ &= \langle n'_+, n'_- | e^{-ifx_0\Delta} | n_+, n_- \rangle \end{aligned} \quad (\text{A.17})$$

with x_0 given by (2.51). Using (2.52) and (2.53), we find

$$\begin{aligned} U(f) &= \frac{1}{|n_+ - n'_+|! |n_- - n'_-|!} \sqrt{\frac{n_+^>! n_-^>!}{n_+^<! n_-^<!}} \left(-i \frac{f\Delta}{2\sqrt{\omega}} \right)^{|n_+ - n'_+| + |n_- - n'_-|} \\ &\quad \cdot {}_1F_1 \left(n_+^> + 1, |n_+ - n'_+| + 1; -\frac{f^2\Delta^2}{4\omega} \right) \\ &\quad \cdot {}_1F_1 \left(n_-^> + 1, |n_- - n'_-| + 1; -\frac{f^2\Delta^2}{4\omega} \right) e^{\frac{f^2\Delta^2}{4\omega}} \end{aligned} \quad (\text{A.18})$$

where

$${}_1F_1(a, b; z) \equiv 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \dots \quad (\text{A.19})$$

is the confluent hypergeometric function, $n_+^> = \max(n_+, n'_+)$, $n_+^< = \min(n_+, n'_+)$, $n_-^> = \max(n_-, n'_-)$ and $n_-^< = \min(n_-, n'_-)$.

For $\Delta = 0$, the matrix elements may be evaluated directly. We obtain

$$\langle n'_+, n'_- | x | n_+, n_- \rangle = 0 \quad (\text{A.20})$$

and

$$\begin{aligned} \langle n'_+, n'_- | y | n_+, n_- \rangle = \frac{f}{2\omega} & \left[(n_+ + n_- + 1) \delta_{n'_+, n_+} \delta_{n'_-, n_-} + \sqrt{n_+ n_-} \delta_{n'_+, n_+ - 1} \delta_{n'_-, n_- - 1} \right. \\ & \left. + \sqrt{(n_+ + 1)(n_- + 1)} \delta_{n'_+, n_+ + 1} \delta_{n'_-, n_- + 1} \right]. \end{aligned} \quad (\text{A.21})$$

APPENDIX B

Equation (3.12) gives the dependence of ϕ_0 on ϕ :

$$\phi_0(\phi) = \phi - \sigma \cos \phi, \quad (\text{B.1})$$

where $\sigma = \lambda g \rho$. In this appendix, we shall derive an explicit expression for the inverse function $\phi(\phi_0)$. Define

$$\chi(\phi_0) = \phi(\phi_0) - \phi_0. \quad (\text{B.2})$$

In terms of χ , (B.1) reads

$$\chi = \sigma \cos(\chi + \phi_0), \quad (\text{B.3})$$

which is invariant under the translation $\phi_0 \rightarrow \phi_0 + 2\pi$. Therefore, $\chi(\phi_0)$ can be expanded into a Fourier series in ϕ_0 ,

$$\chi = \sum_n C_n e^{in\phi_0} \quad (\text{B.4})$$

with

$$C_n = \int_0^{2\pi} \frac{d\phi_0}{2\pi} \chi(\phi_0) e^{-in\phi_0}. \quad (\text{B.5})$$

We discuss separately the two cases: $|\sigma| < 1$ and $|\sigma| > 1$.

B.1. $|\sigma| < 1$

In this case, we are entirely in the region labeled 1 in Figure 4; the mapping from ϕ_0 to ϕ is one-to-one for all ϕ_0 . The function $\chi(\phi_0)$ is single-valued and the integration over ϕ_0 from 0 to 2π in (B.5) can be converted into an integration over ϕ from ϕ' to $2\pi + \phi'$ via (B.1) with ϕ' the solution of (B.1) at $\phi_0 = 0$. According to (B.1)-(B.3), the integrand including the Jacobian of the transformation, written in terms of ϕ , is a periodic function of ϕ and therefore we may integrate from $\phi = 0$ to $\phi = 2\pi$ as well, i.e.

$$C_n = \int_0^{2\pi} \frac{d\phi}{2\pi} (1 + \sigma \sin \phi) \sigma \cos \phi e^{-in(\phi - \sigma \cos \phi)}. \quad (\text{B.6})$$

It follows that

$$C_0 = 0 \quad (\text{B.7})$$

and for $n \neq 0$, after a partial integration,

$$C_n = -i \frac{1}{n} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-in(\phi - \sigma \cos \phi)} = -\frac{i^{n+1}}{n} J_n(n\sigma) \quad (\text{B.8})$$

where $J_n(z)$ is the usual Bessel function. Combining (B.2), (B.4), (B.7) and (B.8) we obtain

$$\phi = \phi_0 - i \sum_n' \frac{i^n}{n} J_n(n\sigma) e^{in\phi_0}, \quad (\text{B.9})$$

where the sum $\sum_n'$ extends over all integers except $n = 0$. Using the formula [18]

$$J_n(n \operatorname{sech} \eta) \sim \frac{e^{-|n|(\eta - \tanh \eta)}}{\sqrt{2\pi|n|\tanh \eta}} \quad (\text{B.10})$$

for large n and $\eta > 0$, we find that the Fourier series (B.9) and its derivatives with respect to ϕ_0 of an arbitrary order converge. Consequently, $\phi(\phi_0)$ is a smooth periodic function of ϕ_0 without singularities.

B.2. $|\sigma| > 1$

In this case, the mapping $\phi_0 \rightarrow \phi$ is no longer unique, at least for certain values of ϕ_0 within $0 \leq \phi_0 < 2\pi$. A number of χ -functions, each being single-valued, have to be introduced to cover the entire multiplet and there is no unique way of doing this. However, a relatively simple expression can be obtained for the solutions within the region Λ of the (x, y) plane, defined by (3.22).

Take the example shown in Figure 3(a) in which there are three copies of ϕ within $\phi_0^I \leq \phi_0 < \phi_0^{II}$. Denote I and II to be the two points on the curve where the derivative,

$$\frac{d\phi_0}{d\phi} = 1 + \sigma \sin \phi \quad (\text{B.11})$$

which differs from the Jacobian (3.5) by a factor g , vanishes. Note that when $\sigma = 1$, $d\phi_0/d\phi = 0$ at $\phi_0 = \phi = \frac{3\pi}{2}$. In Figure 3(a), $\sigma > 1$; we see that the line $\phi_0 = \frac{3\pi}{2}$ cuts the curve consecutively at c' , c , and c'' with $\phi = \frac{3\pi}{2}$ at c . The function $\phi_0(\phi) - \frac{3\pi}{2}$ is odd with respect to the variable $\phi - \frac{3\pi}{2}$ and therefore the area bounded by the straight line $\overline{c'c}$ and the curve $c'IIc$ is equal to the area bounded by $\overline{cc''}$ and the curve cIC'' . According to the minimum principle stated in Section 3, the solution inside Λ corresponds to the one with the largest

Jacobian (and therefore the largest $d\phi_0 / d\phi$) among the members of the multiplet. It follows then that the portion from a' to c' and that from c'' to b' of the curve corresponds to the solution inside Λ .

The above equal area rule is analogous to the Maxwell construction in thermodynamics for a first-order phase transition. Let the function (3.10) for a fixed ρ be

$$\mathcal{I} = \frac{1}{2g^2} [1 + \sigma^2 - (1 + \sigma \sin \phi)^2], \quad (\text{B.12})$$

which is the analog of the Gibbs function with ϕ_0 as the pressure since the ϕ -solution inside Λ corresponds to the one with minimum $\mathcal{I}$. It follows from (B.1) and (B.11) that

$$\frac{d\mathcal{I}}{d\phi_0} = \frac{\phi_0 - \phi}{g^2}. \quad (\text{B.13})$$

When we compare the Gibbs functions $\mathcal{I}$ of different ϕ -solutions for the same ϕ_0 , we need only examine the corresponding area $\int \phi d\phi_0$; this then leads to the equal area rule.

The general situation with an arbitrary number of multiplicity has been analyzed in Section 3. The solutions of (B.1) are the polar angles of the intersections between the curve (3.16) and the circle (3.18). For a given ϕ_0 , according to (3.22), $\Lambda = \{q_1(x_0, y_0)\}$ where q_1 refers to the (one and only one) intersection that lies on the convex branch of (3.16), as shown in Figure 5. When ϕ_0 changes from $-\pi/2$ to $3\pi/2$, q_1 moves continuously from Q to Q' along the circle (3.18), where Q and Q' are the intersections between the boundary of Λ and the circle. For this profile of ϕ versus ϕ_0 , we have

$$C_n = \int_{\Phi}^{\Phi'} \frac{d\phi}{2\pi} (1 + \sigma \sin \phi) \sigma \cos \phi e^{-in(\phi - \sigma \cos \phi)}, \quad (\text{B.14})$$

with Φ and Φ' the polar angles of the points Q and Q' . Because the boundary of Λ is mirror symmetric with respect to the y -axis, we have $\Phi + \Phi' = \pi$. Introducing an angle β ($0 \leq \beta < \pi$) such that

$$\Phi = -\frac{\pi}{2} + \beta \quad (\text{B.15})$$

and

$$\Phi' = \frac{3\pi}{2} - \beta, \quad (\text{B.16})$$

we find

$$C_0 = 0 \quad (\text{B.17})$$

and

$$C_n = -\frac{i^{n+1}}{n} \left(\frac{\sigma}{\pi} \sin \beta + \int_{\beta}^{2\pi-\beta} \frac{d\theta}{2\pi} e^{-in(\theta-\sigma \sin \theta)} \right) \quad (\text{B.18})$$

for $n \neq 0$. The solution inside Λ for arbitrary ϕ_0 is obtained by substituting (B.17) and (B.18) into (B.4). The angle β satisfies

$$\frac{\sin \beta}{\beta} = \frac{1}{\sigma}, \quad (\text{B.19})$$

which is derived from (3.18) and (3.23).

APPENDIX C

In this appendix, we shall evaluate to the one-loop order the propagator $\mathcal{D}_{jj}(t)$ ($j = (x, y)$), which is the expectation value $x_j(t) x_j(0)$ over the path integration

$$\prod_{n=1}^{\mathcal{N}} e^{i\epsilon \mathcal{L}(n)} dx_n dy_n d\xi_n \quad (\text{C.1})$$

with $\mathcal{L}(n)$ given by (6.1). The result will be compared with the exact expression (4.5) and (4.6).

The relevant one-loop diagrams for the propagator $\mathcal{D}_{xx}$ and $\mathcal{D}_{yy}$ are listed in Figure 10. Each solid line represents the free propagator $\mathcal{D}_{\mu\nu}$ given by (6.6)-(6.9) and each dashed line stands for the ghost propagator given by (6.13).

Let a_j, b_j, c_j, d_j, e_j denote the contribution to the propagator $\mathcal{D}_{jj}$ ($j = x$ or y), from the diagrams (a), (b), (c), (d) and (e). We find, in accordance with (6.13)

$$a_x = a_x^{(1)} + a_x^{(2)} + a_x^{(3)} \quad (\text{C.2})$$

where

$$\begin{aligned} a_x^{(1)} = & \int_{-\infty}^{\infty} dt e^{ik_0 t} (-ig)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 [-\dot{D}_{11}(t-t_2) D_{00}(t_2-t_1) D_{22}(t_2-t_1) \dot{D}_{11}(t_1) \\ & - \dot{D}_{11}(t-t_2) D_{00}(t_2-t_1) \dot{D}_{22}(t_2-t_1) D_{11}(t_1) \\ & - D_{11}(t-t_2) D_{00}(t_2-t_1) \dot{D}_{22}(t_2-t_1) \dot{D}_{11}(t_1) \\ & - D_{11}(t-t_2) D_{00}(t_2-t_1) \ddot{D}_{22}(t_2-t_1) D_{11}(t_1)], \end{aligned} \quad (\text{C.3})$$

$$\begin{aligned} a_x^{(2)} = & \int_{-\infty}^{\infty} dt e^{ik_0 t} (-ig)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 [-D_{10}(t-t_2) \ddot{D}_{11}(t_2-t_1) D_{22}(t_2-t_1) D_{01}(t_1) \\ & + 2D_{10}(t-t_2) \dot{D}_{11}(t_2-t_1) \dot{D}_{22}(t_2-t_1) D_{01}(t_1) \\ & - D_{10}(t-t_2) D_{11}(t_2-t_1) \ddot{D}_{22}(t_2-t_1) D_{01}(t_1)], \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned}
a_x^{(3)} = & \int_{-\infty}^{\infty} dt e^{ik_0 t} (-ig)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 [D_{10}(t-t_2) \dot{D}_{10}(t_2-t_1) D_{22}(t_2-t_1) \dot{D}_{11}(t_1) \\
& + \dot{D}_{11}(t-t_2) \dot{D}_{01}(t_2-t_1) D_{22}(t_2-t_1) D_{01}(t_1) \\
& + D_{10}(t-t_2) \dot{D}_{10}(t_2-t_1) \dot{D}_{22}(t_2-t_1) D_{11}(t_1) \\
& + D_{11}(t-t_2) \dot{D}_{01}(t_2-t_1) \dot{D}_{22}(t_2-t_1) D_{01}(t_1) \\
& - D_{10}(t-t_2) D_{10}(t_2-t_1) \dot{D}_{22}(t_2-t_1) \dot{D}_{11}(t_1) \\
& - \dot{D}_{11}(t-t_2) D_{01}(t_2-t_1) \dot{D}_{22}(t_2-t_1) D_{01}(t_1) \\
& - D_{10}(t-t_2) D_{10}(t_2-t_1) \ddot{D}_{22}(t_2-t_1) D_{11}(t_1) \\
& - D_{11}(t-t_2) D_{01}(t_2-t_1) \ddot{D}_{22}(t_2-t_1) D_{01}(t_1)] .
\end{aligned} \tag{C.5}$$

Likewise

$$b_x = b_x^{(1)} + b_x^{(2)} + b_x^{(3)} \tag{C.6}$$

where

$$b_x^{(1)} = \int_{-\infty}^{\infty} dt e^{ik_0 t} ig^2 \int_{-\infty}^{\infty} dt_1 D_{11}(t-t_1) D_{00}(0) D_{11}(t_1), \tag{C.7}$$

$$b_x^{(2)} = \int_{-\infty}^{\infty} dt e^{ik_0 t} ig^2 \int_{-\infty}^{\infty} dt_1 D_{10}(t-t_1) [D_{11}(0) + D_{22}(0)] D_{01}(t_1), \tag{C.8}$$

$$b_x^{(3)} = \int_{-\infty}^{\infty} dt e^{ik_0 t} ig^2 \int_{-\infty}^{\infty} dt_1 [D_{10}(t-t_1) D_{01}(0) D_{11}(t_1) + D_{11}(t-t_1) D_{10}(0) D_{01}(t_1)] \tag{C.9}$$

and

$$c_x = 0, \tag{C.10}$$

$$d_x = \int_{-\infty}^{\infty} dt e^{ik_0 t} (-ig)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 [D_{10}(t-t_1) \dot{D}_{11}(t_1) - \dot{D}_{11}(t-t_1) D_{01}(t_1)] D_{22}(t_2-t_1) \dot{D}_{01}(0), \tag{C.11}$$

$$e_x = \int_{-\infty}^{\infty} dt e^{ik_0 t} (-ig)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 [D_{10}(t-t_1) \dot{D}_{11}(t_1) - \dot{D}_{11}(t-t_1) D_{01}(t_1)] D_{22}(t_2-t_1) i\delta(0), \tag{C.12}$$

where the overall minus signs are due to the ghost loop. Using the momentum representation of $D_{\mu\nu}(t-t')$ and carrying out all integrations over t_1 , t_2 , and t , we have

$$a_x^{(1)} + b_x^{(1)} = \frac{i}{k_0^2 - \omega^2 + i0^+} [-i \Pi_x(k_0)] \frac{i}{k_0^2 - \omega^2 + i0^+} \quad (\text{C.13})$$

where

$$\Pi_x(k_0) = ig^2 \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \left\{ \frac{(p_0 + 2k_0)^2 [(1 + \lambda^2)p_0^2 - \omega^2]}{(p_0^2 - \omega^2 + i0^+)[(p_0 + k_0)^2 - \omega^2 + i0^+]} - \frac{(1 + \lambda^2)p_0^2 - \omega^2}{p_0^2 - \omega^2 + i0^+} \right\}. \quad (\text{C.14})$$

The integrand of (C.14) approaches $2k_0/p_0$ as $p_0 \rightarrow \pm\infty$ and the limit $\pm\infty$ of the integration over p_0 is taken symmetrically, i.e.,

$$\int_{-\infty}^{\infty} dp_0 \equiv \lim_{K \rightarrow \infty} \int_{-K}^K dp_0. \quad (\text{C.15})$$

Therefore, when we replace the integrand in (C.14) by an integration over a large semicircle with its diameter extending from $-K$ to K ($K \rightarrow \infty$) and its arc on the upper or lower half p_0 -plane, in addition to picking up the residue of the pole inside the contour, we have also to subtract the finite contribution from the arc. With this precaution, we obtain

$$\Pi_x(k_0) = g^2 \left[\frac{k_0^2 + \omega^2}{2\omega} + \frac{\lambda^2 (k_0^2 - \omega^2)(k_0^2 + 2\omega^2)}{2\omega (k_0^2 - 4\omega^2)} \right]. \quad (\text{C.16})$$

Similarly, we can show that

$$a_x^{(2)} + b_x^{(2)} = 0 \quad (\text{C.17})$$

and

$$a_x^{(3)} + b_x^{(3)} = \frac{i}{k_0^2 - \omega^2 + i0^+} \left[-i \frac{\lambda^2 g^2}{\omega} k_0^2 \right] \frac{i}{k_0^2 - \omega^2 + i0^+}. \quad (\text{C.18})$$

The contribution from the tadpole diagram, $d_x + e_x$, cancels (C.18) exactly and the final expression for $\mathcal{D}_{xx}$ reads

$$\mathcal{D}_{xx} = \int_{-\infty}^{\infty} \frac{ie^{-k_0 t}}{k_0^2 - \omega^2 + i0^+} \left[1 + \frac{\Pi_x(k_0)}{k_0^2 - \omega^2 + i0^+} + O(g^4) \right]. \quad (\text{C.19})$$

In the case of $\mathcal{D}_{yy}(t)$, only the diagram (a), (b), (c) contributes, where

$$\begin{aligned}
a_y = \int_{-\infty}^{\infty} dt e^{ik_0 t} (-ig)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 [& -D_{22}(t-t_2) \ddot{D}_{11}(t_2-t_1) D_{00}(t_2-t_1) D_{22}(t_1) \\
& - \dot{D}_{22}(t-t_2) \dot{D}_{11}(t_2-t_1) D_{00}(t_2-t_1) \dot{D}_{22}(t_1) \\
& - D_{22}(t-t_2) \dot{D}_{11}(t_2-t_1) D_{00}(t_2-t_1) \dot{D}_{22}(t_1) \\
& - \dot{D}_{22}(t-t_2) \dot{D}_{11}(t_2-t_1) D_{00}(t_2-t_1) D_{22}(t_1) \\
& - D_{22}(t-t_2) \dot{D}_{10}(t_2-t_1) \dot{D}_{01}(t_2-t_1) D_{22}(t_1) \\
& - \dot{D}_{22}(t-t_2) D_{10}(t_2-t_1) D_{01}(t_2-t_1) \dot{D}_{22}(t_1) \\
& - D_{22}(t-t_2) D_{10}(t_2-t_1) \dot{D}_{01}(t_2-t_1) \dot{D}_{22}(t_1) \\
& - \dot{D}_{22}(t-t_2) \dot{D}_{10}(t_2-t_1) D_{01}(t_2-t_1) D_{22}(t_1)] , \tag{C.20}
\end{aligned}$$

$$b_y = \int_{-\infty}^{\infty} dt e^{ik_0 t} ig^2 \int_{-\infty}^{\infty} dt_1 D_{22}(t-t_1) D_{22}(t_1) D_{00}(0) , \tag{C.21}$$

$$c_y = - \int_{-\infty}^{\infty} dt e^{ik_0 t} (i\lambda g)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1 D_{22}(t-t_2) [i\delta(t_2-t_1)]^2 D_{22}(t_1) . \tag{C.22}$$

We find

$$a_y + b_y + c_y = \frac{i}{k_0^2 - \omega^2 + i0^+} [-i\Pi_y(k_0)] \frac{i}{k_0^2 - \omega^2 + i0^+} \tag{C.23}$$

with

$$\Pi_y(k_0) = g^2 \left[\frac{k_0^2 + \omega^2}{2\omega} + \frac{\lambda^2}{2\omega} \frac{(k_0^2 - \omega^2)(k_0^2 + 8\omega^2)}{(k_0^2 - 4\omega^2)} \right] \tag{C.24}$$

and therefore

$$\mathcal{D}_{yy} = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{ie^{-ik_0 t}}{k_0^2 - \omega^2 + i0^+} \left[1 + \frac{\Pi_y(k_0)}{k_0^2 - \omega^2 + i0^+} + O(g^4) \right] . \tag{C.25}$$

Both (C.19) and (C.25), with $\Pi_j(k_0)$ given by (C.16) and (C.24), are found to agree with the exact expressions for $\mathcal{D}_{xx}(k_0)$ and $\mathcal{D}_{yy}(k_0)$, derived via Hamiltonian formulation, to the order of g^2 , as is expected.

APPENDIX D

D.1. Two-loop Diagrams for E_{gd} in the λ -Gauge

To evaluate the seven two-loop diagrams (a), $\dots$, (g) listed in Figure 9, we begin with the path-integration formulation (5.48)-(5.50), but keep ϵ nonzero. These diagrams will be grouped in accordance with (6.27)-(6.28), with the limit $\epsilon \rightarrow 0$ taken in the end. For $U = \frac{1}{2} \omega^2 (x^2 + y^2)$, $L'(n)$ of (5.49) can be written as the following power series in g :

$$L'(n) = L_0(n) + g L_1(n) + g^2 L_2(n) + \frac{g^2}{8} (2 + 3\lambda^2) + O(g^3) \quad (\text{D.1})$$

where

$$L_0(n) = \frac{1}{2} (\dot{x}_n^2 + \dot{y}_n^2) - \frac{\omega^2}{2} (\bar{x}_n^2 + \bar{y}_n^2), \quad (\text{D.2})$$

$$L_1(n) = \lambda (\bar{y}_n \dot{x}_n^2 - \bar{x}_n \dot{x}_n \dot{y}_n) - \frac{i}{\epsilon} \lambda \bar{y}_n, \quad (\text{D.3})$$

$$\begin{aligned} L_2(n) = \frac{1}{2} \left\{ \dot{x}_n^2 [\lambda^2 \bar{x}_n^2 + (\lambda^2 - 1) \bar{y}_n^2] + 2 \bar{x}_n \bar{y}_n \dot{x}_n \dot{y}_n - \bar{x}_n^2 \dot{y}_n^2 \right\} \\ + \frac{i}{2\epsilon} (\bar{x}_n^2 + \bar{y}_n^2) + \frac{i}{2\epsilon} \lambda^2 \bar{y}_n^2, \end{aligned} \quad (\text{D.4})$$

with $\bar{x}_n, \bar{y}_n, \dot{x}_n, \dot{y}_n$ given by (5.15) and the last term $g^2(2 + 3\lambda^2)/8$ is from $-V$, given by (5.4). In (D.4), the last term $i \lambda^2 \bar{y}_n^2 / 2\epsilon$ is due to the $\ell = 2$ term in the expansion (6.17); i.e.,

$$-i \epsilon^{-1} \ln \mathcal{J}' = -i \sum_{\ell=1}^{\infty} \epsilon^{-1} (-\ell^{-1}) G^\ell. \quad (\text{D.5})$$

The other terms in (D.3) and (D.4) that have an explicit factor i/ϵ are all from the $\ell = 1$ term in (D.5), as can be verified by using (6.23). Neglecting $O(g^3)$, we have

$$\begin{aligned} \prod_n e^{i\epsilon L'(n)} = \left\{ 1 + i\epsilon \sum_n [g L_1(n) + g^2 L_2(n) + \frac{g^2}{8} (2 + 3\lambda^2)] \right. \\ \left. + \frac{1}{2} (i\epsilon)^2 g^2 \sum_{n,m} L_1(n) L_1(n+m) \right\} \prod_n e^{i\epsilon L_0(n)}. \end{aligned} \quad (\text{D.6})$$

The shift in groundstate energy is given by the limit $N \rightarrow \infty$ first and then $\epsilon \rightarrow 0$ of

$$E_{\text{gd}} = i \frac{1}{N\epsilon} \ln \frac{\int \chi_0^*(x_{N+1}, y_{N+1}) \prod_{n=1}^N dx_n dy_n e^{i\epsilon L'(n)} \chi_0(x_1, y_1) dx_{N+1} dy_{N+1}}{\int \chi_0^*(x_{N+1}, y_{N+1}) \prod_{n=1}^N dx_n dy_n e^{i\epsilon L_0(n)} \chi_0(x_1, y_1) dx_{N+1} dy_{N+1}} \quad (\text{D.7})$$

where

$$\chi_0(x, y) = \sqrt{\frac{\omega}{\pi}} e^{-\frac{\omega}{2}(x^2+y^2)} \quad (\text{D.8})$$

is the normalized groundstate wave function of a two-dimensional harmonic oscillator.

For any function F of x_m, y_m we define

$$\langle F \rangle \equiv \lim_{N \rightarrow \infty} \frac{\int \chi_0^*(x_{N+1}, y_{N+1}) \prod_{n=1}^N dx_n dy_n e^{i\epsilon L_0(n)} \chi_0(x_1, y_1) F dx_{N+1} dy_{N+1}}{\int \chi_0^*(x_{N+1}, y_{N+1}) \prod_{n=1}^N dx_n dy_n e^{i\epsilon L_0(n)} \chi_0(x_1, y_1) dx_{N+1} dy_{N+1}}. \quad (\text{D.9})$$

Keeping $\epsilon \neq 0$ but taking the limit $N \rightarrow \infty$, we find, neglecting $O(g^3)$, (D.7) to be of the form (6.25):

$$E_{\text{gd}}(g^2) = a + b + c + d + e + f + g - \frac{1}{8} g^2(2 + 3\lambda^2). \quad (\text{D.10})$$

By substituting (D.3)-(D.4) into (D.6)-(D.7) and using $\langle L_1(n) \rangle = 0$, we find

$$\begin{aligned} a = -\frac{i\epsilon}{2} \lambda^2 g^2 \sum_m & \left[2 \langle \dot{x}_n \dot{x}_{n+m} \rangle (\langle \dot{x}_n \dot{x}_{n+m} \rangle \langle \bar{y}_n \bar{y}_{n+m} \rangle \right. \\ & - 2 \langle \dot{x}_n \bar{x}_{n+m} \rangle \langle \bar{y}_n \dot{y}_{n+m} \rangle) \\ & + \langle \dot{y}_n \dot{y}_{n+m} \rangle (\langle \bar{x}_n \bar{x}_{n+m} \rangle \langle \dot{x}_n \dot{x}_{n+m} \rangle \\ & \left. + \langle \bar{x}_n \dot{x}_{n+m} \rangle \langle \dot{x}_n \bar{x}_{n+m} \rangle) \right], \end{aligned} \quad (\text{D.11})$$

$$b = -\frac{1}{2} \lambda^2 g^2 \left(\frac{i}{\epsilon} \right) \langle \bar{y}_n \bar{y}_n \rangle, \quad (\text{D.12})$$

$$\begin{aligned} c = -\frac{1}{2} g^2 & \left\{ -\langle \dot{x}_n \dot{x}_n \rangle \langle \bar{y}_n \bar{y}_n \rangle - \langle \bar{x}_n \bar{x}_n \rangle \langle \dot{y}_n \dot{y}_n \rangle + 2 \langle \bar{x}_n \dot{x}_n \rangle \langle \bar{y}_n \dot{y}_n \rangle \right. \\ & \left. + \lambda^2 \left[\langle \dot{x}_n \dot{x}_n \rangle (\langle \bar{x}_n \bar{x}_n \rangle + \langle \bar{y}_n \bar{y}_n \rangle) + 2 \langle \dot{x}_n \bar{x}_n \rangle^2 \right] \right\}, \end{aligned} \quad (\text{D.13})$$

$$d = -\frac{1}{2} g^2 \left(\frac{i}{\epsilon} \right) (\langle \bar{x}_n \bar{x}_n \rangle + \langle \bar{y}_n \bar{y}_n \rangle), \quad (D.14)$$

$$e = -\frac{i\epsilon}{2} \lambda^2 g^2 \sum_m (\langle \bar{y}_n \bar{y}_{n+m} \rangle \langle \dot{x}_n \dot{x}_n \rangle \langle \dot{x}_{n+m} \dot{x}_{n+m} \rangle + \langle \dot{y}_n \dot{y}_{n+m} \rangle \langle \bar{x}_n \dot{x}_n \rangle \langle \bar{x}_{n+m} \dot{x}_{n+m} \rangle), \quad (D.15)$$

$$f = -\frac{i\epsilon}{2} \lambda^2 g^2 \sum_m \left(\frac{2i}{\epsilon} \right) (- \langle \bar{y}_n \bar{y}_{n+m} \rangle \langle \dot{x}_n \dot{x}_n \rangle + \langle \bar{y}_n \dot{y}_{n+m} \rangle \langle \bar{x}_{n+m} \dot{x}_{n+m} \rangle) \quad (D.16)$$

and

$$g = -\frac{i\epsilon}{2} \lambda^2 g^2 \sum_m \left(\frac{i}{\epsilon} \right)^2 \langle \bar{y}_n \bar{y}_{n+m} \rangle, \quad (D.17)$$

in which

$$b + c + d = -g^2 \langle L_2(n) \rangle \quad (D.18)$$

and

$$a + e + f + g = -\frac{i\epsilon}{2} g^2 \sum_m \langle L_1(n) L_1(n+m) \rangle. \quad (D.19)$$

Expression (D.12) for b is derived by using only the last term $i \lambda^2 \bar{y}_n^2 / 2\epsilon$ in (D.4) for $\langle L_2(n) \rangle$; as noted before, this is in turn from the $\ell = 2$ term in the expansion (D.5), and therefore corresponds to diagram (b) in Figure 9. Likewise, (D.14) is obtained by using only the second-to-last term $i(\bar{x}_n^2 + \bar{y}_n^2) / 2\epsilon$ in (D.4) for $\langle L_2(n) \rangle$, which corresponds to diagram (d). The remaining term in $\langle L_2(n) \rangle$ gives diagram (c). The last term $-i \lambda \bar{y}_n / \epsilon$ in (D.3) for $\langle L_1(n) \rangle$ corresponds to one ghost loop. Substituting (D.3) into $\langle L_1(n) L_1(n+m) \rangle$, one can readily identify (D.11) and (D.15)-(D.17) to be the contributions of diagrams (a), (e), (f) and (g) in Figure 9.

To evaluate (D.11)-(D.17), we introduce

$$\begin{aligned} D_m &\equiv \langle x_n x_{n+m} \rangle, & \bar{D}_m &\equiv \langle \bar{x}_n \bar{x}_{n+m} \rangle, \\ \bar{D}'_m &\equiv \langle \dot{x}_n \bar{x}_{n+m} \rangle & \text{and} & \bar{D}''_m \equiv - \langle \dot{x}_n \dot{x}_{n+m} \rangle. \end{aligned} \quad (D.20)$$

The same expressions apply if all x_n are replaced by y_n . Because of the overall time-translational invariance, $D_m, \bar{D}_m, \bar{D}'_m$ and $\bar{D}''_m$ are independent of n . Since $D_m = D_{-m}$ and $\bar{D}_m = \frac{1}{4} \langle (x_n + x_{n+1})(x_{n+m} + x_{n+m+1}) \rangle = \frac{1}{4} (2D_m + D_{m+1} + D_{m-1})$, we have

$$\bar{D}_m = \bar{D}_{-m}. \quad (D.21)$$

Also, from $\overline{D}'_m = \frac{1}{2\epsilon} \langle (x_{n+1} - x_n)(x_{n+m} + x_{n+m+1}) \rangle = \frac{1}{2\epsilon} (D_{m-1} - D_{m+1})$, it follows that

$$\overline{D}'_m = - \langle \dot{x}_{n+m} \overline{x}_n \rangle = -\overline{D}'_{-m}. \quad (\text{D.22})$$

Likewise,

$$\overline{D}''_m = -\frac{1}{\epsilon^2} (2 D_m - D_{m+1} - D_{m-1}) = \overline{D}''_{-m}. \quad (\text{D.23})$$

When $m = 0$, $\dot{x}_n \overline{x}_n = (x_{n+1}^2 - x_n^2) / 2\epsilon$, and therefore

$$\overline{D}'_0 = 0. \quad (\text{D.24})$$

For $F = \overline{x}_n^2$, or $\dot{x}_n^2$ and neglecting $O(\epsilon)$, (D.9) reduces to

$$\langle F \rangle = \left(\frac{1}{2i\pi\epsilon} \right)^{\frac{1}{2}} \int \chi_0^*(x_{n+1}, y_{n+1}) F e^{i\epsilon L_0(n)} \chi_0(x_n, y_n) dx_n dy_n dx_{n+1} dy_{n+1} \quad (\text{D.25})$$

which leads to

$$\overline{D}_0 = \frac{1}{2\omega} + O(\epsilon) \quad (\text{D.26})$$

and

$$\overline{D}''_0 = -\frac{i}{\epsilon} - \frac{\omega}{2} + O(\epsilon). \quad (\text{D.27})$$

Introduce

$$\overline{D}''_m \equiv \overline{D}''_m + \frac{i}{\epsilon} \delta_{m0}. \quad (\text{D.28})$$

In the limit $\epsilon \rightarrow 0$, $\overline{D}''_m$ is finite. Keeping $m\epsilon = t$ fixed, we have the well-known expressions

$$\mathcal{D}(t) \equiv \lim_{\epsilon \rightarrow 0} \overline{D}_m = \lim_{\epsilon \rightarrow 0} D_m = \frac{1}{2\omega} e^{-i\omega|t|}, \quad (\text{D.29})$$

$$\mathcal{D}'(t) \equiv \lim_{\epsilon \rightarrow 0} \overline{D}'_m = \mathcal{D}(t) \times \begin{cases} -i\omega, & t > 0 \\ 0, & t = 0, \\ i\omega, & t < 0. \end{cases} \quad (\text{D.30})$$

and

$$\mathcal{D}''(t) \equiv \lim_{\epsilon \rightarrow 0} \overline{D}''_m = -\omega^2 e^{-i\omega|t|}, \quad (\text{D.31})$$

in which $\mathcal{D}'(t)$ and $\mathcal{D}''(t)$ are the Dyson-Wick propagators. Note that $\mathcal{D}''(t)$ is different from the second time-derivative $\ddot{\mathcal{D}}(t)$ of the Feynman propagator $\mathcal{D}(t)$. (The function $\mathcal{D}(t)$ is identical to $D_{xx}(t)$, or $D_{yy}(t)$, of (6.14).)

D.2. Evaluation of Diagrams (a)-(g) in Figure 9.

Substituting (D.20) and (D.22) into (D.11)-(D.14) and keeping $\epsilon \neq 0$, we derive

$$a = -\frac{i\epsilon}{2} \lambda^2 g^2 \sum_m 3 \overline{D}_m'' [\overline{D}_m'' \overline{D}_m - (\overline{D}_m')^2], \quad (D.32)$$

$$b = -\frac{i}{2\epsilon} \lambda^2 g^2 \overline{D}_0, \quad d = -\frac{i}{\epsilon} g^2 \overline{D}_0 \quad (D.33)$$

and, on account of $\overline{D}_0' = 0$,

$$c = -g^2(1 - \lambda^2) \overline{D}_0'' \overline{D}_0. \quad (D.34)$$

Hence,

$$b + c + d = -g^2 \overline{D}_0 \left[\frac{i}{\epsilon} + \overline{D}_0'' + \lambda^2 \left(\frac{i}{2\epsilon} - \overline{D}_0'' \right) \right].$$

Using (D.26)-(D.27), we have

$$b + c + d = \frac{g^2}{2\omega} \left[\frac{\omega}{2} - \lambda^2 \left(\frac{3i}{2\epsilon} + \frac{\omega}{2} \right) \right] + O(\epsilon). \quad (D.35)$$

The substitution of (D.24) and (D.28) into (D.32) gives

$$a = -\frac{3i}{2} \epsilon \lambda^2 g^2 \left\{ \left[\frac{i}{\epsilon} + \frac{\omega}{2} + O(\epsilon) \right]^2 \frac{1}{2\omega} + \sum_{m \neq 0} \overline{D}_m'' [\overline{D}_m'' \overline{D}_m - (\overline{D}_m')^2] \right\}. \quad (D.36)$$

Thus,

$$a + b + c + d = \frac{g^2}{2} \left(\frac{1}{2} + \lambda^2 \right) + O(\epsilon) - \frac{3i}{2} \epsilon \lambda^2 g^2 \sum_{m \neq 0} \overline{D}_m'' [\overline{D}_m'' \overline{D}_m - (\overline{D}_m')^2]. \quad (D.37)$$

Taking the limit $\epsilon \rightarrow 0$ and using (D.29)-(D.31), we find the last term in (D.37) to be 0 and

$$a + b + c + d = g^2 \left(\frac{1}{4} + \frac{\lambda^2}{2} \right), \quad (D.38)$$

which is (6.27).

In a similar way, on account of $\overline{D}_0' = 0$, we have

$$e + f + g = -\frac{i\epsilon}{2} \lambda^2 g^2 \left(\overline{D}_0'' + \frac{i}{\epsilon} \right)^2 \sum_m \overline{D}_m.$$

In the limit $\epsilon \rightarrow 0$, because of (D.27) and (D.29), it follows that $\left(\overline{D}_0'' + \frac{i}{\epsilon} \right)^2 = \frac{\omega^2}{4}$ and

$$\epsilon \sum_m \overline{D}_m = \int_{-\infty}^{\infty} \frac{1}{2\omega} e^{-i\omega|t|} dt = \frac{1}{i\omega^2};$$

therefore

$$e + f + g = -\frac{1}{8} \lambda^2 g^2, \quad (\text{D.39})$$

and the sum

$$a + b + c + d + e + f + g = \frac{1}{8} g^2 (2 + 3\lambda^2) \quad (\text{D.40})$$

which confirms (6.29).

Since each individual Feynman diagram in Figure 9 is singular in the limit $\epsilon \rightarrow 0$, to avoid ambiguities it is best to first keep $\epsilon \neq 0$, then group them as $a + b + c + d$ and $e + f + g$, each of which is finite as $\epsilon \rightarrow 0$. In the Feynman rules (6.18)-(6.24), we have already indentified some of the ϵ^{-1} as $\delta(0)$; therefore, these rules lack certain precision in how to cancel the infinities between different diagrams. By starting with the original path-integration formulation (5.48)-(5.50), with $\epsilon \neq 0$ and $x, y, \dot{x}, \dot{y}$ denoting $\overline{x}_n, \overline{y}_n, \dot{x}_n, \dot{y}_n$ of (5.15), we have an unambiguous procedure to carry out the limit $\epsilon \rightarrow 0$. For example, (i) the $i\epsilon^{-1}$ factor associated with each ghost-loop is the same $i\epsilon^{-1}$ in (D.27) for $\overline{D}_0''$ (i.e., the singular part of the second time-derivative of the Feynman propagator $D_{xx}(t)$ and $D_{yy}(t)$) and (ii) in accordance with (D.24), the identity

$$\overline{D}_0''(\overline{D}_0')^2 = 0 \quad (\text{D.41})$$

is exact for any $\epsilon \neq 0$. Thus, it is possible to derive unambiguously the limits (D.38)-(D.40) for $\epsilon \rightarrow 0$.

REFERENCES

1. V. N. Gribov, Nucl.Phys. B139, 1 (1978).
2. D. Zwanziger, Nucl.Phys. B412, 657 (1994).
3. N. H. Christ and T. D. Lee, Phys.Rev. D22, 939 (1980).
4. J. Schwinger, Phys.Rev. 127, 324 (1962); 130, 406 (1963).
5. B. S. DeWitt, Rev.Mod.Phys. 29, 377 (1957).
6. S. F. Edwards and Y. V. Gulyaev. Proc.R.Soc.London A279, 229 (1964).
7. D. W. McLaughlin and L. S. Schulman, J.Math.Phys. 12, 2520 (1971).
8. F. A. Berezin, Theor.Math.Phys. (USSR) 6, 141 (1971).
9. K. S. Chang, J.Math.Phys. 13, 1723 (1972).
10. M. Mizrahi, J.Math.Phys. 16, 2201 (1975).
11. J.-L. Gervais and A. Jevicki, Nucl.Phys. B110, 93 (1976).
12. T. D. Lee and C. N. Yang, Phys.Rev. 128, 885 (1962).
13. L. D. Faddeev and V. N. Popov, Phys.Lett. 25B, 29 (1967).
14. F. J. Dyson, Phys.Rev. 75, 486 (1949).
15. G. C. Wick, Phys.Rev. 80, 268 (1950).
16. R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (New York, McGraw-Hill Book Co., 1965).
17. T. D. Lee, *Particle Physics and Introduction to Field Theory* (Chur, Harwood Academic Publishers, 1981).
18. G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge, Cambridge University Press, 1945), p. 227.

FIGURE CAPTIONS

Figure 1. x vs. α : The curve is (3.1), $x = \rho \cos(\Phi + \alpha)$, and the straight line denotes (3.2), $x = \lambda^{-1}(Z + g^{-1}\alpha)$; their intersection determines the mapping from X, Y, Z in the time-axial gauge into x, y, α in the λ -gauge. In these plots, the parameters are $\Phi = -1$, $g = 1$, $\lambda = 1$ and for (a) $\rho = 1$ and $Z = 1$, for (b) $\rho = 6$ and $Z = 1$, and for (c) $\rho = 6$ and $Z = -2.3$. The mapping is one-to-one in (a), one-to-three in (b) and one-to-five in (c).

Figure 2. (a) The helix, defined by $x = \rho \cos(\Phi + \alpha)$ and $y = \rho \sin(\Phi + \alpha)$ for $\rho = 6$ and $\Phi = -1$, in the (x, y, α) space. (b) In addition to the helix, there are the vertical plane $\mathcal{J} = g^{-1} + \lambda y = 0$ and the slanted plane $x = \lambda^{-1}(Z + g^{-1}\alpha)$, plotted for the parameters $g = \lambda = Z = 1$. The three intersections between the helix and the slanted plane correspond to the same ones as in Figure 1(b).

Figure 3. The curve $\phi = \phi_0 + \lambda g \rho \cos \phi$ for (a) $\lambda g \rho = 2$ and (b) $\lambda g \rho = 10$. In (a), the mapping from ϕ_0 in the space-axial gauge to ϕ in the λ -gauge can be one-to-one, or one-to-three. In (b) the mapping $\phi_0 \rightarrow \phi$ can be one-to-five, or one-to-seven; the seven disconnected pieces between the solid and dashed vertical lines ($\phi_0 = 0$ and $\phi_0 = 2\pi$) can be re-assembled into a continuous smooth curve either from point a' (with $\phi_0 = 0$) to b' (with $\phi_0 = 2\pi$), or from point a (with $\phi = 0$) to b (with $\phi = 2\pi$).

Figure 4. Multiplicity domains in the (x_0, y_0) plane. The alternate white and black areas denote domains of the same multiplicity $\mu = 1, 3, 5, \dots$ in the mapping from (x_0, y_0) in the space-axial gauge to (x, y) in the λ -gauge. The boundaries between these domains are described by the involute P of a circle with center 0 and radius $r = (\lambda g)^{-1} = 1$. (See (3.13).)

Figure 5. The straight line (3.14), $y_0 = x_0 \tan \phi_0$, is shown in (a); it is mapped onto the infinite branches of (3.16), $y = x \tan(\phi_0 + \lambda g x)$, in the λ -gauge with (b) for $\phi_0 = 0$, (c) for $\phi_0 = \pi/4$ and (d) for $\phi_0 = \pi/2$ (solid curves) and $\phi_0 = -\pi/2$ (dashed curve). The shaded region in (d) is Λ . There is a *one-to-one* correspondence between the entire (x_0, y_0) plane and the domain Λ in (x, y) , with its boundary given by the envelope of the two convex branches in (d). (See (3.22)-(3.23).)

Figure 6. Multiplicity domains in Λ , defined by (3.22). The alternate white and black areas denote regions of the same multiplicity $\mu = 1, 3, 5, \dots$; there is a one-to-one correspondence between points inside these areas in Λ and those in (x_0, y_0) , as shown in Figure 4. The boundary of Λ corresponds to a cut in (x_0, y_0) , which runs along the negative y_0 -axis, extending from $y_0 = -(\lambda g)^{-1}$, the point P in Figure 4, to $y_0 = -\infty$. (As in Figure 4, $\lambda g = 1$.)

Figure 7. Multiplicity domains inside the (x, y) plane. The dashed line is the boundary of Λ (shown in Figure 6) inside which every point q_1 within an area labeled μ (or μ') has $(\mu - 1)$ gauge copies $q_2, q_3, \dots, q_\mu$ all located outside Λ , each in a separate area also labeled μ (or μ'). Equation (3.23), $y = -x \cot \lambda g x$, gives the boundary of Λ for $|\lambda g x| \leq \pi$; its other branches are the dotted lines. The horizontal line refers to $\mathcal{J} = g^{-1}(1 + \lambda g y) = 0$. The points A, A' and B, B' define the boundaries of two domains $\Lambda(A)$ and $\Lambda(B)$ described in the remark at the end of Section 3. Like Λ , each point in $\Lambda(A)$, or $\Lambda(B)$, is mapped uniquely to a point in (x_0, y_0) , and *vice versa*. (As in Figures 4 and 6, $\lambda g = 1$.)

Figure 8. The additional interaction $V(x, y)$ of (5.4) due to quantum operator ordering, plotted for $\lambda = g = 1$: (a) is related to (b) by a 180° rotation along the axis $\perp (x, y)$. Note that V is different at different locations of gauge copies, shown in Figure 7. Therefore, V is manifestly non-gauge-invariant.

Figure 9. The seven g^2 -order two-loop diagrams for the groundstate energy. The solid line represents D_{xx} or D_{yy} of (6.14), the dashed line is the ghost propagator (6.18) and the vertices are given by (6.19)-(6.20). The result is given in (6.27)-(6.29), which confirms the validity of the corresponding term in the additional interaction V of (5.4) due to the ordering of quantum operators.

Figure 10. The five g^2 -order one-loop diagrams for the propagators $\mathcal{D}_{xx}$ and $\mathcal{D}_{yy}$. The solid line denotes $D_{\mu\nu}$ given by (6.6)-(6.9), the dashed line is the ghost propagator (6.12) and the vertices are given by (6.13).

Figure 1

(a)

(b)

Figure 2

(a)

(b)

Figure 3

Figure 4

(a)

(b) $\phi_0=0$

(c) $\phi_0=\pi/4$

(d) $\phi_0=\pm\pi/2$

Figure 5

Figure 6

Figure 7

(a)

(b)

Figure 8

(a)

(b)

(c)

(d)

(e)

(f)

(g)

Figure 9

Figure 10