Operator-Formalism Approach to
Two-Dimensional Quantum Gravity in the Lightcone Gauge

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ABSTRACT

Polyakov's two-dimensional "induced" quantum gravity in the lightcone gauge is analyzed by the newly developed method based on the operator formalism, which was quite successful in the corresponding covariant-gauge model. All two-dimensional commutation relations are found, and all Wightman functions are explicitly constructed. The corresponding $\tau$-functions are shown to satisfy Polyakov's recurrence formula. A new type of anomaly, which was discovered in the covariant-gauge case, is shown to exist also in the lightcone-gauge case.
§1. Introduction

In a series of papers,\textsuperscript{1)\textendash{}8) we have investigated two-dimensional quantum gravity in
the covariant (de Donder) gauge on the basis of a new approach based on the operator
formalism and succeeded in constructing the exact solution in compact form. Some of
our remarkable findings are as follows:

1. Our approach is very much simpler than the Feynman diagrammatic method
to obtain the same results in that model.\textsuperscript{6)}

2. All Wightman functions are free of divergence.\textsuperscript{5)}

3. They can be constructed by means of a newly invented diagrammatic method
in the Heisenberg picture.\textsuperscript{7)}\textsuperscript{8)}

4. There appears a new type of anomaly, which is totally irrelevant to any sym-
metry of the action.\textsuperscript{4)}\textsuperscript{5)}

The outline of our method is as follows: Given field equations and canonical
(anti)commutation relations, we set up Cauchy problems for unequal-time
(anti)commutators between all pairs of the primary fields. Then we find their expres-
sions explicitly in terms of a q-number $D$ function.\textsuperscript{1)}\textsuperscript{2)} We calculate all independent
multiple (anti)commutators explicitly.\textsuperscript{3)} Finally, we construct all truncated Wightman
functions so as to be consistent with the above multiple (anti)commutators and the
energy-positivity condition.\textsuperscript{3)\textendash{}5)}

Unfortunately, the above analysis requires very lengthy calculation. Furthermore,
most physicists are unfamiliar to the analysis in the operator formalism. The purpose
of the present paper is to provide an illustrative example of our analysis: We consider
the two-dimensional quantum gravity in the lightcone gauge. Since only one degree of
freedom of $g_{\mu\nu}$ survives in this gauge, all calculations becomes quite simple compared
with the covariant-gauge case. Furthermore, since Polyakov\textsuperscript{9)} already investigated this model from the path-integral approach, we can compare our results with his.

In the present paper, we obtain the following results in the lightcone-gauge two-dimensional quantum gravity:

1. All Wightman functions are obtained explicitly.

2. Their corresponding \( \tau \)-functions satisfy Polyakov's recurrence formula,\textsuperscript{9)} though a minor correction is needed for the latter.

3. The new diagrammatic method is useful also in this model, though a modification of rules is necessary.

4. The new type of anomaly arises quite in the same way as in the covariant-gauge case.

The present paper is organized as follows: In §2, we present field equations and transcribe them into simple form. In §3, after calculating "equal-time" commutation relations, we find two-dimensional commutators. In §4, multiple commutators are explicitly calculated. In §5, all Wightman functions are explicitly constructed so as to be consistent with multiple commutators. In §6, those Wightman functions are shown to be constructed also by the new diagrammatic method. In §7, the \( \tau \)-functions corresponding to our Wightman functions are shown to satisfy Polyakov's recurrence formula. In §8, the new type of anomaly, found previously in the covariant-gauge model, is seen to exist also in the present model. The final section is devoted to discussion. In Appendix A, we confirm that the equivalence between quantizations based on \( x^0 \) and based on \( x^- \). In Appendix B, we briefly summarize the new type of anomaly encountered in the covariant-gauge two-dimensional quantum gravity.
\section{Field equations}

Polyakov\textsuperscript{9)} started with the Lagrangian density

\[ \frac{d}{96\pi} \sqrt{-g} R \frac{1}{\Delta} R. \]  \hspace{1cm} (2.1)

He called his model “induced quantum gravity” because it was arisen from the anomaly consideration of the Polyakov string. But we forget about such “prehistory”. Then the coefficient in (2.1) can be regarded as an arbitrary positive constant. The quantity $1/\Delta$ is nothing but the propagator of a scalar field in the presence of gravity.

In order to have a local Lagrangian density equivalent to (2.1), we introduce a scalar field $\hat{b}$ explicitly and adopt the following Lagrangian density as the fundamental one:

\[ \mathcal{L} = \sqrt{-g} R \hat{b} + \frac{1}{2} \alpha \hat{g}^{\mu \nu} \partial_\mu \hat{b} \cdot \partial_\nu \hat{b}, \]  \hspace{1cm} (2.2)

where $g_{\mu \nu}$ denotes the gravitational field, $g = \det g_{\mu \nu}$, $\hat{g}^{\mu \nu} = \sqrt{-g} g^{\mu \nu}$, $R$ being the scalar curvature, and $\alpha$ is a real parameter.

In the lightcone gauge, we set $g_{00} = h + 1$, $g_{11} = h - 1$, and $g_{01} = g_{10} = h$, where $h$ is Polyakov’s “$h_{++}$”.\textsuperscript{9)} Then (2.2) reduces to

\[ \mathcal{L} = \partial_- h \cdot \partial_- \hat{b} - \frac{1}{2} \alpha h (\partial_- \hat{b})^2 + \frac{1}{2} \alpha \partial_+ \hat{b} \cdot \partial_+ \hat{b}, \]  \hspace{1cm} (2.3)

where we employ the lightcone variables $x^\pm = \frac{x^0 \pm x^1}{2}$, $\partial_\pm = \partial_0 \pm \partial_1$.

In order to take account of the possible renormalization effect, we slightly generalize (2.3) into

\[ \mathcal{L} = \partial_- h \cdot \partial_- \hat{b} - \frac{1}{2} \alpha h (\partial_- \hat{b})^2 + \frac{1}{2} \gamma \alpha \partial_+ \hat{b} \cdot \partial_+ \hat{b}, \]  \hspace{1cm} (2.4)

by introducing an extra parameter $\gamma$. Of course, we can go back to (2.3) any time by setting $\gamma = 1$. 

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The field equations derived from (2.4) are

\[ \partial_-(\partial_- h - \alpha \partial_- \hat{b} \cdot h + \gamma \alpha \partial_+ \hat{b}) = 0, \quad (2.5) \]
\[ \partial_-^2 \hat{b} + \frac{1}{2} \alpha (\partial_- \hat{b})^2 = 0. \quad (2.6) \]

It is convenient to make the following field redefinition:

\[ \rho = \exp \frac{1}{2} \alpha \hat{b}, \quad \hat{b} = \frac{2}{\alpha} \log \rho. \quad (2.7) \]

Then (2.6) is simplified into

\[ \partial_-^2 \rho = 0. \quad (2.8) \]

Hence we may write

\[ \rho(x) = a(x^+) + c(x^+)x^-. \quad (2.9) \]

With \( \rho \), (2.5) becomes

\[ \partial_-(\partial_- h - 2 \rho^{-1} \partial_- \rho \cdot h + 2 \gamma \rho^{-1} \partial_+ \rho) = 0. \quad (2.10) \]

that is,

\[ \partial_- h - 2 \rho^{-1} \partial_- \rho \cdot h + 2 \gamma \rho^{-1} \partial_+ \rho = f \quad (2.11) \]

with \( \partial_- f = 0 \). Hence

\[ \partial_-^2 i \rho \partial_- h - 2 \partial_- \rho \cdot h + 2 \gamma \partial_+ \rho = \partial_-^2 (f \rho). \quad (2.12) \]

Then, by using (2.8), we obtain

\[ \partial_-^3 h = 0. \quad (2.13) \]

as Polyakov obtained in a different way. Following him, therefore, we write

\[ h(x) = j^{(1)}(x^+) - 2j^{(0)}(x^+)x^- + j^{(-1)}(x^+)(x^-)^2. \quad (2.14) \]
It is important to remark that (2.13) is derived from (2.10), but the converse is not true because (2.13) is a third-order differential equation. This fact becomes significant in the consideration of the new type of anomaly.

§3. Commutators

We canonically quantize the theory. We may choose either \( x^0 \) or \( x^- \) as the time variable. Here we employ the latter alternative. Of course, the same results are obtained by choosing the former, as is confirmed in Appendix A.

The canonical conjugates of \( h \) and \( \hat{b} \) are defined by

\[
\pi_h = \frac{\partial \mathcal{L}}{\partial (\partial_- h)} = \partial_- \hat{b},
\]

\[
\pi_{\hat{b}} = \frac{\partial \mathcal{L}}{\partial (\partial_- \hat{b})} = \partial_- h - a h \partial_- \hat{b} + \frac{1}{2} \gamma \alpha \partial_+ \hat{b},
\]

respectively, where \( \mathcal{L} \) is given by (2.4). They can be solved with respect to \( \partial_- \hat{b} \) and \( \partial_- h \). Setting up the canonical commutation relations, therefore, we obtain the following "equal-time" commutation relations:

\[
[h(x), h(y)]_0 = [h(x), \hat{b}(y)]_0 = [\hat{b}(x), \hat{b}(y)]_0 = 0,
\]

\[
[h(x), \partial_- h(y)]_0 = i \alpha b(x^+ - y^+),
\]

\[
[h(x), \partial_- \hat{b}(y)]_0 = i \delta(x^+ - y^+),
\]

\[
[\hat{b}(x), \partial_- \hat{b}(y)]_0 = 0,
\]

where a subscript 0 indicates to set \( x^- = y^- \). We rewrite (3.5) as

\[
[h(x), \partial_- \rho(y)]_0 = \frac{1}{2} i \alpha \rho b(x^+ - y^+).
\]

From (3.3), (3.6) and (2.6), we find that \( \hat{b}(x) \) and \( \hat{b}(y) \) commute two-dimensionally; hence

\[
[\rho(x), \rho(y)] = 0.
\]
Thus, in the present model, $\tilde{b}$ or $\rho$ plays the role of $g_{\mu\nu}$ in the covariant-gauge case.

To calculate the other two-dimensional commutators, we employ (2.9) and (2.14). Since $a$, $c$, $j^{(1)}$, $j^{(0)}$, and $j^{(-1)}$ are independent of $x^-$, they can easily be calculated by using the “equal-time” commutation relations. What is needed in addition to the above formulae is $[h, \partial_-^2 h]_0$. Expressing $\partial_-^2 h$ in terms of $\partial_- h$, $\partial_- \tilde{b}$, etc. by means of (2.5) or (2.10), we obtain

$$[h(x), \partial_-^2 h(y)]_0 = i\alpha \partial_- h \cdot \delta(x^+ - y^+) + i\gamma \alpha \delta'(x^+ - y^+).$$  \hspace{1cm} (3.9)

After some calculation, the following commutation relations are obtained:

$$[j^{(1)}(x^+), a(y^+)] = 0. \hspace{1cm} (3.10)$$

$$[j^{(0)}(x^+), a(y^+)] = \frac{1}{4} i\alpha a \delta(x^+ - y^+). \hspace{1cm} (3.11)$$

$$[j^{(-1)}(x^+), a(y^+)] = -\frac{1}{2} i\alpha c \delta(x^+ - y^+). \hspace{1cm} (3.12)$$

$$[j^{(1)}(x^+), c(y^+)] = \frac{1}{2} i\alpha a \delta(x^+ - y^+). \hspace{1cm} (3.13)$$

$$[j^{(0)}(x^+), c(y^+)] = -\frac{1}{4} i\alpha c \delta(x^+ - y^+). \hspace{1cm} (3.14)$$

$$[j^{(-1)}(x^+), c(y^+)] = 0; \hspace{1cm} (3.15)$$

$$[j^{(1)}(x^+), j^{(-1)}(y^+)] = 0. \hspace{1cm} (3.16)$$

$$[j^{(1)}(x^+), j^{(0)}(y^+)] = -\frac{1}{2} i\alpha j^{(1)} \delta(x^+ - y^+). \hspace{1cm} (3.17)$$

$$[j^{(1)}(x^+), j^{(-1)}(y^+)] = -i\alpha j^{(0)} \delta(x^+ - y^+) + \frac{1}{2} i\gamma \alpha \delta'(x^+ - y^+). \hspace{1cm} (3.18)$$

$$[j^{(0)}(x^+), j^{(0)}(y^+)] = -\frac{1}{4} i\gamma \alpha \delta'(x^+ - y^+). \hspace{1cm} (3.19)$$

$$[j^{(0)}(x^+), j^{(-1)}(y^+)] = -\frac{1}{2} i\alpha j^{(-1)} \delta(x^+ - y^+). \hspace{1cm} (3.20)$$

$$[j^{(-1)}(x^+), j^{(-1)}(y^+)] = 0. \hspace{1cm} (3.21)$$

It is interesting to note that if we define the generators

$$Q^{(k)} = 2(i\alpha)^{-1} \int dx^+ j^{(k)}(x^+), \hspace{1cm} (k = -1, 0, 1) \hspace{1cm} (3.22)$$
they satisfy the commutation relations of $SL(2, \mathbb{R})$:*)

\[ [Q^{(1)}, Q^{(0)}] = -Q^{(1)}, \]
\[ [Q^{(1)}, Q^{(-1)}] = -2Q^{(0)}, \]  \hspace{1cm} (3.23)
\[ [Q^{(0)}, Q^{(-1)}] = -Q^{(-1)}; \] \hspace{1cm} (3.25)

that is, they correspond to Polyakov's $l^{(k)}$. Furthermore, from (3.10)~(3.21), we see that $(a, c)$ behaves like a spinor for $SL(2, \mathbb{R})$.

Substituting (3.10)~(3.21), we find the two-dimensional commutation relations:

\[ [h(x), \rho(y)] = -\frac{1}{2}i\alpha(x^- - y^-)\rho(x)\delta(x^+ - y^+), \] \hspace{1cm} (3.26)
\[ [h(x), h(y)] = -\alpha(x^- - y^-)h(x, y)\delta(x^+ - y^+) \]
\[ + \frac{1}{2}i\gamma\alpha(x^- - y^-)^2\delta'(x^+ - y^+). \] \hspace{1cm} (3.27)

Here $h(x, y)$ is defined only on $x^+ = y^+$ in the following way:

\[ h(x, y) = h(x^-, y^-, x^+ = y^+) \]
\[ = j^{(1)} + j^{(0)}(x^- + y^-) + j^{(-1)}x^-y^-. \] \hspace{1cm} (3.28)

Evidently, $h(x, y) = h(y, x)$ and $h(x, x) = h(x)$.

Finally, we note that there is only one relation among $a, c, j^{(1)}, j^{(0)},$ and $j^{(-1)}$:

\[ c^2 j^{(1)} + 2acj^{(0)} + a^2 j^{(-1)} + \gamma(a\partial_+c - c\partial_+a) = 0. \] \hspace{1cm} (3.29)

*) Helayél-Neto et al.\textsuperscript{10} proposed to take $j^{(-1)} = \text{const}$ so as to have Virasoro algebra. But such a choice is inconsistent with the above commutation relations.
§4. Multiple commutators

From the two-dimensional commutators (3.8), (3.26) and (3.27), we can calculate all multiple commutators explicitly. Evidently, the only nonvanishing \((n-1)\)-ple commutators are those of one \(\rho\) and \((n-1)\) \(h\)'s and those of \(n\) \(h\)'s.

It is straightforward to have

\[
\left[ \cdots \left[ \left[ \rho(x_1), h(x_2) \right], h(x_3) \right], \cdots, h(x_n) \right] = \left( -\frac{1}{2} i \alpha \right)^{n-1} \prod_{j=1}^{n-1} (x_j^- - x_{j+1}^-) \rho(x_n) \prod_{j=1}^{n-1} \delta(x_j^+ - x_{j+1}^+). \tag{4.1} \]

It is more complicated to calculate the multiple commutators of \(n\) \(h\)'s. First, we have

\[
\left[ [h(x_1), h(x_2)], h(x_3) \right] = -i \alpha (x_1^- - x_2^-) \delta(x_1^+ - x_2^+) \\
\left\{ -i \alpha j^{(1)} \left( \frac{x_1^- + x_2^-}{2} - x_3^- \right) + i \alpha j^{(1)} (x_1^- x_2^- - (x_3^-)^2) \right. \\
- i \alpha j^{(-1)} \left( \frac{x_1^- + x_2^-}{2} (x_3^-)^2 + x_1^- x_2^- x_3^- \right) \delta(x_1^+ - x_2^+) \\
+ \frac{1}{2} \gamma \alpha (x_1^- x_2^- + (x_1^- + x_2^-) x_3^- + (x_3^-)^2) \delta'(x_1^+ - x_3^+) \right\}. \tag{4.2} \]

Using identities

\[
x_1^- x_2^- - (x_3^-)^2 = \frac{1}{2} [(x_1^- - x_3^-)(x_2^- + x_3^-) + (x_2^- - x_3^-)(x_1^- + x_3^-)], \tag{4.3} \]

etc., we rewrite (4.2) as

\[
\left[ [h(x_1), h(x_2)], h(x_3) \right] = -\frac{1}{2} \alpha^2 [(x_1^- - x_2^-)(x_1^- - x_3^-)h(x_2, x_3) \\
+ (x_1^- - x_2^-)(x_2^- - x_3^-)h(x_1, x_3)] \delta(x_1^+ - x_2^+) \delta(x_1^+ - x_3^+) \\
+ \frac{1}{2} \gamma \alpha^2 (x_1^- x_2^-)(x_2^- - x_3^-)(x_1^- - x_3^-) \delta(x_1^+ - x_2^+) \delta'(x_1^+ - x_3^+). \tag{4.4} \]

It is noteworthy that we need to introduce no more new quantity.
Hence, it is now straightforward to prove the following general formula by mathematical induction:

\[
\left[ \cdots \left[ h(x_1), h(x_2), h(x_3), \ldots, h(x_n) \right] \right] = \left( -\frac{1}{2} i \alpha \right) \sum_{i < n} (x_{i_1}^- - x_{i_2}^-) \cdots (x_{i_{n-1}}^- - x_{i_n}^-) h(x_{i_1}, x_{i_n}) \prod_{j \neq 2} \delta(x_1^+ - x_j^+) \\
- \gamma \left( -\frac{1}{2} i \alpha \right) \sum_{\text{loop}} (x_{i_1}^- - x_{i_2}^-) \cdots (x_{i_{n-1}}^- - x_{i_n}^-) \prod_{j = 2}^{n-1} \delta(x_1^+ - x_j^+) \cdot \delta'(x_1^+ - x_n^+). \tag{4.5}
\]

Here we use the following notation:

\[
x_{i_1}^- - x_{i_2}^- = x_{i_1}^- - x_{j_1}^- \quad \text{for } i < j,
\]

\[
x_{j_1}^- - x_{i_2}^- = x_{j_1}^- - x_{i_2}^- \quad \text{for } i > j; \tag{4.6}
\]

\{i_1, i_2, \ldots, i_n\} is a permutation of \{1, 2, \ldots, n\} satisfying

\[
i_1 > i_2 > \cdots > i_{k-1} > i_k \equiv 1 < i_{k+1} < \cdots < i_{n-1} < i_n; \tag{4.7}
\]

the number of such linear orderings is \(2^{n-1}\) and that of such cyclic orderings is \(2^{n-2}\).

Since the product is invariant under the inversion \(\{i_1, \ldots, i_n\} \rightarrow \{i_n, \ldots, i_1\}\), any product having the same expression appears twice.

\section*{§5. Wightman functions}

We construct the representation of field operators in a state-vector space. According to Wightman's reconstruction theorem,\(^{11}\) this is done by giving the set of all Wightman functions (i.e., the vacuum expectation values of field-operator products).

It is sufficient to consider \textit{truncated} Wightman functions, which corresponding to \textit{connected} \(\tau\)-functions.
One-point functions \( \langle \rho(x) \rangle \) and \( \langle h(x) \rangle \) are completely arbitrary. We assume that translational invariance is not broken spontaneously; then they are constants, whence they are written as \( \langle \rho \rangle \) and \( \langle h \rangle \). Since \( \mathcal{L} \) is invariant under the transformation \( \hat{b} \rightarrow \hat{b} + \text{const} \), we can always set \( \langle \hat{b} \rangle = 0 \), i.e., \( \langle \rho \rangle = 1 \), but we do not make such specification in the following.

The truncated \( n \)-point Wightman functions must be consistent with the vacuum expectation values of multiple commutators. But the number of former is \( n! \), while that of the independent multiple commutators is \( (n - 1)! \). Thus the number of equations is less than that of unknowns. To determine the latter, we need an additional requirement, which is the energy-positivity condition. For example, \( \{ \phi(x), \phi(y) \} = i \Delta(x - y; m^2) \) is uniquely decomposed into \( \{ \phi(x)\phi(y) \}_\tau = \Delta^{(+)\dagger}(x - y; m^2) \) and \( \{ \phi(y)\phi(x) \}_\tau = \Delta^{(+)\dagger}(y - x; m^2) \) under the energy-positivity requirement.

We first note that

\[
\delta(x^+ - y^+) = \frac{1}{2\pi i} \left( \frac{1}{x^+ - y^+ - i0} + \frac{1}{y^+ - x^+ - i0} \right). \tag{5.1}
\]

Hence it is convenient to introduce

\[
D^{(+)\dagger}(\xi) \equiv \frac{\xi^-}{\xi^+ - i0} = i\xi^- \int_0^\infty dp_+ e^{-p_+\xi^+}. \tag{5.2}
\]

Furthermore, as is obvious from the consideration made in §4, nonvanishing truncated Wightman functions are those which involve at most one \( \rho \).

From (3.26), (3.27), (4.1) and (4.4), nonvanishing truncated two-point and three-point functions are easily seen to be as follows:

\[
\langle \rho(x_1)h(x_2) \rangle_\tau = -\frac{\alpha}{4\pi} \langle \rho \rangle D^{(+)\dagger}(x_1 - x_2), \tag{5.3}
\]

\[
\langle h(x_1)h(x_2) \rangle_\tau = -\frac{\alpha}{2\pi} \langle h \rangle D^{(+)\dagger}(x_1 - x_2) - \frac{\alpha}{4\pi} [D^{(+)\dagger}(x_1 - x_2)]^2; \tag{5.4}
\]

\[
\langle \rho(x_1)h(x_2)h(x_3) \rangle_\tau = \frac{\alpha^2}{16\pi^2} \langle \rho \rangle [D^{(+)\dagger}(x_1 - x_2) + D^{(+)\dagger}(x_1 - x_3)] D^{(+)\dagger}(x_2 - x_3). \tag{5.5}
\]
\begin{align}
\langle h(x_1)h(x_2)\hat{h}(x_3)\rangle_T \\
= \frac{\alpha^2}{8\pi^2} \langle h \rangle \left[ D^{(+)}(x_1 - x_2)D^{(+)}(x_1 - x_3) \\
+ D^{(+)}(x_1 - x_2)D^{(+)}(x_2 - x_3) + D^{(+)}(x_1 - x_3)D^{(+)}(x_2 - x_3) \right] \\
+ \gamma \frac{\alpha^2}{8\pi^2} D^{(+)}(x_1 - x_2)D^{(+)}(x_2 - x_3)D^{(+)}(x_1 - x_3), \tag{5.6} \end{align}

where a subscript T indicates "truncated". In deriving (5.6) from (4.4), one should note that

\begin{align}
\delta(x_1^+ - x_2^+)\delta(x_1^+ - x_3^+) \\
= \delta(x_1^+ - x_2^+) \cdot \frac{1}{2\pi i} \left[ \frac{1}{(x_1^+ - x_3^+ - i0)(x_2^+ - x_3^+ - i0)} \\
- \frac{1}{(x_3^+ - x_1^+ - i0)(x_3^+ - x_2^+ - i0)} \right]. \tag{5.7} \end{align}

From (4.1) and (4.5), nonvanishing truncated n-point functions are seen to be as follows:

\begin{align}
\langle \rho(x_1)h(x_2)\cdots h(x_n)\rangle_T \\
= \left( -\frac{\alpha}{4\pi} \right)^{n-1} \langle \rho \rangle \sum_{P(1, \ldots, \, n)} D^{(+)}(x_1 - x_{i_2})D^{(+)}(x_{i_2} - x_{i_3})\cdots D^{(+)}(x_{i_{n-1}} - x_{i_n}), \tag{5.8} \end{align}

\begin{align}
\langle h(x_1)h(x_2)\cdots h(x_n)\rangle_T \\
= \left( -\frac{\alpha}{4\pi} \right)^{n-1} \left[ \langle h \rangle \sum_{P(1, \ldots, \, n)} D^{(+)}(x_{i_1} - x_{i_2})\cdots D^{(+)}(x_{i_{n-1}} - x_{i_n}) \\
+ \gamma \sum_{C(1, \ldots, \, n)} D^{(+)}(x_{i_1} - x_{i_2})\cdots D^{(+)}(x_{i_{n-1}} - x_{i_n})D^{(+)}(x_{i_n} - x_{i_1}) \right], \tag{5.9} \end{align}

where

\begin{align}
D^{(+)}_<(x_i - x_j) &= D^{(+)}(x_i - x_j) \quad i < j, \\
D^{(+)}(x_j - x_i) &= D^{(+)}(x_j - x_i) \quad i > j, \\
&= 0 \quad i = j; \tag{5.10} \end{align}

P denotes a permutation and C does a cyclic permutation.
§6. New diagrammatic method

As is well known, the Feynman diagrammatic method facilitates the conventional perturbation theory very much. Corresponding to it, we have invented a new diagrammatic method so as to write down easily the results of our approach.\(^7\),\(^8\) In the Heisenberg picture, we do not have the notion of the interaction Lagrangian, and therefore, unlike the Feynman diagram, no prescribed junction at the vertex is available. Our diagrams are constructed from "building blocks", which contain the information about connection. Our new diagrammatic method is quite successful in the covariant-gauge two-dimensional quantum gravity.

Since the general expressions (5.8) and (5.9) are rather simple, there is no appealing reason for introducing the new diagrammatic method into the present model. But we expect that because of its simplicity, it may be instructive for illustrative purpose to do so.

From one-point functions and two-dimensional commutators (3.8), (3.26) and (3.27), we set up the following building blocks:

\[
\begin{align*}
\begin{array}{c}
\bullet \\
\rho \quad \text{(\(\rho\))}, \\
\end{array} & \quad \begin{array}{c}
\bullet \\
h \quad \text{(\(h\))}. \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\rho \\
x \\
\end{array} & \quad \begin{array}{c}
\bullet \\
h \\
y \\
\end{array} & \quad \frac{\alpha}{4\pi} D_{\xi^+}^\rho(x - y), \\
\begin{array}{c}
h \\
x \\
\end{array} & \quad \begin{array}{c}
\bullet \\
h \\
y \\
\end{array} & \quad -\frac{\alpha}{4\pi} D_{\xi^+}^\rho(x - y).
\end{align*}
\]

The diagrammatic rules, which correspond to the Feynman rules, are much simpler than that in the covariant-gauge case. Let \(\{\varphi_1(x_1) \cdots \varphi_n(x_n)\}_T\) be a truncated Wightman function, \(\varphi_j(x)\) being a generic field. The rules are as follows:

1. We prepare \(n\) vertices, named \(\varphi_j\), at \(x_j\) (\(j = 1, 2, \ldots, n\)).
2. By using the building blocks, we construct a connected diagram. Each end point of any building block must be the vertex having the same name as it.

3. At each vertex, at most one white small circle and exactly one black small circle are placed.

4. We add up the contributions from all possible diagrams (with multiplying loop diagrams by a factor $-4\pi\gamma/\alpha$).

The essential difference from the rules in the covariant-gauge case is the treatment of white small circles.

From the above rules, the following diagrams are easily constructed:

\[
\langle \rho(x_1)h(x_2)\cdots h(x_n) \rangle_T = \sum_{P(1_2, \ldots, 1_n)}^{(n-1)!} \rho \begin{array}{c} x_1 \end{array} \begin{array}{c} h \end{array} \begin{array}{c} x_2 \end{array} \begin{array}{c} h \end{array} \begin{array}{c} x_3 \end{array} \cdots \begin{array}{c} h \end{array} \begin{array}{c} x_{n-1} \end{array} \begin{array}{c} h \end{array} \begin{array}{c} x_n \end{array} \] (6.1)

\[
\langle h(x_1)h(x_2)\cdots h(x_n) \rangle_T = \sum_{P(1_2, \ldots, 1_n)}^{n!} \begin{array}{c} h \end{array} \begin{array}{c} x_1 \end{array} \begin{array}{c} h \end{array} \begin{array}{c} x_2 \end{array} \begin{array}{c} h \end{array} \begin{array}{c} x_3 \end{array} \cdots \begin{array}{c} h \end{array} \begin{array}{c} x_{n-1} \end{array} \begin{array}{c} h \end{array} \begin{array}{c} x_n \end{array} \]

\[\left(\frac{-4\pi\gamma}{\alpha}\right)^{\frac{(n-1)!}{n}} \sum_{C(1_1, \ldots, 1_n)} x_{i_1}h \begin{array}{c} h \end{array} \begin{array}{c} x_{i_2} \end{array} \begin{array}{c} h \end{array} \begin{array}{c} x_{i_3} \end{array} \cdots \begin{array}{c} h \end{array} \begin{array}{c} x_{i_n} \end{array} \] (6.2)
§7. Comparison with Polyakov’s formula

The expressions for the Wightman functions can easily be transcribed into those for the \( \tau \)-functions (i.e., the vacuum expectation values of time-ordered products) by replacing \( D_{\xi^+}^+(x_i - x_j) \) by \( D_\tau(x_i - x_j) \), where

\[
D_\tau(\xi) \equiv \frac{\xi^-}{\xi^+ - i0\xi^-} = \xi^-(\frac{\theta(\xi^-)}{\xi^+ - i0} + \frac{\theta(-\xi^-)}{\xi^+ + i0}), \tag{7.1}
\]

which satisfies a differential equation

\[
(2\pi i)^{-1} \partial_{\xi^+}^2 D_\tau(\xi) = \delta^2(\xi). \tag{7.2}
\]

For the \( \tau \)-functions, we can derive the Ward-Takahashi identities in the standard way. From (2.8) and (3.7), we obtain

\[
(\partial_{-x^1})^2 \{ T \rho(x_1) h(x_2) \cdots h(x_n) \}
\]

\[
= \sum_{j=2}^n \delta(x_1^- - x_j^-) \{ T [\partial_{-} \rho(x_1), \ h(x_j)]_0 h(x_2) \cdots \widehat{h(x_j)} \cdots h(x_n) \}
\]

\[
= -\frac{1}{2} i\alpha \sum_{j=2}^n \delta(x_1^- - x_j^-) \{ T \rho(x_j) h(x_2) \cdots \widehat{h(x_j)} \cdots h(x_n) \}, \tag{7.3}
\]

where a hat means the omission. Likewise, from (2.13), (3.4), and (3.9), we have

\[
(\partial_{-x^1})^3 \{ T h(x_1) \cdots h(x_n) \}
\]

\[
= \partial_{-x^1} \left[ \sum_{j=2}^n \delta(x_1^- - x_j^-) \{ T [\partial_{-} h(x_1), \ h(x_j)]_0 \cdots \widehat{h(x_j)} \cdots h(x_n) \} \right]
\]

\[
+ \sum_{j=2}^n \delta(x_1^- - x_j^-) \{ T [\partial_{-}^2 h(x_1), \ h(x_j)]_0 \cdots \widehat{h(x_j)} \cdots h(x_n) \}
\]

\[
= -i\alpha \sum_{j=2}^n \left[ \partial_{-x^1} \delta^2(x_1^- - x_j^-) \{ T h(x_2) \cdots h(x_n) \} \right.
\]

\[
+ \delta^2(x_1^- - x_j^-) \partial_{-x^1} \{ T h(x_2) \cdots h(x_n) \}
\]

\[
- \gamma \partial_{+x^1} \delta^2(x_1^- - x_j^-) \{ T h(x_2) \cdots \widehat{h(x_j)} \cdots h(x_n) \} \right]. \tag{7.4}
\]

We can integrate (7.3) and (7.4) by using (7.2). We find

\[
\{ T \rho(x_1) h(x_2) \cdots h(x_n) \}
\]
\[ \begin{align*} 
&= -\frac{\alpha}{4\pi} \sum_{j=2}^{n} D_{p}(x_{1} - x_{j})(T \rho(x_{1})h(x_{2}) \cdots h(x_{j}) \cdots h(x_{n})) \\
&\quad + (\rho)(T h(x_{2}) \cdots h(x_{n})), \\
&\quad (T h(x_{1})h(x_{2}) \cdots h(x_{n})) \\
&= -\frac{\alpha}{4\pi} \sum_{j=2}^{n} \{ D_{p}(x_{1} - x_{j})[(x_{1}^{-} - x_{j}^{-})\partial_{x_{j}} + 2](T h(x_{2}) \cdots h(x_{n})) \\
&\quad + \gamma[D_{p}(x_{1} - x_{j})]^{2}(T h(x_{2}) \cdots h(x_{j}) \cdots h(x_{n})) \} \\
&\quad + (h)(T h(x_{2}) \cdots h(x_{n})). 
\end{align*} \] (7.5)

Apart from the last term, (7.6) is essentially the same as Polyakov’s recurrence formula.\(^9\)

Unless \( (h) = 0 \), he missed the last term, i.e., an integration constant, without which 3-point \((n \geq 3) \tau\)-function is not fully symmetric under the permutation of \(x_{1}, x_{2}, \ldots, x_{n} \).

If we consider connected \(\tau\)-functions only, recurrence formulae are simplified in the following way:

\[ \begin{align*} 
&\langle T \rho(x_{1})h(x_{2}) \cdots h(x_{n}) \rangle_c \\
&= -\frac{\alpha}{4\pi} \sum_{j=2}^{n} D_{p}(x_{1} - x_{j})(T \rho(x_{1})h(x_{2}) \cdots h(x_{j}) \cdots h(x_{n}))_c \quad \text{for} \quad n \geq 2, \\
&\langle T h(x_{1})h(x_{2}) \cdots h(x_{n}) \rangle_c \\
&= -\frac{\alpha}{4\pi} \sum_{j=2}^{n} D_{p}(x_{1} - x_{j})[(x_{1}^{-} - x_{j}^{-})\partial_{x_{j}} + 2](T h(x_{2}) \cdots h(x_{n}))_c \quad \text{for} \quad n \geq 3,(7.8) 
\end{align*} \]

where a subscript \( C \) means “connected”.

The explicit solutions to the above recurrence formulae are given in the form exhibiting manifest symmetry under permutations:

\[ \begin{align*} 
&\langle T \rho(x_{1})h(x_{2}) \cdots h(x_{n}) \rangle_c \\
&= \left( -\frac{\alpha}{4\pi} \right)^{n-1} \langle \rho \rangle \sum_{P_{1} \neq \cdots \neq P_{n-1}} D_{p}(x_{1} - x_{i_{2}})D_{p}(x_{i_{2}} - x_{i_{3}}) \cdots D_{p}(x_{i_{n-1}} - x_{i_{n}}), \quad (7.9)\\
&\langle T h(x_{1})h(x_{2}) \cdots h(x_{n}) \rangle_c \\
&= \left( -\frac{\alpha}{4\pi} \right)^{n-1} \langle h \rangle \sum_{P_{1} \neq \cdots \neq P_{n}} D_{p}(x_{i_{1}} - x_{i_{2}}) \cdots D_{p}(x_{i_{n-1}} - x_{i_{n}}) 
\end{align*} \]
\[ + \gamma \sum_{C(i_1, \ldots, i_n)} D_F(x_{i_1} - x_{i_2}) \cdots D_F(x_{i_{n-1}} - x_{i_n}) D_F(x_{i_n} - x_{i_1}) \]  

(7.10)

as is transcribed from (5.8) and (5.9), respectively.

Proof is needed only for (7.10). Since the \( n = 1, 2 \) cases are evidently all right, we use (7.8) and prove (7.10) by mathematical induction. The crucial formula is

\[
D_F(x_i - x_j)[(x_j^- - x_i^-) \partial_-^{-x_i} + 1]D_F(x_i - x_j) \\
+ D_F(x_i - x_j)[(x_j^- - x_j^-) \partial_-^{-x_j} + 1]D_F(x_i - x_j) \\
= D_F(x_i - x_i)D_F(x_i - x_j). 
\]

(7.11)

Its proof is elementarily done by substituting the explicit expression for \( D_F \).

First, (7.8) is applied to the loop term of \( \langle T h(x_2) \cdots h(x_n) \rangle_C \), each of which is specified by a cyclic permutation of \( \{2, 3, \ldots, n\} \). Every \( x_j \) (\( j = 2, \ldots, n \)) appears twice in each product of \( D_F \)'s. Hence, according to the Leibniz rule, we write

\[
(x_j^- - x_j^-) \partial_-^{-x_j} + 2 = \left[ (x_j^- - x_j^-) \partial_-^{-x_j} \right]_{on \ first \ right\ D_F} + 1 \\
+ \left[ (x_j^- - x_j^-) \partial_-^{-x_j} \right]_{on \ second \ D_F} + 1. 
\]

(7.12)

Then we apply the formula (7.11). The net effect is to replace \( D_F(x_i - x_j) \) by \( D_F(x_1 - x_i)D_F(x_1 - x_j) \), that is, to insert \( x_i \) into the loop.

Second, (7.8) is applied to the line term of \( \langle T h(x_2) \cdots h(x_n) \rangle_C \). Everything goes in the same way as above except for the end points of the line: If \( x_j \) is the end point, then \( D_F \) involving \( x_j \) is only one. This merely implies that the second part in the right-hand side of (7.12) is replaced by 1. Thus \( x_1 \) is either inserted or added to the end, as it should be.

This completes the proof.
§8. New type of anomaly

A new type of anomaly, which is totally irrelevant to any symmetry of the action, was discovered in the covariant-gauge two-dimensional quantum gravity (see Appendix B).\(^4\)\(^5\) In this section, we show that quite similar anomaly exists also in the lightcone-gauge two-dimensional quantum gravity.

The representation by state vectors has been constructed on the basis of (2.13), i.e.,

\[
\partial_-^3 h = 0, \tag{8.1}
\]

but not of the original field equation (2.10), i.e.,

\[
\mathcal{T} \equiv \partial_-(\partial_- h + 2\gamma \rho^{-1}\partial_+ \rho - 2\rho^{-1}\partial_- \rho \cdot h) = 0. \tag{8.2}
\]

Although (8.1) is derived from (8.2) together with \(\partial_-^2 \rho = 0\), the converse is not true because (8.1) is a differential equation higher than (8.2). We have used \([h, \partial_-^2 h]_b\), which can be calculated only by means of (8.2), but (8.2) itself is taken into account neither in our construction of Wightman functions nor in the derivation of the recurrence formula for \(\tau\)-functions. It is necessary, therefore, to check whether or not (8.2) is respected in the representation.

It is straightforward to see that

\[
\langle \mathcal{T}(x_1)\rho(x_2) \rangle = (\partial_-^x)^2 \langle h(x_1)\rho(x_2) \rangle = 0 \tag{8.3}
\]

by using \(\langle \partial_- \rho \rangle = 0\) and (5.3). We then calculate

\[
\langle \mathcal{T}(x_1)h(x_2) \rangle = (\partial_-^x)^2 \langle h(x_1)h(x_2) \rangle + 2\gamma \partial_-^x \left[ (\rho)^{-1}\partial_+ \rho \langle h(x_1)h(x_2) \rangle \right] \\
- 2\partial_-^x \left[ (\rho)^{-1}(\partial_- \rho \langle h(x_1)h(x_2) \rangle) \bigg|_{zz_1} \right]. \tag{8.4}
\]

From (5.4) and (5.3), we find that the first term and the second one exactly cancel because

\[
(\partial_-^x)^2 \langle h(x_1)h(x_2) \rangle = -\gamma \frac{\alpha}{2\pi} \frac{1}{(x_1^+ - x_2^+ - i0)^2}, \tag{8.5}
\]

\[-17-\]
\begin{equation}
2\gamma \partial_{z_{1}} \left[ (\rho)^{-1}_{\rho} \partial_{\rho}(\rho(x_{1})h(x_{2})) \right] = \frac{\alpha}{2\pi} \frac{1}{(x_{1}^{+} - x_{2}^{+} - i0)^{2}}. \tag{8.6}
\end{equation}

In spite of this fact, the third term of (8.4) does not vanish. Indeed, from (5.5) and (5.3), we see that the third term equals

\begin{align*}
-2\partial_{z_{1}} \left\{ \frac{\alpha^{2}}{16\pi^{2}} \partial_{z_{1}} \left[ D^{(+)}(x_{1} - z)D^{(+)}(z - x_{2}) + D^{(+)}(x_{1} - x_{2})D^{(+)}(z - x_{2}) \right] \right\}_{z = x_{1}} \\
- \frac{\alpha}{4\pi} \partial_{z_{1}} \left[ (h)D^{(+)}(x_{1} - x_{2}) \right] \\
= -\frac{\alpha^{2}}{8\pi^{2}} \frac{1}{(x_{1}^{+} - x_{2}^{+} - i0)^{2}} \neq 0. \tag{8.7}
\end{align*}

We thus conclude that \textit{the field equation (8.2) is violated at the level of the representation!} The situation encountered here is quite similar to that in the covariant-gauge case.

The existence of the above anomaly can also be checked from the path-integral point of view. In general, let \( S \) be the action and \( \varphi \) be any field. As long as the path-integral measure is invariant under the transformation \( \varphi \rightarrow \varphi + \delta \varphi \), we should have

\begin{equation}
i \left\langle T^{\ast} F(x, y, \ldots) \frac{\delta S}{\delta \varphi(z)} \right\rangle + \left\langle T^{\ast} \frac{\delta F(x, y, \ldots)}{\delta \varphi(z)} \right\rangle = 0, \tag{8.8}\end{equation}

where \( F(x, y, \ldots) \) is any function of fields. We apply (8.8) to the present model and set \( F = h \) and \( \varphi = \tilde{b} \). Then we should have

\begin{equation}
\left\langle T^{\ast} h(x)T(z) \right\rangle = 0. \tag{8.9}\end{equation}

Substituting \( \tau \)-functions calculated from the recurrence formulae, one can confirm that (8.9) is really violated.
§9. Discussion

In the present paper, we have fully analyzed the lightcone-gauge two-dimensional quantum gravity from the standpoint of the canonical operator formalism. This model is very suitable for illustrating our approach, by which we have obtained all Wightman functions and confirmed that the corresponding $\tau$-functions satisfy Polyakov's recurrence formula.

This model is interesting in the sense that it has the recently discussed anomaly which is totally irrelevant to any symmetry of the action. We hope that more physicists pay attention to this new phenomenon. The simplicity of the model probably makes the access to the problem facilitate.

It will be possible to extend the present analysis to various quantum-gravity models in noncovariant gauges.

The present authors would like to thank Professor J. Gomis for discussions, especially, on the path-integral approach to the new type of anomaly.
Appendix A. Quantization based on $x^0$

Canonical quantization is usually done by taking $x^0$ as time. In this case, $\pi_{\xi}$ becomes

$$\pi_{\xi} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_0 \hat{b})} = \partial_- h - a h \partial_- \hat{b} + \gamma \alpha \partial_0 \hat{b}$$  \hspace{1cm} (A.1)

instead of (3.2). Then the nonvanishing commutation relations are

$$[h(x), \partial_0 h(y)]|_{x^0 = y^0} = i\alpha (h - \gamma) \delta(x^1 - y^1).$$  \hspace{1cm} (A.2)

$$[h(x), \partial_0 \hat{b}(y)]|_{x^0 = y^0} = i\delta(x^1 - y^1)$$  \hspace{1cm} (A.3)

instead of (3.4) and (3.5), respectively. Based on them, we can derive the same two-dimensional commutation relations as in §3. For example, (3.27) reproduces (A.3) in the following way:

$$[h(x), \partial_0 h(y)]|_{x^0 = y^0} = \frac{i}{2} h \cdot 2 \delta(x^1 - y^1) + \frac{1}{2} \gamma \alpha \cdot 2 \delta(x^1 - y^1) \cdot 4 \delta'(x^1 - y^1)$$

$$= i\alpha (h - \gamma) \delta(x^1 - y^1).$$  \hspace{1cm} (A.4)

Appendix B. New type of anomaly in the covariant-gauge case

For the convenience of comparison, we briefly review the new type of anomaly encountered in the covariant-gauge two-dimensional quantum gravity.\textsuperscript{44,45}

The Lagrangian density is

$$\mathcal{L} = \sqrt{-g} R \hat{b} - \bar{g}^{\mu \nu} (\partial_\mu b_\nu + i \partial_\mu c_\lambda \cdot \partial_\nu b^\lambda) + \mathcal{L}_{\text{matter}}.$$  \hspace{1cm} (B.1)

where $b_\lambda$ is the gravitational B-field, and so on. The field equation derived as the Euler equation with respect to $g_{\mu \nu}$ is

$$T_{\mu \nu} \equiv 2(\nabla_\mu \nabla_\nu - g_{\mu \nu} \Box) \hat{b} - E_{\mu \nu} + \frac{1}{2} g_{\mu \nu} E_{\lambda}^\lambda + T_{\mu \nu} = 0,$$  \hspace{1cm} (B.2)
where

\[ \square \varphi = (\sqrt{-g})^{-1} \partial_{\mu} (\tilde{g}^{\mu \nu} \partial_{\nu} \varphi). \]

[Equation (B.3)]

\[ E_{\mu \nu} \equiv \partial_{\mu} b_{\nu} + i \partial_{\mu} c_{\lambda} \cdot \partial_{\nu} c^{\lambda} + (\mu \leftrightarrow \nu). \]

[Equation (B.4)]

By taking covariant derivative of (B.2), we have

\[ \square b_{\lambda} = 0, \]

[Equation (B.5)]

while the trace of (B.2) yields

\[ \square \tilde{b} = 0. \]

[Equation (B.6)]

The number 3 of the degrees of freedom of (B.2) is equal to that of (B.5) and (B.6), but (B.2) cannot follow from (B.5) and (B.6) because extra differentiation is made to derive (B.5), just as in deriving (2.13) from (2.10).

Two-dimensional commutators are constructed on the basis of (B.5), (B.6), etc., but not of (B.2), which is directly used only for deriving the equal-time commutator \([b_{\mu}(x), \partial_{\nu} b_{\lambda}(y)]\). All Wightman functions are quite satisfactorily constructed so as to be consistent with multiple commutators.\(^5\) Furthermore, they are uniquely determined also by the BRS invariance.\(^8\) However, (B.2) is not respected by the Wightman functions. Indeed, we have the new type of anomaly: If \( \langle g_{\mu \nu} \rangle = \eta_{\mu \nu} \),

\[ \langle T_{\mu \nu}(x)b_{\lambda}(y) \rangle = \frac{1}{2\pi^2} \cdot \frac{1}{(\xi^2 - i0\xi^0)^3} \left[ -4 \xi_{\mu} \xi_{\nu} \xi_{\lambda} + \xi^2 (\eta_{\mu \lambda} \xi_{\nu} + \eta_{\nu \lambda} \xi_{\mu} + \eta_{\mu \nu} \xi_{\lambda}) \right] \]

\[ \neq 0 \quad (\xi^a \equiv x^a - y^a). \]

[Equation (B.7)]

Of course, one can check the \( \tau \)-function version of (B.7) by the Feynman-diagrammatic method.\(^6\)
References


