We prove that the maximal development of any spherically symmetric spacetime with collisionless matter (obeying the Vlasov equation) or a massless scalar field (obeying the massless wave equation) and possessing a constant mean curvature $S^1 \times S^2$ Cauchy surface also contains a maximal Cauchy surface. Combining this with previous results establishes that the spacetime can be foliated by constant mean curvature Cauchy surfaces with the mean curvature taking on all real values, thereby showing that these spacetimes satisfy the closed-universe recollapse conjecture. A key element of the proof, of interest in itself, is a bound for the volume of any Cauchy surface $\Sigma$ in any spacetime satisfying the timelike convergence condition in terms of the volume and mean curvature of a fixed Cauchy surface $\Sigma_0$ and the maximal distance between $\Sigma$ and $\Sigma_0$. In particular, this shows that any globally hyperbolic spacetime having a finite lifetime and obeying the timelike-convergence condition cannot attain an arbitrarily large spatial volume.

I. INTRODUCTION

Given an initial data set for the gravitational field and any matter fields present, what can be said of the spacetime evolved from this initial data?

In the asymptotically flat case, one would like to know such things as how much gravitational energy is radiated to null infinity, the final asymptotic state of the system, whether black holes are formed, the nature of any singularities produced, and whether cosmic censorship is violated. For example, it is known that the maximal development of sufficiently weak vacuum initial data is an asymptotically flat spacetime that is free of singularities and black holes [?]. In this case the gravitational waves are so weak that they cannot coalesce into a black hole; instead they scatter to infinity. Further it is known that an initial data set containing a future trapped surface or a future trapped region must be singular, provided the null-convergence condition holds [?]. In these cases, the gravitational field is already sufficiently strong that collapse is inevitable.

In the cosmological case (spacetimes with compact Cauchy surfaces), the questions one asks are a bit different as one expects these spacetimes to be quite singular. In fact, it is known that spacetimes with compact Cauchy surfaces are singular, provided a genericity condition and the timelike-convergence condition hold [?]. So, here one would like to know such things as the nature of the singularities, if the spacetime has a finite lifetime (in the sense that there is a global upper bound on the lengths of all causal curves therein), whether it expands to a maximal hypersurface and then recollapses or is always expanding (contracting), and whether cosmic censorship is violated. For example, it is known that if the initial data surface is contracting to the future (past), then any development satisfying the timelike-convergence condition must end within a finite time to the future (past) [?]. Can more be said about the behavior of the cosmological spacetimes?

The closed-universe recollapse conjecture asserts that the spacetime associated with the maximal development of an initial data set with compact initial data surface expands from an initial singularity to a maximal hypersurface and then recollapses to a final singularity (all within a finite time), provided that the spatial topology does not obstruct the existence of a maximal Cauchy surface (e.g., $S^3$ or $S^1 \times S^2$) and provided the matter satisfies certain energy and regularity conditions [?]. It has also been conjectured that such spacetimes admit a unique foliation by constant mean curvature (CMC) Cauchy surfaces with the mean curvatures taking on all real values. (See, e.g., conjecture 2.3 of [?] and the weaker conjecture C2 of [?].) Just what energy conditions the matter must satisfy is an open problem. However, in the study of the weak form of this conjecture (which merely asserts that the spacetime has a finite lifetime), the dominant energy and non-negative pressures conditions together have proven sufficient for the cases studied [?]. More subtle is the problem of what regularity conditions the matter needs to satisfy. The difficulty here is that the maximal development of an Einstein-matter initial data set may not contain a maximal hypersurface because of the development of a singularity in the matter fields, such as a shell-crossing singularity in a dust-filled spacetime, before the spacetime has a chance to develop a maximal hypersurface. While not for certain, it is thought that those matter fields that do not develop singularities when evolved in fixed smooth background spacetimes will not lead to the obstruction of a maximal hypersurface.
Here, we study the maximal development of spherically symmetric constant mean curvature initial data sets with $S^1 \times S^2$ Cauchy surfaces and matter consisting of either collisionless particles of unit mass (whose evolution is described by the Vlasov equation) or a massless scalar field (whose evolution is described by the massless wave equation). It has already been established that if the mean curvature is zero on the initial data surface, i.e., it is a maximal hypersurface, then its maximal evolution admits a foliation by CMC Cauchy surfaces with the mean curvature taking on all real values [7]. Further, it is known that the mean curvature of these spacetimes taking on all real values. The nonexistence of a maximal spatial hypersurface and, hence, can be foliated by CMC hyper-surfaces taking on all negative (positive) values [?]. Left unresolved was whether the maximal evolution in the latter two cases actually contains a maximal spatial hypersurface and, hence, can be foliated by CMC hypersurfaces taking on all real values. The nonexistence of a maximal spatial hypersurface would be reasonable if such spacetimes could expand (contract) indefinitely, however, it is known that these spacetimes have finite lifetimes [7]. Therefore, it would seem that their maximal development should contain a maximal Cauchy surface. We show that it does.

**Theorem 1.** The maximal development of any spherically symmetric spacetime with collisionless matter (obeying the Vlasov equation) or a massless scalar field (obeying the massless wave equation) that possesses a CMC $S^1 \times S^2$ Cauchy surface $\Sigma$ admits a unique foliation by CMC Cauchy surfaces with the mean curvature taking on all real values. In particular, it contains a maximal Cauchy surface and its singularities are crushing singularities.

By the maximal development of a globally hyperbolic spacetime, we mean the maximal development of an initial data set induced on a Cauchy surface in the spacetime. This is well-defined as the maximal developments associated with any two Cauchy surfaces are necessarily isometric [?]. Further, recall that a spacetime with compact Cauchy surfaces is said to have a future (past) crushing singularity if the spacetime can be foliated by Cauchy surfaces such that the mean curvature of these surfaces tends to infinity (negative infinity) uniformly to the future (past). That the future and past singularities associated with the spacetimes of theorem 1 are crushing is then a simple consequence of the existence of a CMC foliation taking on all real values.

As a consequence of theorem 1, the maximal development of the spacetimes studied is rather simple. They expand from an initial crushing singularity to a maximal hypersurface and then recollapse to a final crushing singularity—all in a finite time. That is, they satisfy the closed-universe recollapse conjecture in its strongest sense as well as the closed-universe foliation conjecture.

While the maximal development of the spacetimes in theorem 1 is about as complete as one could expect given the existence of a complete CMC foliation, these spacetimes may still be extendible (though there is no globally hyperbolic extension). In other words, theorem 1 does not eliminate the possibility that these spacetimes violate cosmic censorship. In fact, cosmic censorship is violated in the vacuum case. This is easily seen by realizing that the maximal development in this case is either of the of the two regions where $r < 2M$ of an extended Schwarzschild spacetime of mass $M$ ($r$ is the areal radius), modified by identifications so that the Cauchy surface topology is $S^1 \times S^2$. Although the “singularity” corresponding to $r \to 2M$ is a crushing singularity, this is actually a Cauchy horizon. Is this vacuum case exceptional? It is worth noting that if a crushing singularity corresponds to $r \to 0$, then the singularity must in fact be a curvature singularity. This follows easily from the fact that $R_{abcd}R^{abcd} \geq (4m/r^3)^2$, for any spherically symmetric spacetime satisfying the null-convergence condition, and the fact that the mass function $m$ is bounded away from zero by a positive constant in our case [?]. If we could show that $r$ must go to zero (uniformly) at the extremes of our foliation, then the spacetime would indeed be inextendible, thereby satisfying the cosmic censorship hypothesis. Establishing such a result appears to be difficult and the vacuum case shows that such a result will not always hold (though this case may be exceptional).

Using a different approach, Rein has shown that for an open set of initial data, there is a crushing singularity in which $r \to 0$ uniformly, and which, therefore, is a curvature singularity [?]. While this is encouraging, the extent to which the spacetimes of theorem 1 satisfy cosmic censorship remains to be seen.

The proof of theorem 1 involves a combination of three ideas. First, it is known that spherically symmetric spacetimes with $S^1 \times S^2$ or $S^3$ Cauchy surfaces and satisfying the dominant energy and non-negative pressures (or merely “radial” non-negative pressure) conditions have finite lifetimes [?]. Second, using a general theorem (which is independent of symmetry assumptions) established in Sec. III, it follows that the spatial volumes of Cauchy surfaces in the spacetime are bounded above, which allows us to bound various fields describing the spacetime geometry. Third, introducing a new time function to avoid the problems associated with “degenerate” maximal hypersurfaces (i.e., surfaces where the mean curvature cannot be used as a good coordinate), the theorem then follows using the methods developed in [?]. Furthermore, it is worth noting that our method uses only a few properties of the matter fields themselves. Namely, we use the fact that they satisfy the dominant energy and “radial” non-negative pressures conditions and, roughly speaking, the fact that the matter fields are nonsingular as long as the spacetime metric is nonsingular. This latter property has not been given a precise formulation, as it seems difficult to do so, and serves merely as a heuristic principle—the arguments for collisionless matter and the massless scalar field in [?] providing an example of what it means in practice.

In theorem 1 we have restricted ourselves to space-
times with $S^1 \times S^2$ Cauchy surfaces and have not considered similar spacetimes with $S^3$ Cauchy surfaces. The problem with the $S^3$ case is that there exist two time-like curves on which the symmetry orbits degenerate to points. When we then pass to the quotient of our spacetime by the symmetry group, the field equations on the quotient spacetime are singular on boundary points corresponding to the degenerate orbits. Experience has shown that this degeneracy can have nontrivial consequences on the evolution of the spacetime. For example, in the study of the spherically symmetric asymptotically flat solutions of the Einstein-Vlasov equations, it has been shown that if a solution of these equations develops a singularity, then the first singularity (as measured in a particular time coordinate) is at the center [?]. However, currently it is not known how to decide when a central singularity must occur. In the case of asymptotically flat spherically symmetric solutions of the Einstein equations coupled to a massless scalar field, Christodoulou has shown that naked singularities do form in the center of symmetry for certain initial data (and that they can form nowhere else) [?]. Note that the degeneracy of the orbits in these spacetimes is of the same type that occurs in the spherically symmetric spacetimes with $S^3$ Cauchy surfaces. Similar problems occur in the study of the vacuum spacetimes with $U(1) \times U(1)$ symmetry and having $S^3$ or $S^1 \times S^2$ Cauchy surfaces. Here the dimension of the orbits is non-constant and, consequently, this case is much harder to analyze than the $T^3$ case, which has orbits of constant dimension [?]. The spherically symmetric spacetimes with $S^1 \times S^2$ Cauchy surfaces, having no degenerate orbits, avoid these complications.

It would, of course, be preferable to strengthen theorem 1 by removing the requirement that there exist a CMC Cauchy surface in the spacetime. While such a result seems plausible, the methods currently used are not adequate to cover this more general case. Strengthening our results in this direction is a subject for future research.

Our conventions are those of [?], with the notable exception that trace $H$ of the extrinsic curvature $K_{ab}$ of a spatial hypersurface measures the convergence of the hypersurface to the future. Thus, surfaces with negative $H$ are expanding to the future, while those with positive $H$ are contracting to the future.

II. PROOF OF THEOREM 1

Fix a spacetime $(M, g)$ satisfying the conditions of theorem 1. Both classes of spacetimes considered here (the Einstein-Vlasov and massless scalar field spacetimes) satisfy the dominant energy condition (the Einstein tensor $G_{ab}$ satisfies $G_{ab}x^a x^b \geq 0$ for all future-directed timelike vectors $x^a$ and $w^b$) as well as the timelike-convergence condition (the Ricci tensor satisfies $R_{ab} t^a t^b \geq 0$ for all timelike $t^a$). While the Einstein-Vlasov spacetimes also satisfy the non-negative pressures condition ($G_{ab} x^a x^b \geq 0$ for all spacelike $x^a$), in general the massless scalar field spacetimes do not. However, they do satisfy the weaker "radial" non-negative pressures condition ($G_{ab} x^a w^b \geq 0$ for all spatial vectors $x^a$ perpendicular to the spheres of symmetry). It was shown in [?] that the spherically symmetric spacetimes with $S^3$ or $S^1 \times S^2$ Cauchy surfaces satisfying the dominant energy and the non-negative pressures conditions (or merely the "radial" non-negative pressures condition) have a finite lifetime, i.e., the supremum of the lengths of all causal curves is finite. Therefore, our spacetime $(M, g)$ has a finite lifetime. It then follows immediately from lemma 2 (established in Sec. III) that the volumes of all spatial Cauchy surfaces in $(M, g)$ are bounded above.

Denote the mean curvature of the Cauchy surface $\Sigma$ by $t_0$. This initial data surface must be spherically symmetric. In the case $t_0 \neq 0$, this follows from the uniqueness theorem for such hypersurfaces (see, e.g., theorem 1 of [?]) since if a rotation did not leave $\Sigma$ invariant, we would have a distinct CMC Cauchy surface with identical (nonzero) constant mean curvature. The case where $t_0 = 0$ then follows from the fact that there is a neighborhood $N$ of $\Sigma$ in $M$ such that $N$ can be foliated by CMC hypersurfaces, each having a different CMC, and the fact that those with non-zero CMC must be spherically symmetric. As the theorem has already been proven in the case where $t_0 = 0$ ($\Sigma$ is a maximal hypersurface) [?], we shall take $t_0$ to be negative ($\Sigma$ is expanding to the future). The case where the mean curvature is initially positive follows by a time-reversed argument. As was shown in [?], in a neighborhood of the hypersurface $\Sigma$, the spacetime can be foliated by CMC Cauchy surfaces. Define the scalar field $t$ at any point to be the value of the mean curvature of the CMC hypersurface passing through that point, i.e., so level surfaces of $t$ are CMC hypersurfaces and, in particular, the surface $t = t_0$ is $\Sigma$. A further scalar field $x$ can then be introduced so that the spacetime metric $g$ is given by

$$g = -\alpha^2 dt^2 + A^2((dx + \beta dt)^2 + a^2 d\Omega),$$

(2.1)

where $\Omega$ is the natural unit-metric associated with the spheres of symmetry. The functions $\alpha$, $\beta$, and $A$ depend only on $t$ and $x$ (being spherically symmetric) and are periodic in $x$. The function $\alpha$ depends only on $t$. The fields can be chosen so that $\int \beta(t, x) \, dx = 0$ for each $t$, where the integral is taken over one period of a surface of constant $t$.

It was shown in [?] that the initial data induced on $\Sigma$ can be evolved so that $t$ covers the interval $(-\infty, 0)$ and that, if it can be evolved to the closed interval $(-\infty, 0]$, i.e., a maximal hypersurface is attained, the spacetime can be extended and foliated by CMC spatial hypersurfaces taking on all real values. Therefore, our task is to establish the existence of a maximal hypersurface. To this accomplish this, we establish the existence of upper bounds on $a$, $A$, and their inverses on the interval $[t_0, 0)$. 


We then introduce a new time function \( \tau = f(t) \) by introducing a function \( f \) that allows us to avoid the problem associated with \( t \) being a bad coordinate on maximal hypersurfaces. Once this has been accomplished, theorem 1 will follow from an argument similar to that used in [?].

First, we establish upper bounds on the area radius \( r = aA \), the mass function \( m = \frac{1}{2}r(1 - \nabla^a r \nabla_a r) \), the volume \( V(t) \) of level surfaces of \( t \), and their inverses. That \( r \) and \( m^{-1} \) are bounded above follows from the results of [?]. (Note, \( m \) is positive.) Further, the technique introduced in [?] was used in [?] to show that \( m/r \) is bounded above on \( [t_0, 0) \). Therefore, \( m \) and \( r^{-1} \) are also bounded above on \( [t_0, 0) \). (That is, the mass \( m \) cannot become arbitrarily large and \( r \) cannot become arbitrarily small in this portion of the spacetime. This is nontrivial as both \( m \) and \( r^{-1} \) can become arbitrarily large on unbounded intervals, e.g., near an initial or final singularity.) As we have already established that the volume of all spatial Cauchy surfaces are bounded above, \( V(t) \) is bounded above. Using the fact that \( \partial_t V(t) \) is positive on \( [t_0, 0) \), as these hypersurfaces are everywhere expanding, shows that \( V \) is bounded from below by a positive constant, and hence \( V^{-1} \) is bounded above on \( [t_0, 0) \).

Next, that \( a, \dot{A} \), and their inverses are bounded above on \( [t_0, 0) \) now follows easily from the facts that \( r = aA \),

\[
V(t) = 4\pi \int a^2 A^3 \, dx = 4\pi a^{-1} \int r^3 \, dx, \tag{2.2}
\]

and our upper bounds for \( V, r, \) and their inverses.

Next, we bound \( \alpha' \) using the lapse equation

\[-A^{-3}(A\alpha')' + (K_{ab}K^{ab} + R_{ab}n^a n^b)\alpha = 1, \tag{2.3}\]

where \( K_{ab} \) is the extrinsic curvature of the CMC hypersurface, \( n^a \) is a unit timelike normal to the CMC hypersurface, and \( \alpha \) is the lapse function. \( A \) is a derivative by \( \partial_t \). (This is equation (2.4) in [?].) Using the fact that \( K_{ab}K^{ab} \) is manifestly non-negative and \( R_{ab}n^a n^b \geq 0 \) by the timelike convergence condition, it follows that \( (A\alpha')' \geq -A^3 \). Using the fact that \( A \) is bounded above and integrating in a CMC hypersurfaces, we find that \( (A\alpha')'|_p - (A\alpha')'|_q \geq -C_1 \) for some positive constant \( C_1 \) and any two points \( p \) and \( q \) in the hypersurface. Choosing \( q \) where \( \alpha \) is extremal on the surface (so \( \alpha'(q) = 0 \)) and using the fact \( A^{-1} \) is bounded above shows that \( \alpha' \) is bounded from below. Choosing \( p \) where \( \alpha \) is extremal on the surface (so \( \alpha'(p) = 0 \)) and using the fact \( A^{-1} \) is bounded above shows that \( \alpha' \) is bounded from above. Therefore, there exists a constant \( C_2 \) such that \( |\alpha'| \leq C_2 \). Thus, even if \( \alpha \) is unbounded, it must diverge in a way that is uniform in space. For any two points \( p \) and \( q \) in a CMC hypersurface, \( |\alpha(p) - \alpha(q)| = |\int_p^q \alpha' \, dx| \leq \int_p^q |\alpha'| \, dx \leq \pi C_2 \).

If we knew that \( \alpha \) were bounded above on \( [t_0, 0) \), we could then proceed to argue as in [?]. While such a bound can be established rather easily for fields satisfying the dominant energy and non-negative pressures conditions, such an argument fails for the massless scalar field. The difficulty in establishing an upper bound on \( \alpha \) is linked to the possibility that \( dt \) may be zero on a maximal hypersurface, and thus \( t \) being a bad coordinate. Note that this can only occur if \( K_{ab} = 0 \) everywhere on \( \Sigma \) (i.e., \( \Sigma \) is momentarily static) and \( R_{ab}n^a n^b = 0 \) everywhere on \( \Sigma \). If the non-negative energy condition \( (G_{ab} t^a t^b \geq 0 \) for all timelike \( t^a \)) and non-negative sum-pressures condition \( (G_{ab} (t^a t^b + g_{ab}) \geq 0 \) for all unit-timelike \( t^a \)) are satisfied, then \( R_{ab}n^a n^b = 0 \) implies that \( G_{ab} n^a n^b = 0 \) and, hence, by the Hamiltonian constraint equation, the Ricci scalar curvature of the metric induced on \( \Sigma \) must be zero. However, it is easy to show that there are no such spherically symmetric geometries on \( S^1 \times S^2 \). Thus, the Einstein-Vlasov spacetimes do not admit such surfaces. However, it can be shown that there are massless scalar field spacetimes with such “degenerate” maximal hypersurfaces. To avoid this difficulty, we change our time function to one that is guaranteed to be well-behaved even on a maximal hypersurface with \( dt = 0 \).

Fix any inextendible timelike curve \( \gamma \) that is everywhere orthogonal to the CMC hypersurfaces. The length of the segment of \( \gamma \) between any two CMC hypersurfaces \( \Sigma = (t_1, 0) \) and \( \Sigma = (t_2, 0) \) is then simply \( \int_{t_1}^{t_2} \alpha'(u) \, du \). Using the fact that there is a finite upper bound on the lengths of all timelike curves in our spacetime, the integral

\[
\int_{t_1}^{t_0} \alpha'(u) \, du = \lim_{t_2 \to 0} \int_{t_1}^{t_2} \alpha'(u) \, du \tag{2.4}
\]

must exist, i.e., \( \alpha'(t) \) is integrable on any interval of the form \([t_1, t_0) \). Fix some value \( x_0 \) of \( x \) and consider the function \( \alpha(x_0, t) \). Since \( \alpha' \) is bounded there is a constant \( C \) such that \( \alpha'(x_0, t) \leq \alpha'(x_0) + C \). It follows that \( \alpha(x_0, t) \) is also integrable on any interval of the form \( [t_1, t_0) \). Using this fact, define the function \( f \) on \((\infty, 0) \) by setting

\[
f(\lambda) = \lambda - \int_{\lambda}^{0} \alpha'(u, x_0) \, du. \tag{2.5}
\]

Noting that \( f'(\lambda) = 1 + \alpha'(\lambda, x_0) \) and \( \lim_{\lambda \to 0} f(\lambda) = 0 \), we see that \( f \) is an orientation-preserving diffeomorphism from \((\infty, 0) \) to \((\infty, 0) \). Hence,

\[
\tau = f \circ t \tag{2.6}
\]

is a new time function on our spacetime. Note that \( \partial_t \partial_t \tau = 1 + \alpha(t, x_0) \).

The level surfaces of \( \tau \) clearly coincide with those of \( t \) and so are CMC hypersurfaces. As a consequence the field equations for the geometry and the matter written in terms of \( \tau \) look very similar to those written in terms of \( t \). Using \( \tau \) in place of \( t \), the metric has the same form as before

\[
g = -\tilde{\alpha}^2 d\tau^2 + A^2 [ (dx + \tilde{\beta} d\tau)^2 + \alpha^2 \Omega], \tag{2.7}
\]

where the new lapse function \( \tilde{\alpha} \) is given by

\[
\tilde{\alpha} = \alpha \left( \frac{\partial t}{\partial \tau} \right) = \frac{\alpha}{1 + \alpha(t, x_0)}, \tag{2.8}
\]
and similarly for the new shift \( \hat{\beta} \). In terms of our new coordinates (\( \tau \) replacing \( t \)) and new variables (\( \tilde{\alpha} \) and \( \tilde{\beta} \) replacing \( \alpha \) and \( \beta \), respectively), the field equations are the same as in [2] with \( \partial_t \), replacing \( \partial_\tau \), \( \partial_{\tilde{\alpha}} \) replacing \( \partial_t \), \( \partial_{\tilde{\beta}} \) replacing \( \partial_\tau \), and \( \partial t / \partial \tau \) replacing the right-hand side of equation (2.3). Explicit occurrences of \( t \) in the equations are left unchanged, \( t \) being simply considered as a function of \( \tau \), determined implicitly by equation (2.6).

Using equation (2.8), it is straightforward to show that using the right-hand side of equation (2.7),

\[
\alpha(t, x) = \alpha(0, x). 
\]

Using equation (2.9), it is straightforward to show that using the right-hand side of equation (2.7),

\[
A^{-3}(A \tilde{\alpha}')' + (K_{ab} K^{ab} + R_{ab} n^a n^b) \tilde{\alpha}' + D^k \tilde{\alpha} = 0, 
\]

where \( D^k \equiv \partial^k \) denotes the \( k \)-th partial derivative with respect to \( \tau \). Here \( B_k \) is an expression which is already known to be bounded when we are at the step in the inductive argument to bound \( D^k \tilde{\alpha} \) and \( D^k \tilde{\alpha}' \). In lemma 3.4 of [2], \( D^k \alpha \) was bounded by using the fact that \( t \) was bounded away from zero. The analogous procedure is clearly not possible in the present situation, where \( t \) is tending to zero. This kind of argument was also used in [2] to bound time derivatives of higher order spatial derivatives of \( \alpha \), but that is unnecessary, since such bounds can be obtained directly by differentiating the lapse equation once the time derivatives of \( \alpha \) and \( \alpha' \) have been bounded. The same argument applies here, so all we need to do is to prove the boundedness of \( D^k \tilde{\alpha} \) and \( D^k \tilde{\alpha}' \) using equation (2.10) under the hypothesis that \( B_k \) is bounded. This follows by simply noting that equation (2.10) has the same form for each value of \( k \) and the following lemma.

**Lemma 1.** Consider the differential equation

\[
(u')' = bu + c + du(x_0) 
\]

where \( a, b, c, d, \) and \( u \) are \( 2\pi \)-periodic functions on the real line and \( x_0 \) is a point therein. Suppose that \( a > 0 \), \( b > 0, \) \( d > 0, \) and that \( d \) is not identically zero. Then \( |u| \) and \( |u'| \) are bounded by constants depending only on the quantities

\[
K_1 = \max\{|a^{-1}(x)| \geq 2\pi k, \}
\]

and similarly for the field equations.

Thus, by equation (2.8), \( \tilde{\alpha} \) and \( \tilde{\beta} \) are bounded.

Therefore, by equation (2.8), it is straightforward to show that

\[
A^{-3}(A \tilde{\alpha}')' + (K_{ab} K^{ab} + R_{ab} n^a n^b) \tilde{\alpha}' + D^k \tilde{\alpha} = 0, 
\]

which follows from the fact that \( \alpha \) is known to be bounded when we are at the step in the inductive argument to bound \( \alpha \) and \( \alpha' \).

**Proof.** First, if \( u(x_0) > 2\pi K_1 K_2 \), then \( u > 0 \) everywhere. To see this, suppose otherwise and let \( x_1 \) be a point where \( u \) achieves its maximum, so \( u(x_1) \geq u(x_0) > 2\pi K_1 K_2 \) and let \( x_2 \) be that number such that \( u > 0 \) on \([x_1, x_2]\) and \( u(x_2) = 0 \) (so \( x_1 < x_2 < x_1 + 2\pi \)). Then, on the interval \([x_1, x_2]\), we have \( (au)' \geq c \), from which it follows that \( u' \geq -K_1 K_2 \) on \([x_1, x_2]\). Integrating this and using the fact that \( u(x_2) = 0 \), we find that \( u(x_1) \leq 2\pi K_1 K_2 \), contradicting the fact that \( u(x_1) > 2\pi K_1 K_2 \). Therefore, as \( u \) is everywhere positive, it follows that \( (au)' \geq c \). Integrating this inequality starting (or ending) at a point where \( u' = 0 \) shows that \( |u'| \leq k_1 K_2 \). Integrating equation (2.11) from 0 to \( 2\pi \) and using the fact that \( u \) is positive gives \( u(x_0) \int_0^{2\pi} d(x) \) \( dx \) \( \leq \int_0^{2\pi} |c(x)| \) \( dx \), and hence, \( |u(x_0)| \leq k_2 K_3^{-1} \). Using this and the fact that \( |u'| \leq k_1 K_2 \) shows that \( |u| \leq k_2 K_3^{-1} + 2\pi K_1 K_2 \). Second, if \( u(x_0) < -2\pi K_1 K_2 \), a similar argument shows that \( u \) is everywhere negative and we again obtain the same bounds on \( |u'| \) and \( |u| \). Third, suppose that \( |u(x_0)| \leq 2\pi K_1 K_2 \). If \( \max(u) > 2\pi K_1 K_2(1 + 2\pi K_1 K_3) \), using the inequality \( (au)' \geq c + du(x_0) \), we can argue much as before to see that \( u \) is everywhere positive and again obtain the same bounds on \( |u'| \) and \( |u| \). Similarly, if \( \min(u) < -2\pi K_1 K_2(1 + 2\pi K_1 K_3) \), it follows that \( u \) is everywhere negative and again we recover the same bounds on \( |u'| \) and \( |u| \). Next, if \( |u| \leq 2\pi K_1 K_2(1 + 2\pi K_1 K_3) \) everywhere, \( |u| \) is already bounded, and to bound \( |u'| \), we note that we have bounds for all terms on the right-hand side of equation (2.11), so it suffices to integrate it starting from a point where \( u' \) is zero to bound \( |u'| \).

At this stage, we have indicated how all geometric and matter quantities, expressed in terms of the new time coordinate \( \tau \), can be bounded, together with all their derivatives. In particular, this means that all these quantities are uniformly continuous on any interval of the form \([\tau_i, 0]\), where \( \tau_i \) is finite. It follows that all these quantities have smooth extensions to the interval \([\tau_i, 0]\). Restricting them to the hypersurface \( \tau = 0 \) gives an initial data set for the Einstein-matter equations with zero mean curvature. By the standard uniqueness theorems for the Cauchy problem, the spacetime which, in the old coordinates, was defined on the interval \((\infty, 0)\) is isometric to a subset of the maximal development of this new initial data set. It follows that the original spacetime has an extension which contains a maximal hypersurface.

Lastly, that the foliation is unique now follows from the fact that compact CMC Cauchy surfaces with non-zero mean curvature are unique [2] and that the spacetime is indeed maximal follows from the fact that any spacetime admitting a complete foliation by compact CMC Cauchy surfaces is maximal [2].
III. A BOUND FOR THE VOLUME OF SPACE

It is well known that as we transport an “infinitesimal” spacelike surface \( S \) along the geodesics normal to itself, the ratio \( \nu \) of its volume of to its original volume is governed by the Raychaudhuri equation

\[
\frac{d^2}{dt^2} \nu^{1/3} + \frac{1}{3} (R_{ab} t^a t^b + \sigma_{ab} \sigma^{ab}) \nu^{1/3} = 0, \tag{3.1}
\]

where \( t \) is the proper time measured along the geodesics normal to \( S \), \( R_{ab} \) is the Ricci tensor, and \( \sigma_{ab} \) is the shear tensor associated with the geodesic flow \([?,?,?]\). (This equation is usually written in terms of the divergence of the geodesic flow \( \theta = \nu^{-1} dv/dt \).) On the surface \( S \), \( \nu \) satisfies the initial condition \( \nu = 1 \) and \( dv/dt = -H(p) \), where \( H(p) \) is the trace of the extrinsic curvature of \( S \) at the point \( p \) where the geodesic intersects \( S \). Therefore, if the spacetime satisfies the timelike-convergence condition \( (R_{ab} t^a t^b \geq 0 \) for all timelike \( t^a \)) it follows that as long as \( \nu \) remains non-negative,

\[
\frac{d^2}{dt^2} \nu^{1/3} \leq 0, \tag{3.2}
\]

from which we find that

\[
\nu(t) \leq \left[ 1 - \frac{1}{3} H(p)(t - t_0) \right]^3. \tag{3.3}
\]

This equation bounds the growth of the volume of a local spatial region in the spacetime.

Using this result, it is not difficult to show that, in a spacetime satisfying the timelike-convergence condition, if we fix a Cauchy surface \( \Sigma_0 \) and construct from it a second Cauchy surface \( \Sigma \) by transporting \( \Sigma_0 \) to the future along the flow determined by the geodesics normal to \( \Sigma_0 \), as long as these flow lines do not self-intersect (which will be true if \( \Sigma \) is sufficiently close to \( \Sigma_0 \)), then

\[
\text{vol}(\Sigma) \leq \text{vol}(\Sigma_0) \left[ 1 + \frac{1}{3} \sup_{\Sigma_0} (-H) T \right]^3, \tag{3.4}
\]

where \( \text{vol}(S) \) denotes the three-volume of a Cauchy surface \( S \) and \( T \) is the “distance” between the two surfaces measured by the lengths of the geodesics normal to \( \Sigma_0 \) (which will be independent of which geodesic is chosen by the construction of \( \Sigma \)). Therefore, we have a bound on the volume of \( \Sigma \) in terms of the volume of \( \Sigma_0 \), the extrinsic curvature of \( \Sigma_0 \), and the distance between \( \Sigma_0 \) and \( \Sigma \). Does a similar result hold for more general Cauchy surfaces \( \Sigma \)? For instance, a more general hypersurface \( \Sigma \) may not be everywhere normal to the geodesics from \( \Sigma_0 \), some geodesics normal to \( \Sigma_0 \) may intersect another between \( \Sigma_0 \) and \( \Sigma \), and parts of \( \Sigma \) may lie to the future of \( \Sigma_0 \) while other parts may lie to the past. Can the simple bound given by equation (3.4) be modified to cover these cases? That it can is the subject of the following lemma.

Lemma 2. Fix an orientable globally hyperbolic spacetime \((M, g_{ab})\) satisfying the timelike-convergence condition \( (R_{ab} t^a t^b \geq 0 \) for all timelike \( t^a \)) and a smooth spacelike Cauchy surface \( \Sigma_0 \) therein. Then, for any smooth spacelike Cauchy surface \( \Sigma \),

\[
\text{vol}(\Sigma) \leq \text{vol}(\Sigma_0) \left[ 1 + \frac{1}{3} \sup_{\Sigma_0} (-H) \Delta(\Sigma_0, \Sigma) \right]^3, \tag{3.5}
\]

where \( \text{vol}(S) \) denotes the three-volume of a Cauchy surface \( S \), \( H \) is the trace of the extrinsic curvature of \( \Sigma_0 \) (using the convention that \( H \) measures the convergence of the future-directed timelike normals to a spacelike surface), and \( \Delta(\Sigma_0, \Sigma) \) is the least upper bound to the lengths of causal curves connecting \( \Sigma_0 \) to \( \Sigma \) (either future or past directed). Further, for any Cauchy surface \( \Sigma \subset D^+(\Sigma_0) \),

\[
\text{vol}(\Sigma) \leq \text{vol}(\Sigma_0) \left[ 1 + \frac{1}{3} \sup_{\Sigma_0} (-H) \Delta(\Sigma_0, \Sigma) \right]^3. \tag{3.6}
\]

Note that for \( p, q \in M \), \( \Delta(p, q) \) is not quite the distance function \( d(p, q) \) as used in [?] as \( d(p, q) = 0 \) if \( q \in J^-(p) \). Instead, \( \Delta(p, q) \) does not distinguish between future and past: \( \Delta(p, q) = \Delta(q, p) = d(p, q) + d(q, p) \).

From lemma 2, we see that for a spacetime satisfying the timelike-convergence condition, possessing compact Cauchy surfaces, and having a finite lifetime (in the sense that \( d(p, q) \) [equivalently \( \Delta(p, q) \)] is bounded above by a constant independent of \( p \) and \( q \)), then the volume of a Cauchy surface therein cannot be arbitrarily large. Further, we see that if the spacetime admits a maximal Cauchy surface \( \Sigma_0 \) (\( H = 0 \) thereon), we reproduce the result that there is no other Cauchy surface having volume larger than \( \Sigma_0 \) (though there may be surfaces of equal volume) [?].

In the following, \( df \) denotes the derivative map associated with a differentiable map \( f \) between manifolds. When viewed as a pull-back, we denote \( df \) by \( f^* \) and, when viewed as a push-forward, we denote \( df \) by \( f_* \). For a map \( f : A \to B \), \( f(A) \) denotes the image of \( A \) in \( B \). Lastly, \( A \setminus B \) denotes the set of elements in \( A \) that are not in \( B \).

A. Proof of lemma 2

To begin the proof of lemma 2, for each point \( p \in \Sigma_0 \), let \( \gamma_p \) denote the unique inextendible geodesic containing \( p \) and intersecting \( \Sigma_0 \) orthogonally. Parameterize \( \gamma_p \) by \( t \) so that the tangent vector to \( \gamma_p \) is future-directed unit-timelike and \( \gamma_p(0) = p \). Then, define the map \( f : \Sigma_0 \to \Sigma \), by

\[
f(p) = \gamma_p \cap \Sigma. \tag{3.7}
\]

Note that for each \( p \in \Sigma_0 \), \( f \) is well defined since \( \gamma_p \) intersects \( \Sigma \) at precisely one point as \( \Sigma \) is a spacelike Cauchy surface for the spacetime.