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Rigorous Bounds for Critical Temperatures of $O(N)$ Spin Models by Transformations of Block Spin Type

K. R. Ito *

*Department of Mathematics and Physics, Setunan University,
Neyagawa 572, Ikeda, Japan*

T. Kugo and H. Tamura †

*Department of Mathematics, Faculty of Science, Kanazawa University,
Kanazawa 920-11, Japan*

Abstract

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We obtain new upper bounds of critical temperatures of N -vector (Heisenberg) models. We apply a transformation of block spin type to random walk representations of the spin models, which was developed by Fröhlich et al more than a decade ago. Though the transformation is applied just one time, the upper bounds are considerably improved.

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*also at : Division of Mathematics, College of Human and Environmental Studies,
Kyoto University, Kyoto 606, Japan
E-mail : ito@kurims.kyoto-u.ac.jp

†E-mail: tamura@kappa.s.kanazawa-u.ac.jp

I. INTRODUCTION

We develop a method of finding an upper bound for the inverse temperature β below which the system exhibits exponential clustering. To be concrete, the model we investigate is the ν dimensional Heisenberg model with $O(N)$ symmetry (N-vector model):

$$\langle \cdot \rangle \equiv \frac{1}{Z_L(\beta)} \int (\cdot) \exp\left[\frac{\beta}{2} \sum_{|x-y|=1} \mathbf{S}(x) \cdot \mathbf{S}(y)\right] \prod d\mathbf{S}(x), \quad (1.1)$$

where $\mathbf{S}(x) \in S^{N-1}$ for all lattice points $x \in L(\subset \mathbf{Z}^\nu)$, $\beta > 0$ is the inverse temperature and the integration $d\mathbf{S}(x)$ is over the $N - 1$ dimensional sphere S^{N-1} . The normalization constant $Z_L(\beta)$ is chosen so that $\langle 1 \rangle = 1$.

Among the latest results on this model, the following is most related to our work:

Theorem 1 [1,3,4]. (1) *If $\beta < (\frac{2\nu-1}{N} + \frac{1}{N+2})^{-1}$, there exist strictly positive constants m and C such that $\langle \mathbf{S}(0) \cdot \mathbf{S}(x) \rangle \leq C \exp[-m|x|]$.*

(2) *The free energy $\alpha(\beta) = \lim_{L \uparrow \mathbf{Z}^\nu} \log(Z_L)/|L|$ is represented in terms of the truncated correlation functions of random walks. The free energy is analytic in β in the region $\Omega(N, \nu) = \{\beta; |\beta| < \beta_c\}$ where $\beta_c \equiv (1 - \varepsilon)N/2\nu$ and $\varepsilon > 0 \rightarrow 0$ as $\nu \rightarrow \infty$ uniformly in N .*

These results were obtained by the random walk representations of spin models developed in [1,2]. It seemed hard to improve these bounds since it implies that we must estimate accurately the number of the random walks generated by the models. This may be done by a technique which is similar to the block-spin transformation, but it also seemed hard.

We have succeeded in the first step of the transformations which reduce random walks to simpler ones. In fact we can calculate in a closed form the effects of the smallest loops of type $i \rightarrow i + e_\mu \rightarrow i \rightarrow i + e_\mu \rightarrow \dots \rightarrow i$, where $i \in \mathbf{Z}^\nu$ and e_μ is the unit vector in the μ direction. This means that we can eliminate these small loops from random walks, so this technique may be called a trimming or a block spin transformation of random walks (at distance scale 1). Thus we are left with the random walks on the dilute lattice $\mathbf{Z}_e^\nu =$

$\{(i_1, \dots, i_\nu) \in \mathbf{Z}^\nu; |i| = \text{even}\}$, where $|i| = |i_1| + \dots + |i_\nu|$. The numerical estimate using this idea considerably improves the previous results.

II. MATRIX INVERSE AND RANDOM WALKS

Let L be a finite subset of $\mathbf{Z}^\nu$ and let J, Λ be the matrices with index set L whose components satisfy for any $i, j \in L$

$$J_{ij} \geq 0,$$

$$\Lambda_{ij} = \lambda_i \delta_{ij}, \quad \text{with some } \lambda_i \neq 0.$$

Let us recall the fundamental facts on random walk representations of the inverse and the determinant of the matrix $\Lambda - J$ in the following lemmas. These lemmas are the base of our argument of the present letter. Let ω be a walk on L , i.e., ω is a sequence of ordered pairs;

$$\omega = \{(i_1, i_2), (i_2, i_3), \dots, (i_{M-1}, i_M) | i_1, \dots, i_M \in L, \forall k \in \{1, \dots, M\}\}. \quad (2.1)$$

Let $n(k, \omega)$ be the number of times ω hits $k \in L$, i.e., $n(k, \omega)$ is the number of times k appears in $\{i_1, \dots, i_M\}$. We call M in (2.1) the length of ω , and write it $|\omega|$. Then the equality $|\omega| = \sum_{k \in L} n(k, \omega)$ holds. We use $||$ for the number of ordered pair which ω consists of, i.e., $||\omega|| = |\omega| - 1$. The first lemma is a consequence of the expansion to Neumann series.

Lemma 1 [1].

$$(\Lambda - J)_{ij}^{-1} = \sum_{\omega: i \rightarrow j} \left(\prod_{s \in \omega} J_s \right) \prod_{k \in L} \lambda_k^{-n(k, \omega)}$$

If the right hand side converges absolutely, the left hand side exists and the equality holds.

In Lemma 1, we mean by $s \in \omega$ that s is one of the ordered pairs in ω . We call s a step of ω .

Let A_i be the set of walks beginning and ending at the same point $i \in L$. We put $A = \bigcup_{i \in L} A_i$. We say that $\omega_1 = \{(i_1, i_2), (i_2, i_3), \dots, (i_{N-1}, i_1)\} \in A$ and $\omega_2 =$

$\{(j_1, j_2), (j_2, j_3), \dots, (j_{M-1}, j_1)\} \in A$ are equivalent if and only if ω_1 is equal to ω_2 as a set and the order of steps of ω_2 is a cyclic permutation of the order of steps of ω_1 . The equivalence classes made by the equivalence relation is called loops. Note that we have $|\omega| = \|\omega\|$ for a loop ω . For a walk ω (loop ω), we use the following notation:

$$J^\omega \equiv \prod_{s \in \omega} J_s, \quad J^\omega \equiv \prod_{s \in \omega} J_s.$$

For two walks $\omega_1 = \{(i_1, i_2), (i_2, i_3), \dots, (i_{Q-1}, i_Q)\}$ and $\omega_2 = \{(j_1, j_2), (j_2, j_3), \dots, (j_{R-1}, j_R)\}$, satisfying $i_Q = j_1$, $\omega_1 \cdot \omega_2$ denotes the walk $\{(i_1, i_2), (i_2, i_3), \dots, (i_{Q-1}, i_Q), (j_1, j_2), \dots, (j_{R-1}, j_R)\}$. Then the following lemma holds.

Lemma 2 [1].

$$\det(\Lambda - J)^{-1} = \left(\prod_{i \in L} \lambda_i \right)^{-1} \exp \left\{ \sum_{\omega} \frac{1}{\tau_{\omega}} J^{\omega} \prod_{i \in L} \lambda_i^{-n(i, \omega)} \right\}$$

The summation on the exponential is taken over all loops. Here τ_{ω} is the maximum number k with which representative ω of ω is expressed by $\omega = \omega_0^k$ with some $\omega_0 \in A$.

III. RANDOM WALK REPRESENTATION ON $\mathbf{Z}_e^{\nu}$

Now we consider a model of classical statistical mechanics of lattice spin system having nearest neighbor coupling only. Let J be the matrix whose components are

$$J_{ij} = \begin{cases} \beta & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{S}_i = (S_i^{(1)}, \dots, S_i^{(N)})$ be a vector-valued spin at site $i \in L$. The Hamiltonian of our lattice system is

$$H \equiv -\frac{1}{2} \sum_{i, j \in L} \sum_{\alpha=1, \dots, N} S_i^{(\alpha)} J_{ij} S_j^{(\alpha)}. \quad (3.1)$$

For a function of spins F , the unnormalized expectation is given by

$$[F] \equiv \int_{\mathbf{R}^{|L|N}} F e^{-H} \prod_{i \in L} g_i(\mathbf{S}_i^2) dS_i^{(1)} \dots dS_i^{(N)}.$$

The normalized expectation is

$$\langle F \rangle \equiv [F]/Z,$$

where $Z = [1]$ is the partition function. Here g_i are assumed to fall off faster than exponentially at ∞ .

We represent g_i by the Fourier transformation

$$g_i(\mathbf{S}_i^2) = \int_{\Gamma} \tilde{g}_i(a_i) e^{-\sqrt{-1}a_i \mathbf{S}_i^2} da_i, \quad (3.2)$$

where $\Gamma = \{a \in \mathbf{C}; \Im a = -\lambda\}$ and the positive number λ will be chosen sufficiently large, and specified later. Using (3.2) and performing the integration with respect to the spin variables $S_i^{(\alpha)}$, we get

$$\begin{aligned} Z &= \int_{\mathbf{R}^{|L|N} \times \Gamma^{|L|}} \exp \left(-H - \sqrt{-1} \sum_{j \in L} a_j \mathbf{S}_j^2 \right) \prod_{i \in L} (d\mathbf{S}_i \tilde{g}_i(a_i) da_i) \\ &= (2\pi)^{|L|N/2} \int_{\Gamma^{|L|}} \det^{-N/2} (2\sqrt{-1}A - J) \prod_{i \in L} \tilde{g}_i(a_i) da_i, \end{aligned} \quad (3.3)$$

where A is the diagonal matrix whose components is $a_i \delta_{ij}$ for each $i, j \in L$.

We now apply transformations of renormalization group type to random walks. To do so, we divide L in two parts, $L = L_o \cup L_e$, where $L_o \equiv \{i \in L | i : \text{odd}\}$ and $L_e \equiv \{i \in L | i : \text{even}\}$, and i is called odd (resp. even) if $|i| \equiv \sum_{\mu=1}^{\nu} |i_{\mu}|$ is odd (resp. even).

We integrate out $\{a_i\}_{i \in L_e}$ first. Let us define the matrices B, C, K ;

$$\begin{aligned} B_{ij} &= 2\sqrt{-1}a_i \delta_{ij}, \\ C_{kl} &= 2\sqrt{-1}a_k \delta_{kl}, \\ K_{ik} &= \begin{cases} \beta & \text{if } |i - k| = 1 \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (3.4)$$

for any $i, j \in L_o, k, l \in L_e$. With these notation, the matrix in (3.3) can be represented as

$$2\sqrt{-1}A - J = \begin{pmatrix} B & -K \\ -K^T & C \end{pmatrix}. \quad (3.5)$$

The following identity plays the key role in the present work:

$$\begin{pmatrix} B & -K \\ -K^T & C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -K^T B^{-1} & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & C - K^T B^{-1} K \end{pmatrix} \begin{pmatrix} 1 & -B^{-1} K \\ 0 & 1 \end{pmatrix}. \quad (3.6)$$

Therefore we have the following identities:

$$\det(2\sqrt{-1}A - J) = \det B \det(C - K^T B^{-1} K), \quad (3.7)$$

$$\begin{pmatrix} B & -K \\ -K^T & C \end{pmatrix}^{-1} = \begin{pmatrix} 1 & B^{-1} K \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & (C - K^T B^{-1} K)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K^T B^{-1} & 1 \end{pmatrix}. \quad (3.8)$$

We put

$$\begin{aligned} (K_2)_{kl} &= (K^T B^{-1} K)_{kl} (1 - \delta_{kl}), \\ M_{kl} &= \delta_{kl} M_k \end{aligned} \quad (3.9)$$

with

$$M_k = C_{kk} - (K^T B^{-1} K)_{kk},$$

where $k, l \in L_e$. Then

$$C - K^T B^{-1} K = M - K_2 \quad (3.10)$$

holds. Note that $|(K_2)_{kl}| \leq \beta^2/\lambda$ for $|k - l| \leq 2$, $(K_2)_{kl} = 0$ for $|k - l| > 2$, and $|M_k| \geq 2\lambda - \nu\beta^2/\lambda$. So, Lemma 1 and Lemma 2 are applicable to $M - K_2$ if λ is so large that $2\nu^2\beta^2/(2\lambda^2 - \nu\beta^2) < 1$ holds.

Applying Lemma 2 to $\det^{-N/2}(M - K_2)$, we get

$$\begin{aligned} Z &= (2\pi)^{|L|N/2} \int_{\Gamma|L_o|} \det^{-N/2} B \left(\prod_{j \in L_o} \tilde{g}_j(a_j) da_j \right) \int_{\Gamma|L_e|} \det^{-N/2}(M - K_2) \prod_{k \in L_e} \tilde{g}_k(a_k) da_k \\ &= (2\pi)^{|L|N/2} \int_{\Gamma|L_o|} \left(\prod_{j \in L_o} (2\sqrt{-1}a_j)^{-N/2} \tilde{g}_j(a_j) da_j \right) \\ &\quad \times \int_{\Gamma|L_e|} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2} \right)^n \sum_{\nu_1, \dots, \nu_n} \left(\prod_{l=1}^n \frac{K_2^{\nu_l}}{\tau_{\nu_l}} \right) \prod_{k \in L_e} M_k^{-n(k, \nu_1, \dots, \nu_n)} \tilde{g}_k(a_k) da_k, \end{aligned} \quad (3.11)$$

where $n(k, \nu_1, \dots, \nu_n) = n(k, \nu_1) + \dots + n(k, \nu_n) + N/2$, $K_2^{\nu_l} = \prod_{s \in \nu_l} (K_2)_s$. It should be noted that ν denotes a loop on L_e , not on L in (3.11).

Now we concentrate to the Heisenberg model, i.e.,

$$g_i(\mathbf{S}^2) = \delta(\mathbf{S}^2 - 1) \quad (3.12)$$

for all $i \in L$. In this case $\tilde{g}(a) = e^{\sqrt{-1}a}/2\pi$ and we have

$$\int_{\Gamma} \frac{\tilde{g}(a)}{(2\sqrt{-1}a - x)^n} da = \frac{e^{x/2}}{2^n(n-1)!} \quad (3.13)$$

Here and hereafter we use the notation $n!$ not only for integers but also for half integers n (namely $n!$ stands for $\Gamma(n+1)$). From (3.9) and (3.13), we get

$$\prod_{k \in L_e} \int_{\Gamma} da_k \tilde{g}(a_k) M_i^{-n(k, \nu_1, \dots, \nu_n)} = \prod_{k \in L_e} \frac{\exp \left(\sum_{\substack{j \in L_o \\ |k-j|=1}} \frac{\beta^2}{4\sqrt{-1}a_j} \right)}{2^{n(k, \nu_1, \dots, \nu_n)} (n(k, \nu_1, \dots, \nu_n) - 1)!}$$

Therefore, (3.11) can be written as

$$\begin{aligned} Z &= (2\pi)^{|L|N/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2} \right)^n \sum_{\nu_1, \dots, \nu_n} \\ &\quad \times \int_{\Gamma^{|L_o|}} \left(\prod_{l=1}^n \frac{K_2^{\nu_l}}{\tau_{\nu_l}} \right) \left(\prod_{j \in L_o} \exp \left(\sqrt{-1}a_j + \frac{\nu\beta^2}{2\sqrt{-1}a_j} \right) \frac{1}{(2\sqrt{-1}a_j)^{N/2}} \frac{da_j}{2\pi} \right) \\ &\quad \times \prod_{k \in L_e} (2^{n(k, \nu_1, \dots, \nu_n)} (n(k, \nu_1, \dots, \nu_n) - 1)!)^{-1}. \end{aligned} \quad (3.14)$$

For any nearest neighbor random walk ω (resp. random loop ω) on the original lattice L starting and ending at even sites, we define a random walk $\iota(\omega)$ (resp. random loop $\iota(\omega)$) on the even lattice L_e such that each step of $\iota(\omega)$ (resp. $\iota(\omega)$) is a composition of successive two steps of ω (resp. ω). For example if $\omega = \{(i_0, i_1), (i_1, i_2), \dots, (i_{2m-1}, i_{2m})\}$ where $i_0, i_{2m} \in L_e, |i_j - i_{j-1}| = 1, j = 1, \dots, 2m$, $\iota(\omega) = \{(i_0, i_2), (i_2, i_4), \dots, (i_{2m-2}, i_{2m})\}$. We used the symbol v (resp. ν) for a random walk (resp. a random loop) on L_e . By $v \subset \omega$ (resp. $\nu \subset \omega$), we mean $\iota(\omega) = v$ (resp. $\iota(\omega) = \nu$). For one random walk v (resp. random loop ν) on L_e , there may exist more than one random walks ω satisfying $v \subset \omega$ (resp. random loops ω satisfying $\nu \subset \omega$) on L . Applying (3.4) to (3.14), we get

$$\begin{aligned} Z &= (2\pi)^{|L|N/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2} \right)^n \sum_{\nu_1, \dots, \nu_n} \sum_{\omega_1 \supset \nu_1, \dots, \omega_n \supset \nu_n} \left(\prod_{l=1}^n \frac{\beta^{2\|\nu_l\|}}{\tau_{\nu_l}} \right) \\ &\quad \times \prod_{j \in L_o} \int_{\Gamma} \exp \left(\sqrt{-1}a_j + \frac{\nu\beta^2}{2\sqrt{-1}a_j} \right) \frac{1}{(2\sqrt{-1}a_j)^{n(j, \omega_1, \dots, \omega_n)}} \frac{da_j}{2\pi} \\ &\quad \times \prod_{k \in L_e} 2^{-n(k, \nu_1, \dots, \nu_n)} (n(k, \nu_1, \dots, \nu_n) - 1)!)^{-1}, \end{aligned} \quad (3.15)$$

where $\|\psi_l\|$ is the number of steps of ψ_l . Using (3.13) and the expansion to Taylor series, the integration with respect to a_j in (3.15) can be represented as

$$\int_{\Gamma} \exp \left[\sqrt{-1}a + \frac{(\sqrt{\nu}\beta)^2}{2\sqrt{-1}a} \right] \frac{da}{2\pi(2\sqrt{-1}a)^n} = \sum_{k=0}^{\infty} \frac{(\sqrt{\nu}\beta)^{2k}}{k!} \frac{1}{2^{n+k}(n+k-1)!} \quad (3.16)$$

$$\equiv f_n(\sqrt{\nu}\beta). \quad (3.17)$$

The function f_n is represented by the n -th modified Bessel function:

$$f_n(x) = \frac{I_{n-1}(\sqrt{2}x)}{2(\sqrt{2}x)^{n-1}}. \quad (3.18)$$

Thus we have

$$\begin{aligned} Z &= (2\pi)^{|L|N/2} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2} \right)^n \sum_{\psi_1, \dots, \psi_n} \sum_{\varphi_1 \supset \psi_1, \dots, \varphi_n \supset \psi_n} \left(\prod_{l=1}^n \frac{\beta^{2\|\psi_l\|}}{\tau_{\psi_l}} \right) \\ &\quad \times \prod_{j \in L_o} f_{n(j, \varphi_1, \dots, \varphi_n)}(\sqrt{\nu}\beta) \\ &\quad \times \prod_{k \in L_e} 2^{-n(k, \psi_1, \dots, \psi_n)} (n(k, \psi_1, \dots, \psi_n) - 1)!^{-1}. \end{aligned}$$

Now let us estimate the correlation function

$$\langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle = \frac{1}{Z} \int_{\mathbf{R}^{|L|N}} S_p^{(\alpha)} S_q^{(\alpha)} e^{-H} \prod_{i \in L} g(\mathbf{S}_i^2) dS_i^{(1)} \dots dS_i^{(N)}.$$

Using Fourier transformation of $g(\mathbf{S}_i^2)$ and integrating with respect to $S_i^{(\alpha)}$, we get

$$\langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle = \frac{(2\pi)^{|L|N/2}}{Z} \int_{\Gamma^{|L|}} (2\sqrt{-1}A - J)_{pq}^{-1} \det^{-N/2}(2\sqrt{-1}A - J) \prod_{i \in L} \tilde{g}(a_i) da_i.$$

First, we consider the case of even k, l . Using (3.5), (3.6), (3.7) and (3.10), we get

$$\langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle = \frac{(2\pi)^{|L|N/2}}{Z} \int_{\Gamma^{|L|}} (M - K_2)_{pq}^{-1} \det^{-N/2} B \det^{-N/2} (M - K_2) \prod_{i \in L} \tilde{g}(a_i) da_i,$$

where $\tilde{g}(a) = e^{\sqrt{-1}a}/2\pi$. From Lemma 1 and Lemma 2, it follows that

$$\begin{aligned} \langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle &= \frac{(2\pi)^{|L|N/2}}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2} \right)^n \sum_{v: p \rightarrow q} \sum_{\psi_1, \dots, \psi_n} \int_{\Gamma^{|L_o|}} \left(\prod_{j \in L_o} \frac{\tilde{g}(a_j)}{(2\sqrt{-1}a_j)^{N/2}} da_j \right) \\ &\quad \times \int_{\Gamma^{|L_e|}} K_2^v \left(\prod_{l=1}^n \frac{1}{\tau_{\psi_l}} K_2^{\psi_l} \right) \left(\prod_{k \in L_e} M_k^{-n(k, \psi_1, \dots, \psi_n, v)} \tilde{g}(a_k) da_k \right) \end{aligned}$$

Integrating with respect to a_k ($k \in L_e$) by the use of (3.13) and (3.17), we get

$$\begin{aligned}
\langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle &= \frac{(2\pi)^{|L|N/2}}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2}\right)^n \sum_{\mathbf{v}_1, \dots, \mathbf{v}_n} \sum_{\omega_1 \supset \mathbf{v}_1, \dots, \omega_n \supset \mathbf{v}_n} \sum_{v: p \rightarrow q} \sum_{\omega \supset v} \left(\beta^{2\|\mathbf{v}\|} \prod_{l=1}^n \frac{\beta^{2\|\mathbf{v}_l\|}}{\tau_{\omega_l}} \right) \\
&\quad \times \prod_{j \in L_o} f_{n(j, \omega_1, \dots, \omega_n, \omega)}(\sqrt{\nu}\beta) \\
&\quad \times \prod_{k \in L_e} 2^{-n(k, \mathbf{v}_1, \dots, \mathbf{v}_n, v)} ((n(k, \mathbf{v}_1, \dots, \mathbf{v}_n, v) - 1)!)^{-1}, \tag{3.19}
\end{aligned}$$

as in the case of Z .

IV. SPLITTING OF WALKS AND CORRELATION INEQUALITIES

We set

$$\varphi_n(x) = \frac{f_{n+1}(x)}{f_n(x)}. \tag{4.1}$$

Since the inequality $\varphi_n \geq \varphi_{n+1}$ holds (for a proof, see Appendix), we have

$$f_{n(j, \omega_1, \dots, \omega_n, \omega)}(\sqrt{\nu}\beta) \leq \prod_{m=0}^{n(j, \omega)-1} \varphi_{(N+2m)/2}(\sqrt{\nu}\beta) f_{n(j, \omega_1, \dots, \omega_n)}(\sqrt{\nu}\beta) \tag{4.2}$$

$$\leq \left(\varphi_{N/2}(\sqrt{\nu}\beta) \right)^{n(j, \omega)} f_{n(j, \omega_1, \dots, \omega_n)}(\sqrt{\nu}\beta). \tag{4.3}$$

Note also the inequalities

$$\begin{aligned}
&2^{-n(k, \mathbf{v}_1, \dots, \mathbf{v}_n, v)} ((n(k, \mathbf{v}_1, \dots, \mathbf{v}_n, v) - 1)!)^{-1} \\
&\leq 2^{-n(k, \mathbf{v}_1, \dots, \mathbf{v}_n)} ((n(k, \mathbf{v}_1, \dots, \mathbf{v}_n) - 1)!)^{-1} \prod_{m=0}^{n(k, v)-1} (N + 2m)^{-1} \tag{4.4}
\end{aligned}$$

$$\leq 2^{-n(k, \mathbf{v}_1, \dots, \mathbf{v}_n)} ((n(k, \mathbf{v}_1, \dots, \mathbf{v}_n) - 1)!)^{-1} N^{-n(k, v)}. \tag{4.5}$$

Substituting (4.3) and (4.5) to (3.19), we obtain

$$0 \leq \langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle \leq \sum_{\omega: p \rightarrow q} \sum_{\omega \supset v} \frac{\beta^{2\|\mathbf{v}\|}}{N^{|\mathbf{v}|}} \left(\varphi_{N/2}(\sqrt{\nu}\beta) \right)^{|\omega| - |\mathbf{v}|}. \tag{4.6}$$

Now let us estimate the number of walks ω on L constructed via walks v on L_e . Each step of v , say (k, l) is corresponding to two steps of $\omega(\supset v)$, say (k, j) and (j, l) where $k, l \in L_e, j \in L_o$ satisfy $k \neq l, |k - j| = |j - l| = 1$. Hence, the number of such walks ω of $2m$ steps, which come from walks on L_e of m steps, is not greater than $(2\nu(2\nu - 1))^m$. Thus setting

$$\tau_0 = \frac{2\nu(2\nu-1)\beta^2}{N} \varphi_{N/2}(\sqrt{\nu}\beta), \quad (4.7)$$

we have

$$0 \leq \langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle \leq \frac{1}{N} \sum_{m \geq |p-q|/2} \tau_0^m \quad (4.8)$$

for $p, q \in L_e$ ($p \neq q$). Similarly, for $r \in L_o, q \in L_e$, we have

$$0 \leq \langle S_r^{(\alpha)} S_q^{(\alpha)} \rangle \leq \frac{\beta \varphi_{N/2}(\sqrt{\nu}\beta)}{N} \sum_{\substack{k \in L_e \\ |r-k|=1}} \sum_{m \geq |k-q|/2} \tau_0^m \quad (4.9)$$

and for $p \in L_e, r \in L_o$ and for $r, s \in L_o$ ($r \neq s$), we can obtain similar bounds which are left to the reader.

At this point, we remark that τ_0^m in the inequalities (4.8) and (4.9) can be replaced by $\tau_0 \tau_1^{m-1}$, where

$$\tau_1 = \left[\frac{\varphi_{(N+2)/2}(\sqrt{\nu}\beta)}{N+2} + \frac{2\nu-2}{N} \varphi_{(N+2)/2}(\sqrt{\nu}\beta) + \frac{(2\nu-1)^2}{N} \varphi_{N/2}(\sqrt{\nu}\beta) \right] \beta^2. \quad (4.10)$$

In fact, assume that the walk has arrived at $l \in L_e$ through the steps $(k, j), (j, l)$ where $k, l \in L_e, j \in L_o, |k-l|=2$ and $|k-j|=|j-l|=1$. There exist $2\nu-1$ ways to leave l for points $k' \in L_e$ ($|k'-l|=2$) through j , and one of them revisits k . As is seen from ineqs. (4.2) and (4.4), these steps yield $(2\nu-2)\varphi_{(N+2)/2}(\sqrt{\nu}\beta)/N$ and $\varphi_{(N+2)/2}(\sqrt{\nu}\beta)/(N+2)$, respectively.

In these inequalities, the right hand sides are absolutely convergent if $\tau_0 < 1$ or $\tau_1 < 1$. (Of course the latter gives better bounds.) Since β_c is the infimum over values of β which $\langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle$ do not tend to zero exponentially, as $|p-q| \rightarrow \infty$, we obtain the following theorem.

Theorem 2.

In the ν -dimensional Heisenberg model (3.1) and (3.12) , the critical inverse temperature β_c satisfies

$$\begin{aligned}\beta_c &\geq \inf \{ \beta \geq 0 \mid \tau_1 \geq 1 \} \\ &\geq \inf \{ \beta \geq 0 \mid \tau_0 \geq 1 \} .\end{aligned}$$

As a simple consequence, we get the following corollary which yields bounds better than those in [1] ;

Corollary 1

$$\begin{aligned}\beta_c &\geq \left[\frac{1}{(N+2)^2} + \frac{2\nu-2}{N(N+2)} + \frac{(2\nu-1)^2}{N^2} \right]^{-1/2} \\ &\geq \frac{N}{\sqrt{2\nu(2\nu-1)}} .\end{aligned}$$

Proof

Since

$$2nf_{n+1}(\sqrt{\nu}\beta) = \sum_{k=0}^{\infty} \frac{(\sqrt{\nu}\beta)^{2k}}{k!} \frac{1}{2^{n+k}(n+k-1)!} \frac{n}{n+k} \leq f_n(\sqrt{\nu}\beta),$$

we have

$$\varphi_n(\sqrt{\nu}\beta) \leq \frac{1}{2n}. \quad \square$$

V. DISCUSSIONS

We applied a block-spin like transformation to random walk representations for the spin models. In this paper, we calculated the first step of the transformation. As is seen in Table I in which our results are compared with previous ones, our method improves the previous rigorous bounds in [1] by 15% ($\nu=2$) to 8 % ($\nu=4$), and our results are equal to the real values (Monte Carlo simulations) within the differences of 15 % ($\nu=2$) to 5 % ($\nu=4$) (this also depends on N).

Let us consider the point of our transformation. First of all we took the full effects of the smallest loops of type $k \rightarrow k + e_\mu \rightarrow k \rightarrow k + e_\mu \cdots \rightarrow k$, $k \in L_e$, and $i \rightarrow i + e_\mu \rightarrow i \rightarrow i + e_\mu \cdots \rightarrow i$, $i \in L_o$ (more than 1 round in this case). In fact we could integrate over $a_i, i \in L_o$ in a closed form, and this effect is completely absorbed by the modified Bessel functions. This is a trimming of random walks at distance scale 1, and is the first step of the transformations of block spin type. (See [6,7] on the trimming of random walks.)

Finally as for the smallest loops of type $i \rightarrow i + e_\mu \rightarrow i$, $i \in L_o$, which we did not eliminate, we take the effect of the second visits into our consideration. This significantly improves Theorem 1(1). (This idea is extended to extreme in [5].)

Random walks cross with themselves with probability one in lower dimensions. Therefore our method must be effective especially in lower dimensions if we can continue the transformations. However the direct extension of the present method seems difficult. We need good ideas! A similar but slightly complicated idea is applied in [6] to improve β_c of the free energy in Theorem 1 (2), in which one can continue the transformations of random walks of the block spin type.

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TABLES

TABLE I. Comparison of our results with MC Simulations

		$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$			$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$
$N = 1$	β_0	0.7500	0.3000	0.1875	0.1364	$N = 3$	β_0	1.8753	0.8333	0.5357	0.3947
	β_1	0.9487	0.3198	0.1945	0.1399		β_1	2.5725	0.9232	0.5694	0.4122
	β_2	0.8483	0.3057	0.1889	0.1369		β_2	2.8275	0.9575	0.5818	0.4185
	β_3	1.2705	0.3415	0.2018	0.1435		β_3	3.5581	1.0232	0.6053	0.4305
	β_c	∞	0.4408	0.2217	0.1503		β_c	∞		0.6930	
$N = 2$	β_0	1.3000	0.5714	0.3636	0.2667	$N = 4$	β_0	2.4000	1.0909	0.7059	0.5271
	β_1	1.7889	0.6247	0.3831	0.2767		β_1	3.3282	1.2184	0.7544	0.5472
	β_2	1.8272	0.6287	0.3841	0.2771		β_2	3.8275	1.2885	0.7804	0.5605
	β_3	2.4632	0.6838	0.4038	0.2871		β_3	4.6141	1.3606	0.8063	0.5738
	β_c	∞	1.06	0.4542			β_c	∞		0.9360	0.6090

$\beta_0 = \left[\frac{2\nu-1}{N} + \frac{1}{N+2} \right]^{-1}$: the lower bound obtained in [1].

$\beta_1 = \left[\frac{1}{(N+2)^2} + \frac{2\nu-2}{N(N+2)} + \frac{(2\nu-1)^2}{N^2} \right]^{-1/2}$

$\beta_2 = \inf\{\beta \geq 0 \mid \tau_0 \geq 1\}$

$\beta_3 = \inf\{\beta \geq 0 \mid \tau_1 \geq 1\}$

β_c : data obtained by Monte Carlo simulations except for that of the 2 dimensional Ising model which is exactly soluble. Data are taken from [8–14].

APPENDIX A: PROOF OF THE INEQUALITY

In this appendix, we prove the inequalities

$$\varphi_n(x) \geq \varphi_{n+1}(x)$$

for all $n \in \frac{1}{2}\mathbb{N}$ and $x > 0$. By (4.1), it is enough to show $f_{n+1}^2(x) \geq f_n(x)f_{n+2}(x)$. From (3.18), we get

$$\begin{aligned} & (f_{n+1}(x))^2 - f_n(x)f_{n+2}(x) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{x^{2(k+l)}}{2^{2n+k+l+2}} \frac{1}{k!l!} \left(\frac{1}{(n+k)!(n+l)!} - \frac{1}{(n+k-1)!(n+l+1)!} \right) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{x^{2(k+l)}}{2^{2n+k+l+2}} \frac{1}{(k+l)!} \frac{1}{(2n+k+l)!} \binom{k+l}{k} \left(\binom{2n+k+l}{n+k} - \binom{2n+k+l}{n+k-1} \right) \\ &= \sum_{m=0}^{\infty} \frac{x^{2m}}{2^{2n+m+2}} \frac{1}{m!} \frac{1}{(2n+m)!} \sum_{k=0}^m \binom{m}{k} \left(\binom{2n+m}{n+k} - \binom{2n+m}{n+k-1} \right). \end{aligned}$$

Now let us see that the summation over k is nonnegative;

$$\sum_{k=0}^m \binom{m}{k} \left(\binom{2n+m}{n+k} - \binom{2n+m}{n+k-1} \right) \tag{A1}$$

$$= \sum_{k=0}^m \binom{m}{k} \left(\binom{2n+m}{n+k} - \binom{2n+m}{n+k+1} \right) \tag{A2}$$

$$\begin{aligned} &= \frac{1}{2} \sum_{k=1}^m \left(\binom{m}{k} - \binom{m}{k-1} \right) \left(\binom{2n+m}{n+k} - \binom{2n+m}{n+k-1} \right) \\ &+ \binom{m}{m} \left(\binom{2n+m}{n+m} - \binom{2n+m}{n+m+1} \right) \end{aligned} \tag{A3}$$

$$\begin{aligned} &= \frac{1}{2} \sum_{k=1}^m (m-2k+1)^2 \frac{m!}{k!(m-k+1)!} \frac{(2n+m)!}{(n+k)!(n+m-k+1)!} \\ &+ \frac{(2n+m)!(m+1)}{(n+m+1)!n!} \geq 0. \end{aligned}$$

Here, in the first equality we replace $m-k$ with k . We set $(A3) = ((A1) + (A2))/2$ in the second equality.

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