Accurate Bounds for Critical Temperatures of $O(N)$ Spin Models by Renormalized Random Walk Representations

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Abstract

We obtain new accurate bounds of critical temperatures of N -vector (Heisenberg) models. We apply a block spin type transformation to random walk representations of the spin models and we rigorously estimate the effects of longer random walks accurately with the help of computer. The bounds are equal to the results by Monte Carlo simulations within the differences of 6.3 % (dimension=3) to 3.3 % (dimension=4).

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I. INTRODUCTION

We have developed a method of finding a lower bound for the inverse critical temperature β_c below which the system exhibits exponential clustering. What we investigate is the ν dimensional Heisenberg model with $O(N)$ symmetry (N-vector model) where the Gibbs measure restricted to the finite region $L \subset \mathbf{Z}^\nu$ is defined by

$$\langle F \rangle \equiv \frac{1}{Z_L(\beta)} \int F(\mathbf{S}) \exp[-H_L(\mathbf{S})] \prod_i g(\mathbf{S}_i^2) d\mathbf{S}(x). \quad (1.1)$$

Here $\mathbf{S}_i = (S_i^{(1)}, \dots, S_i^{(N)})$ is the vector valued spin at $i \in L$, Z_L is the partition function defined so that $\langle 1 \rangle = 1$ and $g(x) = \delta(x - 1)$. Moreover H_L is the Hamiltonian given by

$$H_L \equiv -\frac{1}{2} \sum_{i,j \in L} \sum_{\alpha=1, \dots, N} S_i^{(\alpha)} J_{ij} S_j^{(\alpha)}. \quad (1.2)$$

where J is the matrix whose components are given by $J_{i,j} = \beta \delta_{1,|i-j|}$. Here β is the inverse temperature and $|i| = \sum_{j=1}^\nu |i_j|$. Our latest result in [5] is

Theorem 1 [5]. *If*

$$\beta < \left[\frac{1}{(N+2)^2} + \frac{2\nu-2}{N(N+2)} + \frac{(2\nu-1)^2}{N^2} \right]^{-1/2}$$

there exist strictly positive constants m and C such that $\langle \mathbf{S}(0)\mathbf{S}(x) \rangle \leq C \exp[-m|x|]$.

This is an extension of the results in [1,2] which were obtained by the random walk representations of the spin model (see also [3]). We applied a block spin type argument to the random walk representation to simplify the random walks. Namely we can take in a closed form (modified Bessel functions I_k) the full effects of the smallest loops of type $p \rightarrow q \rightarrow p \rightarrow q \rightarrow \dots \rightarrow p$ ($|p-q| = 1$). The advantage of this transformation is two folds: (1) the number of random walks is drastically decreased, (2) $I_{(N/2)+k}/I_{N/2}$ has better properties than $\prod_{i=0}^{k-1} (N+2i)^{-1}$ (see [1,5]).

We here improve the bounds to extreme by considering longer walks which were not considered in [5]. For example we obtain 0.2077 (resp. 0.1453) as the rigorous lower bound of the 3D (resp. 4D) Ising model. This is only 6.3 (resp. 3.3)% less than the experimental data 0.2217 (resp. 0.1503).

II. SHORT VISIT TO THE RENORMALIZED RANDOM WALK REPRESENTATION

Since $\delta(\mathbf{S}^2 - 1) = \int \exp[-ia(\mathbf{S}^2 - 1)] da / 2\pi$, we have

$$Z = C_{L,N} \int \det^{-N/2}(2iA - J) \prod_{j \in L} e^{ia_j} \frac{da_j}{2\pi}, \quad (2.1)$$

$$A_{ij} = a_i \delta_{ij} \quad (i, j \in L). \quad (2.2)$$

Here $C_{LN} = (2\pi)^{|L|N/2}$ and the contour is chosen $\{x - i\lambda | x \in \mathbf{R}\}$ where $\lambda > 0$ is sufficiently large. To renormalize loops which come from the expansion of the determinant, we set $L = L_o \cup L_e$, where $L_o \equiv \{j \in L | j : \text{odd}\}$ and $L_e \equiv \{j \in L | j : \text{even}\}$, and j is called odd (resp. even) if $|j| \equiv \sum_{\mu=1}^{\nu} |j_{\mu}|$ is odd (resp. even).

Let P_e (resp. P_o) be the projection operator to L_e (resp. L_o) and define $B = P_o(2iA)P_o$, $C = P_e(2iA)P_e$ and $K = P_oJP_e$:

$$B_{kl} = 2ia_k \delta_{kl}, \quad C_{pq} = 2ia_p \delta_{pq},$$

$$K_{kp} = \begin{cases} \beta & \text{if } |k - p| = 1 \\ 0, & \text{otherwise,} \end{cases} \quad (2.3)$$

($k, l \in L_o, p, q \in L_e$). Then we have $2iA - J = \begin{pmatrix} B & -K \\ -K^T & C \end{pmatrix}$ and the Hamiltonian matrix can be blockwise diagonalized by triangular matrices [7] :

$$\begin{pmatrix} B & -K \\ -K^T & C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -K^T B^{-1} & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & C - K^T B^{-1} K \end{pmatrix} \begin{pmatrix} 1 & -B^{-1} K \\ 0 & 1 \end{pmatrix}. \quad (2.4)$$

Therefore

$$\det(2iA - J) = \det B \det(C - K^T B^{-1} K), \quad (2.5)$$

$$\begin{pmatrix} B & -K \\ -K^T & C \end{pmatrix}^{-1} = \begin{pmatrix} 1 & B^{-1} K \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & (C - K^T B^{-1} K)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K^T B^{-1} & 1 \end{pmatrix}. \quad (2.6)$$

We put

$$C - K^T B^{-1} K = M - K_2 \quad (2.7)$$

$$M_{kl} \equiv C_{kl} - (K^T B^{-1} K)_{kl} \delta_{kl} \quad (2.8)$$

$$(K_2)_{kl} \equiv (K^T B^{-1} K^T)_{kl} (1 - \delta_{kl}). \quad (2.9)$$

Then M is diagonal and $(K_2)_{kl} \neq 0$ only for $|k - l| \leq 2$:

$$M_{kl} = \delta_{kl} M_k = \delta_{kl} (2ia_k - \sum_{j:|k-j|=1} \frac{\beta^2}{2ia_j}) \quad (2.10)$$

Note that $|(K_2)_{kl}| \leq \beta^2/\lambda$ for $|k - l| \leq 2$, $(K_2)_{kl} = 0$ for $|k - l| > 2$, and $|M_k| \geq 2\lambda - \nu\beta^2/\lambda$. Then $\det(2iA - J) = \det B \det M \det(1 - K_2/M)$ admits the loop expansion [1,5] if λ is large :

$$\begin{aligned} Z = C_{LN} \int & \left[\prod_{j \in L_o} (2ia_j)^{-N/2} e^{ia_j} \frac{da_j}{2\pi} \right] \\ & \times \int \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2} \right)^n \sum_{\check{v}_1, \dots, \check{v}_n} \left(\prod_{l=1}^n \frac{K_2^{\check{v}_l}}{\tau_{\check{v}_l}} \right) \prod_{k \in L_e} M_k^{-n(k, \check{v}_1, \dots, \check{v}_n)} \frac{e^{ia_k} da_k}{2\pi}, \end{aligned} \quad (2.11)$$

where $\tau_{\check{v}}$ is the winding number of the loop $\check{v}$ [1,4], and denoting the visiting number of the random loop $\check{v}$ at $k \in L_e$ by $n(k, \check{v})$, we used the notation $n(k, \check{v}_1, \dots, \check{v}_n) = n(k, \check{v}_1) + \dots + n(k, \check{v}_n) + N/2$, $K_2^{\check{v}_l} = \prod_{s \in \check{v}_l} (K_2)_s$.

Note that $\int e^{ia} (2ia - x)^{-n} da = 2\pi e^{x/2} / [2^n \Gamma(n)]$, and we use the notation $n! = \Gamma(n + 1)$ not only for integers but also for half integers. Thus from (2.10), we get

$$\prod_{k \in L_e} \int \frac{e^{ia_k} da_k}{2\pi} M_k^{-n(k, \check{v}_1, \dots, \check{v}_n)} = \prod_{k \in L_e} \frac{\exp \left[\sum_{\substack{j \in L_o \\ |k-j|=1}} \frac{\beta^2}{4ia_j} \right]}{2^{n(k, \check{v}_1, \dots, \check{v}_n)} (n(k, \check{v}_1, \dots, \check{v}_n) - 1)!}$$

and then

$$\begin{aligned} Z = C_{LN} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2} \right)^n \sum_{\check{v}_1, \dots, \check{v}_n} \int & \left[\prod_{l=1}^n \frac{K_2^{\check{v}_l}}{\tau_{\check{v}_l}} \right] \left[\prod_{j \in L_o} \exp \left[ia_j + \frac{\nu\beta^2}{2ia_j} \right] \frac{1}{(2ia_j)^{N/2}} \frac{da_j}{2\pi} \right] \\ & \times \prod_{k \in L_e} (2^{n(k, \check{v}_1, \dots, \check{v}_n)} (n(k, \check{v}_1, \dots, \check{v}_n) - 1)!)^{-1}. \end{aligned} \quad (2.12)$$

For any nearest neighbor random walk ω on the original lattice L starting and ending at even sites, we define a random walk $\iota(\omega)$ on the even lattice L_e such that

each step of $\iota(\omega)$ is a composition of successive two steps of ω . For example if $\omega = \{(i_0, i_1), (i_1, i_2), \dots, (i_{2m-1}, i_{2m})\}$ where $i_0, i_{2m} \in L_e, |i_j - i_{j-1}| = 1, j = 1, \dots, 2m$, $\iota(\omega) = \{(i_0, i_2), (i_2, i_4), \dots, (i_{2m-2}, i_{2m})\}$. We used the symbol v for a random walk on L_e . By $v \subset \omega$ we mean $\iota(\omega) = v$. For one random walk v on L_e , there exist many random walks ω satisfying $v \subset \omega$ on L . The same notation will be used for random loops $\check{\omega}$ on L and random loops $\check{v}$ on L_e . Substituting (2.3) to (2.12), we get

$$\begin{aligned} Z &= C_{LM} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2}\right)^n \sum_{\check{v}_1, \dots, \check{v}_n} \sum_{\check{\omega}_1 \supset \check{v}_1, \dots, \check{\omega}_n \supset \check{v}_n} \left(\prod_{l=1}^n \frac{\beta^{2\|\check{v}_l\|}}{\tau_{\check{v}_l}} \right) \\ &\times \prod_{j \in L_o} \int \exp \left[ia_j + \frac{\nu \beta^2}{2ia_j} \right] \frac{1}{(2ia_j)^{n(j, \check{\omega}_1, \dots, \check{\omega}_n)}} \frac{da_j}{2\pi} \\ &\times \prod_{k \in L_e} 2^{-n(k, \check{v}_1, \dots, \check{v}_n)} (n(k, \check{v}_1, \dots, \check{v}_n) - 1!)^{-1}, \end{aligned} \quad (2.13)$$

where $\|\check{v}_l\|$ is the number of steps of $\check{v}_l$. The integration with respect to a_j in (2.13) then yields

$$\begin{aligned} \int \exp \left[ia + \frac{(\sqrt{\nu}\beta)^2}{2ia} \right] \frac{da}{2\pi(2ia)^n} &= \sum_{k=0}^{\infty} \frac{(\sqrt{\nu}\beta)^{2k}}{k!} \frac{1}{2^{n+k}(n+k-1)!} \\ &\equiv f_n(\sqrt{\nu}\beta). \end{aligned} \quad (2.14)$$

The function f_n is essentially the n -th modified Bessel function:

$f_n(x) = I_{n-1}(\sqrt{2}x)/2(\sqrt{2}x)^{n-1}$. Thus we have

$$\begin{aligned} Z &= C_{LN} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2}\right)^n \sum_{\check{v}_1, \dots, \check{v}_n} \sum_{\check{\omega}_1 \supset \check{v}_1, \dots, \check{\omega}_n \supset \check{v}_n} \left(\prod_{l=1}^n \frac{\beta^{2\|\check{v}_l\|}}{\tau_{\check{v}_l}} \right) \\ &\times \prod_{j \in L_o} f_{n(j, \check{\omega}_1, \dots, \check{\omega}_n)}(\sqrt{\nu}\beta) \prod_{k \in L_e} 2^{-n(k, \check{v}_1, \dots, \check{v}_n)} (n(k, \check{v}_1, \dots, \check{v}_n) - 1!)^{-1}. \end{aligned}$$

Similarly for the correlation function $\langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle$ ($p, q \in L_e$), using (2.4), (2.5) and (2.9), we get

$$\begin{aligned} \langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle &= \frac{C_{LN}}{Z} \int (M - K_2)_{pq}^{-1} \det^{-N/2} B \det^{-N/2} (M - K_2) \prod_{j \in L} \frac{e^{ia_j} da_j}{2\pi} \\ &= \frac{C_{LN}}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2}\right)^n \sum_{v: p \rightarrow q} \sum_{\omega \supset v} \left[\sum_{\check{v}_1, \dots, \check{v}_n} \sum_{\check{\omega}_1 \supset \check{v}_1} \dots \sum_{\check{\omega}_n \supset \check{v}_n} \right] \end{aligned} \quad (2.15)$$

$$\begin{aligned}
& \times \left(\beta^{2\|v\|} \prod_{l=1}^n \frac{\beta^{2\|\check{v}_l\|}}{\tau_{\check{\omega}_l}} \right) \prod_{j \in L_o} f_{n(j, \check{\omega}_1, \dots, \check{\omega}_n, \omega)}(\sqrt{\nu}\beta) \\
& \times \prod_{k \in L_e} 2^{-n(k, \check{v}_1, \dots, \check{v}_n, v)} ((n(k, \check{v}_1, \dots, \check{v}_n, v) - 1)!)^{-1},
\end{aligned} \tag{2.16}$$

as in the case of Z . We have similar expressions for other points in L .

III. ESTIMATE OF THE CRITICAL TEMPERATURES

We set $\varphi_n(x) = f_{n+1}(x)/f_n(x)$. Since $\varphi_n \geq \varphi_{n+1}$ (see [5]), we have

$$f_{n(j, \check{\omega}_1, \dots, \check{\omega}_n, \omega)}(\sqrt{\nu}\beta) \leq \frac{f_{(N+2n(j, \omega))/2}(\sqrt{\nu}\beta)}{f_{N/2}(\sqrt{\nu}\beta)} f_{n(j, \check{\omega}_1, \dots, \check{\omega}_n, \omega)}(\sqrt{\nu}\beta). \tag{3.1}$$

We also have the trivial inequalities

$$\begin{aligned}
& 2^{-n(k, \check{v}_1, \dots, \check{v}_n, v)} ((n(k, \check{v}_1, \dots, \check{v}_n, v) - 1)!)^{-1} \\
& \leq 2^{-n(k, \check{v}_1, \dots, \check{v}_n)} ((n(k, \check{v}_1, \dots, \check{v}_n) - 1)!)^{-1} \prod_{m=0}^{n(k, v)-1} (N + 2m)^{-1}.
\end{aligned} \tag{3.2}$$

Substituting eqs. (3.1) and (3.2) to eq. (2.16), we get

$$\langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle \leq \sum_{v: p \rightarrow q} \sum_{\omega \supset v} (\beta^{\|\omega\|}) \left(\prod_j \frac{f_{(N+2n(j, \omega))/2}(\sqrt{\nu}\beta)}{f_{N/2}(\sqrt{\nu}\beta)} \right) \prod_k \prod_{m=0}^{n(k, \omega)-1} (N + 2m)^{-1}. \tag{3.3}$$

We count the number of the walks ω on L constructed via walks v on L_e . Each step of v , say (k, l) is corresponding to two steps of $\omega(\supset v)$, say (k, j) and (j, l) where $k, l \in L_e, j \in L_o$ satisfy $k \neq l, |k - j| = |j - l| = 1$. Therefore the random walks ω satisfy the following conditions:

1. if ω has visited at $k \in L_e$, then the next step of ω can go to any nearest neighbor point $j \in L_o, |k - j| = 1$.
2. if ω has visited at $j \in L_o$, through the step (k, j) , then the next step of ω can go to any nearest neighbor point $l \in L_e, |j - l| = 1, l \neq k$.

Then the number of such walks ω of $2m$ steps equals $(2\nu(2\nu - 1))^m$. Consider a random walk ω starting $p \in L_e$ and ending at $q \in L_e$. We cut ω into strings $\omega_1, \omega_2, \dots$ of length ℓ . To get

the bound (3.3), we take the sum over all possible ω_1 with the fixed end point (= starting point of ω_2) and with free starting points. Let $\tilde{\omega} = \omega - \omega_1$. We count the self-crossing points of ω_1 as follows ($\#$ stands for the number of the elements):

$$a_e(\omega_1, n) = \# \{n\text{-ple self-crossing points} \in L_e \setminus \text{supp } \tilde{\omega}\} \quad (3.4)$$

$$a_o(\omega_1, n) = \# \{n\text{-ple self-crossing points} \in L_o \setminus \text{supp } \tilde{\omega}\} \quad (3.5)$$

$$b_e(\omega_1, n) = \# \{n\text{-ple self-crossing points} \in L_e \cap \text{supp } \tilde{\omega}\} \quad (3.6)$$

$$b_o(\omega_1, n) = \# \{n\text{-ple self-crossing points} \in L_o \cap \text{supp } \tilde{\omega}\} \quad (3.7)$$

In (3.3), we can factorize ω_1 with the factor

$$\begin{aligned} \tilde{W}(\omega_1) &= \prod_{n \geq 1} \left[\prod_{m=0}^{n-1} \frac{1}{N+2m} \right]^{a_e(n)} \prod_{n \geq 1} \left[\frac{f_{(N+2n)/2}(\sqrt{\nu}\beta)}{f_{N/2}(\sqrt{\nu}\beta)} \right]^{a_o(n)} \\ &\quad \times \prod_{n \geq 1} \left[\prod_{m=1}^{n-1} \frac{1}{N+2m} \right]^{b_e(n)} \prod_{n \geq 1} \left[\frac{f_{(N+2+2n)/2}(\sqrt{\nu}\beta)}{f_{(N+2)/2}(\sqrt{\nu}\beta)} \right]^{b_o(n)}. \end{aligned} \quad (3.8)$$

Set

$$W(\beta, \ell) = \max_{\tilde{\omega}} \left(\sum_{\omega_1} \tilde{W}(\omega_1) \beta^\ell \right) \quad (3.9)$$

then we have

$$0 \leq \langle S_p^{(\alpha)} S_q^{(\alpha)} \rangle \leq \text{const} \sum_{m > |p-q|/\ell} W^m(\beta, \ell) \quad (3.10)$$

for $p, q \in L_e$ ($p \neq q$). We have similar bounds for other combinations of the points.

Theorem 2. *In the ν -dimensional Heisenberg model, the critical inverse temperature β_c satisfies*

$$\beta_c \geq \inf \{ \beta \geq 0 \mid W(\beta, \ell) \geq 1 \}.$$

For example, for $\ell=2$ we have

$$W = \left[\frac{\varphi_{(N+2)/2}(\sqrt{\nu}\beta)}{N+2} + \frac{2\nu-2}{N} \varphi_{(N+2)/2}(\sqrt{\nu}\beta) + \frac{(2\nu-1)^2}{N} \varphi_{N/2}(\sqrt{\nu}\beta) \right] \beta^2. \quad (3.11)$$

In fact if the walk has arrived at $l \in L_e$ through the steps $(k, j), (j, l)$ where $k, l \in L_e, j \in L_o$, $|k-l| = 2$ and $|k-j| = |j-l| = 1$, there exist $2\nu-1$ ways to leave l for points $k' \in L_e$

($|k' - l| = 2$) through j , and one of them revisits k . Ineqs. (3.1) and (3.2), mean that these steps yield $(2\nu - 2)\varphi_{(N+2)/2}(\sqrt{\nu}\beta)/N$ and $\varphi_{(N+2)/2}(\sqrt{\nu}\beta)/(N + 2)$, respectively. Since $\varphi_n(x) < 1/2n$ (see (2.14)), we obtain our previous result in the introduction.

The bound for the critical temperature becomes accurate for large ℓ 's. The number of such random walks increases exponentially in ℓ (i.e. $(2\nu(2\nu - 1))^{\ell/2}$), and is already astronomical for $\ell \geq 10$. Also the number of the weight functions $\tilde{W}(\omega)$ increases faster than any polynomial of ℓ . However if $\nu = 2$, it is 3 for $\ell = 2$, 11 for $\ell = 4$, 44 for $\ell = 6$, 155 for $\ell = 8$, 467 for $\ell = 10$ and 1354 for $\ell = 12$. So for $\ell < 16$, for example, we can calculate $W(\beta, \ell)$ exactly by computer.

We count exactly the number of random walks ω_1 with the types of the self-crossing points. The program written in C [8] consists of the following steps:

1. Generate $(2\nu(2\nu - 1))^{\ell/2}$ random walks ω_1 under the influence of $\tilde{\omega}$ which is taken as the straight walk from the origin to $(0, \dots, 0, \infty)$.
2. For each random walk, make a table of the self-crossing points, and classify the random walks by the tables. This is most easily done by using binary trees [8].
3. Finally scanning the binary tree, we calculate $W = W(\beta)$ and increase β until $W < 1$ is violated.

Remark. For small ℓ , it is easy to see that the straight walk $\tilde{\omega} \equiv (0, \dots, 0) \rightarrow (0, 0, \dots, \infty)$ gives the maximum in (3.9). But we do not have a proof for large ℓ 's and the numerical values are insensitive for $\tilde{\omega}$ if ℓ is large.

IV. ANALYSIS OF RESULTS AND DISCUSSIONS

Our results are compared in Tables I and II, with the critical temperatures obtained by Monte Carlo simulations. Considerably satisfactory results are given already by the $\ell = 2$ calculations. The bounds are less than the experimental data by 23 % ($\nu=2$, $N = 1$) to

5 % ($\nu=4$, $N = 1$). By increasing the length of the walks, we see that the bounds are slowly improved and converge. We note that the number of self avoiding random walks of length ℓ is more than ν^ℓ . These are random walks which do not have the suppressing factors due to self-crossing, and then $\beta_2 < \beta_4 < \dots < \infty$ even if $\nu = 2$, at the finite times of renormalization.

[2D] (1) Since most random walks have self crossing points in two dimensions, the improvements are most explicit. Choosing the random walks of length 16 improves the bound for the critical temperature of 2D Ising model by 8.9 %. Similarly this increases the bound for the 2D XY model by 7.8 %, the bound for the 2D Heisenberg model by 6.7 %, the bound for the 2D $O(5)$ model by 5.3 %.

(2) If the conjecture $\beta_c = \infty$ for $N \geq 3$ holds, we may expect that β_ℓ of the $O(N)$ model ($N > 2$) increases in ℓ in a different way from $N = 1, 2$. However, at the renormalization at the present distance scale, we do not see any significant deviation in β_ℓ .

[3,4D] (1) For higher dimensional models, the improvement is slow. In fact, the self-crossing points are fewer than those in two dimensional models and the $\ell = 2$ results are already close to the experimental data.

(2) Choosing the random walks of length 10 improves the bound for the critical temperature of the 3D Ising model about 2.9 %. Similarly this increases the bound for the 3D XY model just by 2.4 %, the bound for the 3D Heisenberg model by 2.0 %, the bound for the 3D $O(4)$ model by 1.8 %, the bound for the 3D $O(5)$ model by 1.6 %.

As a conclusion, the improvement is most drastic for small N and ν (i.e. 2D Ising model) and the bounds are most close to the experimental data for large ν and small N (e.g. 4D Ising model). As was explained already, this work has been motivated by the following points (a) the number of random walks is drastically decreased, (b) f_{n+k}/f_n is better than $\prod_{i=0}^{k-1} (N + 2i)^{-1}$ since $\prod_{i=0}^{k-1} (N + 2i)^{-1} > f_{N/2+k}/f_{N/2} \rightarrow 0$ as $\beta \rightarrow \infty$.

However the improvement at the renormalization at this distance scale is moderate. So we need to renormalize larger loops. Our method may be called a trimming of random walks at distance scale 1, and is the first step of the transformations of block spin type. (See [9,10])

on the trimming of random walks.)

We are now trying to extend the present method to renormalize larger loops [6]. A similar but slightly complicated idea [4] will be applied in [9] to improve β_c , in which one can in principle continue the transformations of random walks.

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TABLES

TABLE I. Rigorous Bounds by Random walks ($\nu = 2$)

N	β_2	β_{10}	β_{12}	β_{14}	β_{16}	β_c
$N = 1$	0.3415	0.3663	0.3686	0.3705	0.3720	0.4408
$N = 2$	0.6836	0.7260	0.7304	0.7339	0.7368	1.06
$N = 3$	1.0232	1.0772	1.0832	1.0881	1.0921	
$N = 4$	1.3606	1.4231	1.4303	1.4362	1.4412	
$N = 5$	1.6969	1.7656	1.7738	1.7806	1.7863	
$N = 6$	2.0325	2.1059	2.1150	2.1224	2.1288	

β_ℓ : rigorous bounds by walks of length ℓ .

β_c : 2DXY model data is from [12]. The 2D Ising model is exactly soluble.

TABLE II. Rigorous Bounds by Random walks ($\nu \geq 3$)

N	ν	β_2	β_6	β_8	β_{10}	β_c
$N = 1$	$\nu = 3$	0.2018	0.2058	0.2069	0.2078	0.2217
	$\nu = 4$	0.1435	0.1450	0.1453		0.1503
$N = 2$	$\nu = 3$	0.4038	0.4104	0.4122	0.4135	0.4542
	$\nu = 4$	0.2871	0.2895	0.2901		
$N = 3$	$\nu = 3$	0.6053	0.6136	0.6159	0.6177	0.6930
	$\nu = 4$	0.4305	0.4335	0.4343		
$N = 4$	$\nu = 3$	0.8063	0.8136	0.8186	0.8206	0.9360
	$\nu = 4$	0.5738	0.5773	0.5782		0.6090
$N = 5$	$\nu = 3$	1.0071	1.0176	1.0206	1.0229	
	$\nu = 4$	0.7170	0.7208	0.7218		
$N = 6$	$\nu = 3$	1.2077	1.2189	1.2221	1.2246	
	$\nu = 4$	0.8601	0.8641	0.8652		

β_ℓ : rigorous bounds by walks of length ℓ .

β_c : estimates by MC simulations [11–17].

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