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THE MASSIVE THIRRING
MODEL IN LABORATORY COORDINATES

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Abstract

We demonstrate a simple procedure for finding the squared eigenfunctions of an integrable system and illustrate the method by deriving the squared eigenfunctions of the massive Thirring model. We give the formula for the inner products between the direct squared eigenfunction problem and its adjoint. We also demonstrate closure and express the variations of the potentials in terms of variations of the scattering data.

1 Introduction

Besides being a simple integrable model in field theory, the massive Thirring model [1] also plays a certain role in nonlinear optics in a problem of wave propagation in a periodically modulated nonlinear fiber [2].

Naturally, in such situations, one at times needs to understand how perturbations can effect such a system. For this, one needs to know the squared eigenfunctions for the appropriate eigenvalue problem and their closure. In this paper, we solve that problem.

There are two standard methods for finding squared eigenfunctions. One method, the first one discovered [5], involves obtaining the equations for perturbations solving the adjoint problem and then determining the inner products between the original eigenfunctions and the adjoint eigenfunctions. The main part of this technique consists of finding the equation for the squared eigenfunctions, which can require much of trial and error, and thus the success of the method significantly depends on the researcher's ingenuity. On the other hand, there is a procedure based on the Riemann-Hilbert method that will always work [3]. However, in the interesting case when one is interested in the perturbation on the background of a soliton (or a N -soliton solution), this method becomes technically very complex.

We have found a simple regular procedure that allows one to eliminate the trial and error from the former method. We do not start by seeking an equation for the squared eigenfunctions, but instead assume a particular general structure for the Wronskian relation between certain eigenstates related to the original eigenvalue problem. Then one proceeds to determine the coefficients in the general Wronskian relation. The method is simple to use and is straightforward to apply. There seems to be no serious restrictions on the structure of the original eigenvalue problem. In particular, it would work even in the case where eigenvalues can occur in an arbitrary polynomial form inside the eigenvalue problem. We note, however, that at the moment our method does not allow one to find the equation for the squared eigenfunctions; we discuss this point later in Sections 3

and 5.

In Section 2, we review the direct and inverse scattering problems for the massive Thirring model. In Section 3, we solve the above for variations of the scattering data in terms of variations of the potentials. We then discuss the problem of inverting this relation in order to obtain the variations of the potentials in terms of variations of the scattering data. Then in Section 4, we solve this problem by a new and general method. We start from a general Wronskian relation and use the original and adjoint eigenvalue problems to fully determine all coefficients in it. This results in obtaining the specific Wronskian relation required for the massive Thirring model. Now it becomes trivial to obtain the inner products and a bit less trivial to verify closure. Then we give the variations of the potentials in terms of variations of the scattering data. In Section 5 we make some concluding remarks. In the Appendix the method based on the Riemann-Hilbert formulation is briefly considered.

2 The Direct and Inverse Scattering Problems

We consider the equations of the Massive Thirring Model (MTM) in the laboratory coordinates:

$$\begin{aligned} i(u_x + u_t) + v + u|v|^2 &= 0, \\ i(-v_x + v_t) + u + v|u|^2 &= 0. \end{aligned} \tag{1}$$

Here x and t are the space and time coordinates, respectively. The initial conditions for the fields u and v are:

$$u(x, t = 0) = u_0(x), \quad v(x, t = 0) = v_0(x) \tag{2}$$

on the whole x -axis $\mathbf{R}$, and $|u_0(x)|$, $|v_0(x)|$ are assumed to be integrable on $\mathbf{R}$ and sufficiently smooth (e.g., from the Schwartz class). Equations (1) are the compatibility condition for the following linear system ([1]):

$$\left(\frac{\partial}{\partial x} - L\right)\psi = 0, \tag{3a}$$

$$\left(\frac{\partial}{\partial t} - A\right)\psi = 0, \tag{3b}$$

where ψ is a 2-vector, L and A are 2×2 matrices. Here we will only need the expression for L :

$$L = \frac{i}{4}(|u|^2 - |v|^2)\sigma_3 - \frac{i\lambda}{2} \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} + \frac{i}{4}(\lambda^2 - \frac{1}{\lambda^2})\sigma_3, \quad (4)$$

where we have used the notations for the Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. An asterisk in (4) and everywhere below stands for complex conjugation. Notice that the notations in (4) are slightly different from those in [1], where the representation (3) was first found.

The problem that we solve in this paper is the following: what quadratic combinations of the eigenfunctions (we will term them “squared eigenfunctions” for brevity) of (3a) does one need in order to be able to expand over them the variations δu , δu^* , δv , δv^* of the potentials u and v of the operator L . Before passing to the solution of the problem stated we will give a brief summary of the results on the direct and inverse scattering problems for the operator L , obtained in [1].

Let us introduce two pairs of the Jost solutions of (3a):

$$\begin{aligned} \chi(\lambda) &\rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(-ikx), & x \rightarrow -\infty \\ \bar{\chi}(\lambda) &\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(ikx), & x \rightarrow -\infty \end{aligned} \quad (5a)$$

$$\begin{aligned} \phi(\lambda) &\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(ikx), & x \rightarrow +\infty \\ \bar{\phi}(\lambda) &\rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} \exp(-ikx), & x \rightarrow +\infty \end{aligned} \quad (5b)$$

where $k \equiv k(\lambda) = \frac{1}{4}(\lambda^2 - \lambda^{-2})$. The operator L satisfies the relation:

$$\sigma_2 L^*(\lambda^*) \sigma_2 = L(\lambda). \quad (6)$$

So if $\psi(x, \lambda)$ is a solution of (3a), then

$$\bar{\psi}(x, \lambda) = i\sigma_2 \psi^*(x, \lambda^*) \quad (7)$$

is another solution. In particular, the solutions $\bar{\chi}$ and $\bar{\phi}$ are related by involution (7) to χ and ϕ , respectively. Also,

$$\bar{\psi}(x, \lambda) = \pm \sigma_3 \psi(x, -\lambda) , \quad (8)$$

is yet another solution of (3a). If ψ is any of the Jost solutions, then it is invariant under transformation (8), with the sign in the r.h.s. of (8) being chosen in accordance with asymptotics (5). The Wronskian

$$W(f, g) = \det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}$$

of any two solutions f and g of (3a) with the same λ is independent of x . This yields that $\bar{\psi}(\lambda)$ and $\psi(\lambda)$ are linearly independent (i.e. their Wronskian is non-zero), where the ψ denotes any of the Jost solutions.

Now let us introduce the scattering matrix for $\text{Im}\lambda^2 = 0$:

$$\begin{aligned} \phi &= a(\lambda) \bar{\chi} + b(\lambda) \chi \\ \bar{\phi} &= -\bar{a}(\lambda) \chi + \bar{b}(\lambda) \bar{\chi} . \end{aligned} \quad (9a)$$

From (7) it follows that

$$\bar{a}(\lambda) = a^*(\lambda^*) \quad \bar{b}(\lambda) = b^*(\lambda^*) . \quad (10)$$

In a standard way one shows that $\chi(\lambda)e^{ikx}$, $\phi(\lambda)e^{-ikx}$ are analytic where $\text{Im}\lambda^2 \geq 0$, and $\bar{\chi}(\lambda)e^{-ikx}$, $\bar{\phi}(\lambda)e^{ikx}$ are analytic where $\text{Im}\lambda^2 \leq 0$. From (9a) one finds:

$$a = W(\phi, \chi) , \quad b = W(\bar{\chi}, \phi) , \quad (11)$$

whence $a(\lambda)$ is analytic for $\text{Im}\lambda^2 > 0$, and accordingly $\bar{a}(\lambda)$ is analytic for $\text{Im}\lambda^2 < 0$; $b(\lambda)$ and $\bar{b}(\lambda)$ are, in general, defined only for $\text{Im}\lambda^2 = 0$. From (11), (8), and (5) it follows that

$$a(-\lambda) = a(\lambda) , \quad b(-\lambda) = -b(\lambda) \quad (12)$$

and also that, for $\text{Im}\lambda^2 = 0$,

$$a\bar{a} + b\bar{b} = 1 . \quad (13)$$

From (9a) and (13) one determines the inverse of (9a):

$$\begin{aligned}\chi &= -a\bar{\phi} + \bar{b}\phi \\ \bar{\chi} &= \bar{a}\phi + b\bar{\phi}.\end{aligned}\tag{9b}$$

At the points of the discrete spectrum λ_n ($\text{Im}\lambda_n^2 > 0$) and $\lambda_n^* \equiv \lambda_{\bar{n}}$ such that $a(\lambda_n) = \bar{a}(\lambda_{\bar{n}}) = 0$, one has

$$\begin{aligned}\chi_n &= \bar{b}_n\phi_n, & \bar{\chi}_{\bar{n}} &= b_{\bar{n}}\bar{\phi}_{\bar{n}}, \\ \phi_n &= b_n\chi_n, & \bar{\phi}_{\bar{n}} &= \bar{b}_{\bar{n}}\bar{\chi}_{\bar{n}},\end{aligned}\tag{9c}$$

where $\phi_n = \phi(x, \lambda_n)$, etc. Note that these points appear in pairs: if λ_n is an eigenvalue, then so is $-\lambda_n$, since $a(\lambda)$ is an even function. The asymptotic values of $a(\lambda)$ can be obtained from (3a), (11):

$$\left. \begin{aligned}\lambda \rightarrow \infty & & a(\lambda) & \rightarrow \exp(-i\hat{Q}) \\ \lambda \rightarrow 0 & & a(\lambda) & \rightarrow \exp(i\hat{Q})\end{aligned} \right\} \text{Im}(\lambda^2) \geq 0, \tag{14}$$

where $\hat{Q} = \frac{1}{4} \int_{-\infty}^{\infty} (|u|^2 + |v|^2) dx$. The asymptotics of the Jost solutions for large and small λ are the following (for $\text{Im}\lambda_n^2 \geq 0$):

$$\begin{aligned}\lambda \rightarrow \infty \quad \chi &\rightarrow \begin{pmatrix} v^*\lambda^{-1} \\ 1 \end{pmatrix} (1 + \mathcal{O}(\lambda^{-1})) \exp(-ikx - iQ^-(x)), \\ \phi &\rightarrow \begin{pmatrix} 1 \\ -v\lambda^{-1} \end{pmatrix} (1 + \mathcal{O}(\lambda^{-1})) \exp(ikx - iQ^+(x)),\end{aligned}\tag{15a}$$

$$\begin{aligned}\lambda \rightarrow 0 \quad \chi &\rightarrow \begin{pmatrix} \lambda u^* \\ 1 \end{pmatrix} (1 + \mathcal{O}(\lambda)) \exp(-ikx + iQ^-(x)), \\ \phi &\rightarrow \begin{pmatrix} 1 \\ -\lambda u \end{pmatrix} (1 + \mathcal{O}(\lambda)) \exp(ikx + iQ^+(x)),\end{aligned}\tag{15b}$$

where

$$Q^\pm(x) = \pm \frac{1}{4} \int_x^{\pm\infty} (|u|^2 + |v|^2) dx'.$$

To complete the review of the discrete and inverse scattering problems for the operator L , we present the singular integral equation which allows one to reconstruct the Jost solutions from known scattering data $S = \{\rho(\lambda) = b/a \mid \lambda^2 \text{ is real; } \lambda_n, c_n = b_n/\bar{a}_n \mid n = -N, \dots, N, n \neq 0\}$:

$$Y(x, \lambda) = \sum_{n=-N, n \neq 0}^N \frac{c_n \chi_n e^{-ik_n x}}{\lambda - \lambda_n} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{iQ^-(x)} + \frac{1}{2\pi i} \int_{\Gamma} \frac{\rho(\lambda') \chi(x, \lambda') e^{-ik' x}}{\lambda' - \lambda} d\lambda' , \quad (16a)$$

where

$$Y(x, \lambda) = \begin{cases} (\phi/a) e^{-ikx} , & \text{Im} \lambda^2 > 0 \\ \bar{\chi} e^{-ikx} , & \text{Im} \lambda^2 < 0 . \end{cases} \quad (16b)$$

Above we have used the following notations: $\dot{a}_n = (\frac{da}{d\lambda})|_{\lambda=\lambda_n}$, $k_n = k(\lambda_n)$, $k' = k(\lambda')$, $\lambda_{-n} = -\lambda_n$, and the contour Γ is shown in Fig.1.

Now let us pass to the solution of the problem stated in the beginning. We have four variations of the potentials: δu , δu^* , δv , and δv^* . In order to map them one-to-one into variations of the scattering data, we need four independent components of the latter. If we ignore for now the discrete spectrum, then we seem to have only two functions: $\rho(\lambda)$ and $\bar{\rho}(\lambda) = \rho^*(\lambda^*)$ on Γ . However, in reality we have exactly four of these. Indeed, the natural spectral parameter for (3a) is not the λ but rather $k(\lambda)$. On Γ , $k(\lambda)$ is four-fold degenerate, since for each value of k , there are four different possible values of λ , namely, λ itself, $-\lambda$, and $\pm i\lambda^{-1}$. As λ moves along each of the 4 rays of which the contour Γ is composed, k ranges from $-\infty$ to ∞ . From (12) one deduces the relation $\rho(-\lambda) = -\rho(\lambda)$ (and consequently, $\bar{\rho}(-\lambda) = -\bar{\rho}(\lambda)$). Thus, although the scattering data at $-\lambda$ is always related to the scattering data at λ , there is no such a relation connecting the scattering data at $\pm i\lambda^{-1}$ with that at λ . Therefore, the four independent functions in the set of scattering data are $\rho(\lambda)$ and $\bar{\rho}(\lambda)$ on contour Γ_+ (see Fig.1).

3 Variations of potentials and scattering data

Before going on to the solution of the problem, we note the following. As it was pointed out in Introduction, there exist two equivalent ways of finding variations of the potentials via the components of the Jost solutions. The first way (see, for example, [3]) employs the Riemann–Hilbert formulation of the corresponding scattering problem. This method would readily give us the answer if no discrete

spectrum was present, but it becomes technically very cumbersome when one seeks to obtain the contribution to the variations of the potentials from the variation of the discrete spectrum. Therefore, here we have chosen to use the other method which is based on calculating the variations of the scattering data via the variations of the potentials, and then inverting this relation. This latter method was used, for example, in [5–9]. In the Appendix we give, for the sake of completeness of presentation, a brief exposition of the former method.

Thus, we first find the variation of $\rho(\lambda)$ corresponding to variations of the potentials u and v . Then in a standard way (see, for example, [4]) we find from (3a), (11), (5), and (9):

$$\begin{aligned}\delta\rho &= \frac{i}{4a^2} < Z(\phi(x, \lambda)) | \delta p(x) >, \\ \delta\bar{\rho} &= \frac{i}{4\bar{a}^2} < \bar{Z}(\phi(x, \lambda)) | \delta p(x) >, \end{aligned} \tag{17}$$

where

$$\begin{aligned} < f | g > \equiv \int_{-\infty}^{\infty} dx f(x) g(x), \\ \delta p &= (\delta u, \delta u^*, \delta v, \delta v^*)^T, \\ Z &= (Z_1, Z_2, Z_3, Z_4), \quad \bar{Z}(\phi) \equiv Z(\bar{\phi}); \\ Z_1(\phi) &= 2(-\frac{1}{\lambda}\phi_1^2 + u^*\phi_1\phi_2), \\ Z_2(\phi) &= 2(\frac{1}{\lambda}\phi_2^2 + u\phi_1\phi_2), \\ Z_3(\phi) &= 2(\lambda\phi_1^2 - v^*\phi_1\phi_2), \\ Z_4(\phi) &= 2(-\lambda\phi_2^2 - v\phi_1\phi_2), \end{aligned} \tag{18}$$

and ϕ_i in above is the i th ($i = 1, 2$) component of ϕ , the Jost solutions defined in (5b). We omit arguments of functions wherever those are obvious or unimportant. In what follows it will be convenient for us to use the following notations:

$$V_i(fg) = f_i g_i, \quad i = 1, 2; \quad V_3(fg) = f_1 g_2 + f_2 g_1, \tag{19}$$

and $V_i(f) = V_i(ff)$, $i = 1, 2, 3$.

Let us now show how (17) should be inverted to find $\delta p(x)$. To simplify the subsequent calculations, let us suppose for the moment that L has no discrete spectrum. Then $\delta\rho, \delta\bar{\rho}$ are its only scattering data. In order to invert this, we would require a set of adjoint functions, $Z^A(x, \lambda)$, orthogonal to the set, Z , in (18). Suppose for the moment that we have such a set where

$$\frac{i}{4} \int_{\Gamma} (Z^A(x', \lambda), \bar{Z}^A(x', \lambda)) \begin{pmatrix} Z(x, \lambda)/a^2(\lambda) \\ \bar{Z}(x, \lambda)/\bar{a}^2(\lambda) \end{pmatrix} d\lambda = \delta(x' - x) \cdot I_4, \quad (20)$$

where I_n is the $n \times n$ unit matrix. Let us use the notation $\hat{Z}$ for the matrix $\begin{pmatrix} Z/a^2 \\ \bar{Z}/\bar{a}^2 \end{pmatrix}$ and $\hat{Z}^A$ for the matrix $(Z^A, \bar{Z}^A)$. Using (20), we could then obtain from (17):

$$\delta p(x) = \int_{\Gamma} d\lambda \hat{Z}^A(x, \lambda) \begin{pmatrix} \delta\rho(\lambda) \\ \delta\bar{\rho}(\lambda) \end{pmatrix}. \quad (21)$$

One way to find the form of $\hat{Z}^A$ would be to start with the expression dual to (20). Namely, if one multiplies (21) by $\hat{Z}$, integrates over x , and then takes into account (17), then one would obtain:

$$\frac{i}{4} \langle \hat{Z}(x, \lambda) | \hat{Z}^A(x, \lambda') \rangle = \delta(\lambda - \lambda') \cdot I_2. \quad (22)$$

Once one has such a set of orthogonality relations as in (22), then one could identify $\hat{Z}_A$ from the integrands. This is how we shall proceed.

In earlier works [5–9], the explicit form of $\hat{Z}^A$ was found in the following way. First, find the integro-differential equation for Z of the form:

$$\Lambda Z = 0; \quad (23a)$$

with the integro-differential operator Λ being of the form:

$$\Lambda = \Lambda_0 + \lambda \quad \text{or} \quad \Lambda = \Lambda_0 + \lambda^2, \quad \text{etc.}, \quad (23b)$$

where Λ_0 is a λ -independent part. For the cases in [5–9], it was always found that this could be done even when the original L -operator contained both positive and negative powers of λ (see [9]). Once (23) had been found, it was then straightforward to determine Z^A as a solution of the adjoint equation (hence the superscript ‘A’) and the inner product between Z and Z^A . The apparent

drawback of this procedure was that the form of Λ had always been found by trial and error, which sometimes was a non-trivial matter [6–9]. In the case of the MTM in laboratory coordinates, we have been unable to easily find such an analogue of representation (23). The most apparent difficulty in doing this is that upon taking a derivative of Z_i , $i = 1, \dots, 4$, one then has derivatives of the potentials u, u^*, v, v^* as multiplying factors, which is a complicating feature. Nevertheless, we have found that it is still possible to find the explicit form of Z^A and determined its inner product with Z by means of another technique. Below we will describe this technique in detail since we believe that it will be useful in other related problems.

First, from (3a), for the vector $\vec{V} = (V_1, V_2, V_3)^T$ one straightforwardly obtains:

$$\left[I_3 \frac{\partial}{\partial x} - 2i(w + k) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & q \\ 2q & 2r & 0 \end{pmatrix} \right] \vec{V} = 0, \quad (24a)$$

where $k \equiv k(\lambda)$ and $w = \frac{1}{4}(|u|^2 - |v|^2)$, $r = \frac{i}{2}(\lambda v^* - u^*/\lambda)$, $q = \frac{i}{2}(\lambda v - u/\lambda)$.

Defining the solution $\vec{V}^A$ of the adjoint problem as the vector satisfying

$$\left[-I_3 \frac{\partial}{\partial x} - 2i(w + k) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2q \\ 0 & 0 & 2r \\ r & q & 0 \end{pmatrix} \right] \vec{V}^A = 0, \quad (24b)$$

one finds that $\vec{V}^A = (V_2, V_1, -V_3/2)$. Thus the solution of the adjoint problem is related to that of the original one by a similarity transformation.

4 A New Method

Now we make the observation that, in all the previous cases, one was able to obtain from the corresponding integro-differential equation, (23), a Wronskian relation from the analogue of equation (24) in the following form:

$$\begin{aligned} \frac{\partial}{\partial x} [\alpha(\lambda, \lambda') V_1^A V_1 + \beta(\lambda, \lambda') V_2^A V_2 + \gamma(\lambda, \lambda') V_3^A V_3] = \\ h(\lambda, \lambda') (Z_1^A Z_1 + Z_2^A Z_2 + Z_3^A Z_3 + Z_4^A Z_4). \end{aligned} \quad (25)$$

In (25), the V 's and Z 's are functions of x and λ , the adjoint states V^A 's and Z^A 's are functions of x and λ' , and the coefficients α, β, γ , and h are independent of x . So, if this is the form which always results, is it possible to start with (25), then use all the above relations and uniquely determine α, β, γ, h , and Z^A ? Indeed, such is the case.

We start with (24) and the definition of $\vec{V}^A$. Then (25) becomes of the form (summation over repeated indices is implied):

$$h(\lambda, \lambda') Z_i^A Z_i = V_j' M_{js} V_s, \quad (26a)$$

with

$$M = \begin{pmatrix} 0 & -2i\beta(k - k') & q'\gamma - q\beta \\ 2i\alpha(k - k') & 0 & r'\gamma - r\alpha \\ q\gamma - q'\alpha & r\gamma - r'\beta & 0 \end{pmatrix}, \quad (26b)$$

where $V_j \equiv V_j(\lambda)$, $V_j' \equiv V_j(\lambda')$, etc. From (18) one has:

$$Z_i = m_{is} V_s \quad (27a)$$

with

$$m = \begin{pmatrix} -2\lambda^{-1} & 0 & u^* \\ 0 & 2\lambda^{-1} & u \\ 2\lambda & 0 & -v^* \\ 0 & -2\lambda & -v \end{pmatrix}. \quad (27b)$$

Substituting (27a) into the l.h.s. of (26a), one sees that the resulting equation is linear in V_s . Then, after formally differentiating it with respect to V_s , one obtains:

$$h(\lambda, \lambda') Z_i^A m_{is} = V_j' M_{js}. \quad (28)$$

One cannot invert it to find Z^A because m is not a square matrix. However, if Z^A exists, then it is seen from (28) that it must be of the form:

$$Z_i^A = m_{ij}^A V_j', \quad (29)$$

for some yet unknown 4×3 matrix m^A . The key to determining m^A lies in the fact that the entries of m^A may depend on λ' , but not on λ . Upon substituting (29) into (28) and formally differentiating with respect to V_j' , one obtains:

$$h m_{ij}^A m_{is} = M_{js}. \quad (30)$$

Let us introduce the following notations (see (26b), (27b)):

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \quad m = (D, E), \quad m^A = (F, G),$$

where A, B, C are 2×2 , 2×1 , and 1×2 matrices, respectively. The (22)-entry of M is the number zero, D and F are 4×2 matrices, and E and G are 4×1 matrices. Notice that A and D are independent of the potentials. With these notations, (30) rewrites as follows:

$$\begin{aligned} h F^T \cdot D &= A, & h F^T \cdot E &= B, \\ h G^T \cdot D &= C, & G^T \cdot E &= 0. \end{aligned} \quad (31)$$

Consider the first matrix equation in (31). Using (26b), (27b) and the explicit form of q, r , one obtains:

$$-\frac{1}{\lambda} F_{11} + \lambda F_{31} = 0; \quad (32a)$$

$$h \left(-\frac{1}{\lambda} F_{12} + \lambda F_{32} \right) = i(k - k')\beta; \quad (32b)$$

$$h \left(-\frac{1}{\lambda} F_{21} + \lambda F_{41} \right) = i(k - k')\alpha; \quad (32c)$$

$$-\frac{1}{\lambda} F_{22} + \lambda F_{42} = 0. \quad (32d)$$

Let $F_{ij} = F_{ij}^0 + F_{ij}^p$, where F_{ij}^0, F_{ij}^p are the potential-independent and potential-dependent parts, respectively. Then one easily finds that

$$F_{1j}^p = \lambda^2 F_{3j}^p, \quad F_{2j}^p = \lambda^2 F_{4j}^p, \quad j = 1, 2. \quad (33)$$

Since the matrix m^A is independent of λ , then from (33) it follows that the potential-dependent part of F must be zero.

Using the second equation in (31), one obtains:

$$h \left(u^* F_{11} + u F_{21} - v^* F_{31} - v F_{41} \right) = \frac{i}{2} \left[v(\gamma\lambda' - \beta\lambda) + u \left(\frac{\beta}{\lambda} - \frac{\gamma}{\lambda'} \right) \right], \quad (34a)$$

$$h \left(u^* F_{12} + u F_{22} - v^* F_{32} - v F_{42} \right) = \frac{i}{2} \left[v^*(\gamma\lambda' - \alpha\lambda) + u^* \left(\frac{\alpha}{\lambda} - \frac{\gamma}{\lambda'} \right) \right]. \quad (34b)$$

Since F has been found to be potential-independent, one immediately sees that

$$F_{11} = F_{31} = F_{22} = F_{42} = 0; \quad (35a)$$

$$h F_{21} = \frac{i}{2} \left(\frac{\beta}{\lambda} - \frac{\gamma}{\lambda'} \right); \quad (35b)$$

$$h F_{41} = \frac{i}{2} (\beta \lambda - \gamma \lambda'); \quad (35c)$$

$$h F_{12} = \frac{i}{2} \left(\frac{\alpha}{\lambda} - \frac{\gamma}{\lambda'} \right); \quad (35d)$$

$$h F_{41} = \frac{i}{2} (\alpha \lambda - \gamma \lambda'). \quad (35e)$$

Substituting (35b,c) into (32c) and (35d,e) into (32b), one obtains two linear equations for α , β , γ , from which one straightforwardly deduces:

$$\alpha = \beta, \quad \gamma = \alpha \cdot \frac{\lambda^2 + \lambda'^2}{2\lambda\lambda'}. \quad (36)$$

Without loss of generality, one can set $\alpha = 1$.

Let $G = (g_1, g_2, g_3, g_4)^T$. Then from the third equation in (31) and equation (36) it follows:

$$h \left(-\frac{1}{\lambda} g_1 + \lambda g_3 \right) = \frac{i}{4} \left[\lambda v \frac{\lambda^2 - \lambda'^2}{2\lambda\lambda'} + \frac{u}{\lambda} \frac{\lambda^2 - \lambda'^2}{2\lambda\lambda'} \right]; \quad (37a)$$

$$h \left(\frac{1}{\lambda} g_2 - \lambda g_4 \right) = \frac{i}{4} \left[\lambda v^* \frac{\lambda^2 - \lambda'^2}{2\lambda\lambda'} + \frac{u^*}{\lambda} \frac{\lambda^2 - \lambda'^2}{2\lambda\lambda'} \right]. \quad (37b)$$

Since the entries of G are to be independent of λ , then it must be:

$$\begin{aligned} g_1 &= -\frac{u}{2} f(\lambda'), & g_2 &= \frac{u^*}{2} f(\lambda'), \\ g_3 &= \frac{v}{2} f(\lambda'), & g_4 &= -\frac{v^*}{2} f(\lambda'), \\ h &= \frac{i(\lambda^2 - \lambda'^2)}{4\lambda\lambda' f(\lambda')}, \end{aligned} \quad (38)$$

where $f(\lambda')$ is an arbitrary function. Then the fourth equation in (31) is satisfied identically. As one can see from (35) and (38), the function $f(\lambda')$ simply multiplies the matrix m^A by an arbitrary function of λ' . In order to have a simple form for Z^A , we take $f(\lambda') = 1$. Finally, using (35), (36), (38), one obtains the desired Wronskian relation:

$$\begin{aligned} \frac{\partial}{\partial x} [V_1^A V_1 + V_2^A V_2 + \frac{\lambda^2 + \lambda'^2}{2\lambda\lambda'} V_3^A V_3] = \\ \frac{i(\lambda'^2 - \lambda^2)}{8\lambda\lambda'} \cdot (Z_1 Z_2' + Z_2(-Z_1') + Z_3 Z_4' + Z_4(-Z_3')), \end{aligned} \quad (39)$$

where $Z_j \equiv Z_j(\lambda)$, $Z'_j \equiv Z_j(\lambda')$, $j = 1, \dots, 4$. With this we can define the inner product that we need. We note that a formula, which proves to be equivalent to (39), was given (without derivation) in [1].

From (39), it follows that we may take

$$Z^A = \frac{i}{2\pi}(Z_2, -Z_1, Z_4, -Z_3)^T, \quad \bar{Z}^A = -\frac{i}{2\pi}(\bar{Z}_2, -\bar{Z}_1, \bar{Z}_4, -\bar{Z}_3)^T,$$

and we choose to evaluate the Z - and Z^A -states using the Jost solutions ϕ and χ , respectively. Then, from (5) and (39), we obtain, for $\text{Im}\lambda^2 = 0$:

$$\frac{i}{4} \langle \hat{Z} | \hat{Z}'^A \rangle = 2\lambda\lambda' \delta(\lambda^2 - \lambda'^2) \cdot I_2. \quad (40)$$

Similarly, the inner products involving the discrete spectrum are:

$$\langle \hat{Z}_n | \hat{Z}_m^A \rangle = 0, \quad (41a)$$

$$\langle \hat{D}_n | \hat{D}_m^A \rangle = \begin{pmatrix} \frac{2}{\pi} \lambda_n \dot{a}_n \ddot{a}_n (\delta_{n,m} + \delta_{n,-m}) & 0 \\ 0 & \frac{2}{\pi} \lambda_{\bar{n}} \dot{\bar{a}}_{\bar{n}} \ddot{\bar{a}}_{\bar{n}} (\delta_{\bar{n},\bar{m}} + \delta_{\bar{n},-\bar{m}}) \end{pmatrix} \quad (41b)$$

$$\langle \hat{D}_n | \hat{Z}_m^A \rangle = \langle \hat{Z}_n | \hat{D}_m^A \rangle = \begin{pmatrix} \frac{2}{\pi} \lambda_n \dot{a}_n^2 (\delta_{n,m} + \delta_{n,-m}) & 0 \\ 0 & \frac{2}{\pi} \lambda_{\bar{n}} \dot{\bar{a}}_{\bar{n}}^2 (\delta_{\bar{n},\bar{m}} + \delta_{\bar{n},-\bar{m}}) \end{pmatrix}, \quad (41c)$$

for $m, n = -N, \dots, N$; $m, n \neq 0$. In (41), we have used the following shorthand notations:

$$\hat{Z}_n = (Z^T(x, \lambda_n), \bar{Z}^T(x, \lambda_{\bar{n}}))^T, \quad \hat{D}_n = \left(\frac{\partial}{\partial \lambda} Z^T(x, \lambda)|_{\lambda_n}, \frac{\partial}{\partial \lambda} \bar{Z}^T(x, \lambda)|_{\lambda_{\bar{n}}} \right)^T,$$

$$\hat{Z}_n^A = (Z^A(x, \lambda_n), \bar{Z}^A(x, \lambda_{\bar{n}})), \quad \hat{D}_n^A = \left(\frac{\partial}{\partial \lambda} Z^A(x, \lambda)|_{\lambda_n}, \frac{\partial}{\partial \lambda} \bar{Z}^A(x, \lambda)|_{\lambda_{\bar{n}}} \right).$$

Let us now derive a generalization of (20) to the case when the L -operator (3a) has a discrete spectrum. Taking into account (40), consider the following principal value integral:

$$\frac{i}{8} \oint_{\Gamma} \frac{d\lambda}{\lambda} \hat{Z}^A(\chi(x)) \cdot \hat{Z}(\phi(y)) =$$

$$= \begin{cases} -\frac{i}{8} \left[\left(\int_{C_0^+} + \int_{C_\infty^+} \right) \frac{d\lambda}{\lambda a^2} Z^A(\chi(x)) Z(\phi(y)) + \right. \\ \left. + \left(\int_{C_0^-} + \int_{C_\infty^-} \right) \frac{d\lambda}{\lambda \bar{a}^2} \bar{Z}^A(\chi(x)) \bar{Z}(\phi(y)) \right] - \frac{\pi}{4} (\text{D.Sp.}) , \\ \text{for } y \geq x ; \\ \\ -\frac{i}{8} \left[\left(\int_{C_0^+} + \int_{C_\infty^+} \right) \frac{d\lambda}{\lambda a^2} (2Z^A(\chi\phi(x)) Z(\chi\phi(y)) - Z^A(\phi(x)) Z(\chi(y))) + \right. \\ \left. + \left(\int_{C_0^-} + \int_{C_\infty^-} \right) \frac{d\lambda}{\lambda \bar{a}^2} (2\bar{Z}^A(\chi\phi(x)) \bar{Z}(\chi\phi(y)) - \bar{Z}^A(\phi(x)) \bar{Z}(\chi(y))) \right] - \frac{\pi}{4} (\text{D.Sp.}) , \\ \text{for } y \leq x ; \end{cases} \quad (42a)$$

where, for instance, $Z(\chi\phi(y))$ is evaluated for the functions $\chi(y, \lambda)$ and $\phi(y, \lambda)$ as in (19); $Z_n(\phi(y)) = Z(\phi(y, \lambda_n))$, etc. Now in (42a), each case contains the term (D.Sp.), which is the contribution to (42a) of the discrete spectrum. This term is given by

$$\begin{aligned} (\text{D.Sp.}) = & \sum_{n=-N, n \neq 0}^N \left\{ -\left(\frac{\dot{a}_n + \lambda_n \ddot{a}_n}{\dot{a}_n^3 \lambda_n^2} \right) Z_n^A(\chi(x)) Z_n(\phi(y)) + \right. \\ & + \frac{1}{\dot{a}_n^2 \lambda_n} [D_n^A(\chi(x)) Z_n(\phi(y)) + Z_n^A(\chi(x)) D_n(\phi(y))] \} \\ & + \{ \text{same quantities with the bar} \} . \end{aligned} \quad (42b)$$

The contours $C_0^\pm$, $C_\infty^\pm$ are shown in Fig.1. In deriving (42), we have used the analyticity of the Jost functions in their corresponding domains, and also we made use of relations (9) to obtain the representation of the l.h.s. in the regions $y \geq x$ and $y \leq x$ (see [6]). Now, one can use the asymptotics (14), (15) to evaluate the continuous spectrum part in the r.h.s. of (42a) and then “pull back” onto the contour Γ , after deleting those terms which give vanishing contributions at infinity. In this way, upon also using the identities

$$\oint_{\Gamma} \frac{d\lambda}{\lambda^3} e^{ik(\lambda)x} = \oint_{\Gamma} \lambda d\lambda e^{ik(\lambda)x} = 8\pi\delta(x) ,$$

we arrive at the formula which is the main result of this paper:

$$\begin{aligned}
& \frac{i}{8} \oint_{\Gamma} \frac{d\lambda}{\lambda} \hat{Z}^A \cdot \hat{Z} + \frac{\pi}{4} \sum_{n=-N, n \neq 0}^N \left\{ -\hat{Z}_n^A \cdot \begin{pmatrix} (\dot{a}_n + \lambda_n \ddot{a}_n) Z_n / (\dot{a}_n^3 \lambda_n^2) \\ (\dot{\bar{a}}_{\bar{n}} + \lambda_{\bar{n}} \ddot{\bar{a}}_{\bar{n}}) \bar{Z}_{\bar{n}} / (\dot{\bar{a}}_{\bar{n}}^3 \lambda_{\bar{n}}^2) \end{pmatrix} + \right. \\
& \quad \left. + \hat{D}_n^A \cdot \begin{pmatrix} Z_n / (\dot{a}_n^2 \lambda_n) \\ \bar{Z}_{\bar{n}} / (\dot{\bar{a}}_{\bar{n}}^2 \lambda_{\bar{n}}) \end{pmatrix} + \hat{Z}_n^A \cdot \begin{pmatrix} D_n / (\dot{a}_n^2 \lambda_n) \\ \bar{D}_{\bar{n}} / (\dot{\bar{a}}_{\bar{n}}^2 \lambda_{\bar{n}}) \end{pmatrix} \right\} = \\
& \quad = \delta(x - y) \cdot I_4 .
\end{aligned} \tag{43}$$

In (43) each Z^A and D^A are to be evaluated for $\chi(x, \lambda)$ and each Z and D are to be evaluated for $\phi(y, \lambda)$. Notice that, making use of symmetry (8), (12), one may restrict the summation in (42b), (43), and formulae (45), (46) below to be over $n = 1, \dots, N$ (one should then take the overall coefficient in front of the sum sign to be a factor of 2 larger), and at the same time, replace $\oint_{\Gamma}$ by $2 \oint_{\Gamma+}$.

Now we can produce formulae for the expansions of both the potential $p(x) = (u, u^*, v, v^*)^T$ and $\delta p(x)$ over the complete set $\{Z^A\}$. In the standard way (see [10]) one obtains:

$$\delta \lambda_n = \frac{i}{4 b_n \dot{a}_n} < Z_n(\phi) | \delta p > , \tag{44a}$$

$$\delta c_n = \frac{i}{4 \dot{a}_n^2} \left[-\left(\frac{\ddot{a}_n}{\dot{a}_n} \right) < Z_n(\phi) | \delta p > + < D_n(\phi) | \delta p > \right] ; \tag{44b}$$

the quantity c_n was defined before formula (16). Note that in the derivation of (44), one must take into account the fact that $b_n, \dot{a}_n$ depend on the potential $p(x)$ both implicitly, as functionals, and explicitly, through their dependence on λ_n . Equations (17), (44), (43) now lead to the following expansion formula for δp :

$$\begin{aligned}
\delta p = & \frac{1}{2} \oint_{\Gamma} \frac{d\lambda}{\lambda} \hat{Z}^A(\chi) \begin{pmatrix} \delta \rho \\ \delta \bar{\rho} \end{pmatrix} - i\pi \sum_{n=-N, n \neq 0}^N \left[\hat{Z}_n^A \begin{pmatrix} \delta(c_n/\lambda_n) \\ -\delta(\bar{c}_{\bar{n}}/\lambda_{\bar{n}}) \end{pmatrix} + \right. \\
& \quad \left. + \hat{D}_n^A \begin{pmatrix} (c_n/\lambda_n) \delta \lambda_n \\ -(\bar{c}_{\bar{n}}/\lambda_{\bar{n}}) \delta \lambda_{\bar{n}} \end{pmatrix} \right] .
\end{aligned} \tag{45}$$

From the third row in (24a) one easily finds the inner products between the

Z -states and the potential $p(x)$. Whence one obtains:

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} p(x) = \frac{1}{2} \oint_{\Gamma} \frac{d\lambda}{\lambda} \hat{Z}^A(\lambda) \begin{pmatrix} -\rho \\ \bar{\rho} \end{pmatrix} + i\pi \sum_{n=-N, n \neq 0}^N \hat{Z}_n^A(\lambda) \begin{pmatrix} c_n/\lambda_n \\ \bar{c}_n/\lambda_n \end{pmatrix}. \quad (46)$$

Note that the D_n^A states are absent in (46).

For possible applications, we present below the exact expressions of the one-soliton potentials $u_s(x)$, $v_s(x)$, the corresponding scattering data, and the Jost solutions $\chi(x, \lambda)$ and $\chi(x, \lambda_1)$, from which the Z - and Z^A -states may be computed as needed:

$$u_s(x, t) = \sin Q \operatorname{sech}(\Theta - \frac{iQ}{2}) e^{i\beta}, \quad (47)$$

$$v_s(x, t) = -\sin Q \operatorname{sech}(\Theta + \frac{iQ}{2}) e^{i\beta};$$

$$0 \leq Q \leq \pi, \quad \Theta = x \sin Q, \quad \beta = -t \cos Q,$$

$$\lambda_1 = \exp(iQ/2), \quad c_1 = \sin Q \cdot e^{i\beta}, \quad (48)$$

$$a(\lambda) = \frac{\lambda^2 - \lambda_1^2}{\lambda^2 - \lambda_1^{*2}} e^{-iQ}, \quad b(\lambda) = 0 \quad \text{for } \operatorname{Im} \lambda^2 = 0;$$

$$\chi(x, \lambda_1) = \frac{1}{2} \begin{pmatrix} i \operatorname{sech}(\Theta + \frac{iQ}{2}) \cdot \exp(-i\beta + iA(x) + (\Theta - iQ)/2) \\ \operatorname{sech}(\Theta - \frac{iQ}{2}) \cdot \exp(-iA(x) - (\Theta - iQ)/2) \end{pmatrix}, \quad (49)$$

where

$$A(x) = \arctan(\tan(\frac{Q}{2}) \tanh \Theta) + \frac{Q}{2}.$$

$$\chi(x, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-iA(x) - ik(\lambda)x} - \frac{\exp(-ik(\lambda)x) \cdot \sin Q}{\lambda^2 - \lambda_1^{*2}} \cdot \begin{pmatrix} \lambda \operatorname{sech}(\Theta + \frac{iQ}{2}) \cdot \exp(-i\beta - iQ + iA(x)) \\ i \operatorname{sech}(\Theta - \frac{iQ}{2}) \cdot \exp(\Theta - iA(x) - iQ/2) \end{pmatrix}. \quad (50)$$

Formulae (49), (50) were obtained from (16) and (48).

5 Concluding Remarks

The main difference between the result of the present work and those of the earlier ones ([4–9]) on this subject is a new method for finding squared eigenfunctions. Previously, one has been able to find both the exact form of the

“squared eigenfunctions” and the operator Λ_0 (see (23)), whose eigenfunctions were the “squared eigenfunctions”. The knowledge of that operator allowed one to construct a hierarchy of integrable evolutionary equations, related to a given L -operator, and their infinite series of conservation laws. Let us note, however, that the infinite series of the conservation laws for equations (1) was obtained in [1] by standard “elementary” means, and thus the whole hierarchy of integrable evolutionary equations of which the Massive Thirring Model in laboratory coordinates is a member, is in principle known. We have proposed a regular procedure by which the structure of the inner product, and thus the explicit form of Z^A , can be determined. This procedure gave us the result for equations (1), and we believe that it will be useful in other related problems.

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7 Appendix

Here we demonstrate how the form of the “squared eigenfunctions” can be found from the Riemann–Hilbert (RH) formulation of the inverse scattering problem for operator (4). Below we will assume that the coefficients $a(\lambda)$, $\bar{a}(\lambda)$ are non-zero in their respective domains of analyticity.

The second equations in (9a,b) can be rewritten in the form:

$$\Psi^+ = \Psi^- G, \quad \lambda \in \Gamma, \quad (A.1a)$$

where the 2×2 matrices $\Psi^\pm$, G are:

$$\Psi^+ = \left(\frac{\phi}{a} e^{-ikx}, \chi e^{ikx} \right), \quad \Psi^- = \left(\bar{\chi} e^{-ikx}, -\frac{\bar{\phi}}{\bar{a}} e^{ikx} \right), \quad (A.1b)$$

$$G = \begin{pmatrix} 1 + \rho \bar{\rho} & \bar{\rho} e^{2ikx} \\ \rho e^{-2ikx} & 1 \end{pmatrix}.$$

In deriving (A.1), equation (13) was used. $\Psi^\pm$ are analytic where $\text{Im } \lambda^2 \gtrless 0$, respectively. If the potentials in (4) vary, then $\Psi^\pm$, G will vary according to the equation:

$$\delta\Psi^+(\Psi^+)^{-1} = \delta\Psi^-(\Psi^-)^{-1} + \Psi^-\delta G(\Psi^+)^{-1}, \quad \lambda \in \Gamma, \quad (\text{A.2})$$

where we have used: $G = \Psi^+(\Psi^-)^{-1}$. Matrices $(\Psi^\pm)^{-1}$ are non-singular (see (11)), thus $\delta\Psi^\pm(\Psi^\pm)^{-1}$ are analytic in the corresponding domains of the λ -plane. Then (A.2) constitutes a simple RH problem, whose solution is:

$$\delta\Psi(x, \lambda) = \delta\Psi^0(x)\Psi(x, \lambda) + \frac{1}{2i\pi} \oint_\Gamma \frac{\Phi(x, \xi)d\xi}{\xi - \lambda} \Psi(x, \lambda), \quad (\text{A.3})$$

where $\Psi(x, \lambda)$ is the solution of (A.1). In (A.3),

$$\Phi = \Psi^-\delta G(\Psi^+)^{-1}, \quad (\text{A.4a})$$

or in the component form:

$$\begin{aligned} \Phi_{11} &= -\Phi_{22} = \delta\rho \chi_1 \chi_2 - \delta\bar{\rho} \bar{\chi}_1 \bar{\chi}_2, \\ \Phi_{12} &= -\delta\rho \chi_1^2 + \delta\bar{\rho} \bar{\chi}_1^2, \quad \Phi_{21} = \delta\rho \chi_2^2 - \delta\bar{\rho} \bar{\chi}_2^2. \end{aligned} \quad (\text{A.4b})$$

Matrix $\delta\Psi^0(x)$ in (A.3) should be determined from the asymptotics at $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$. Taking into account symmetry (8), (12), it is convenient to rewrite (A.3) as follows:

$$\begin{aligned} \delta\Psi(x, \lambda) &= \delta\Psi^0(x)\Psi(x, \lambda) + \\ &+ \frac{1}{i\pi} \int_{\Gamma^+} \frac{d\xi}{\xi^2 - \lambda^2} \left[\xi \begin{pmatrix} \Phi_{11} & 0 \\ 0 & -\Phi_{11} \end{pmatrix} + \lambda \begin{pmatrix} 0 & \Phi_{12} \\ \Phi_{21} & 0 \end{pmatrix} \right] \Psi(x, \lambda). \end{aligned} \quad (\text{A.5})$$

The desired variations δu , δu^* , δv , δv^* are found from the asymptotics of (A.5) for $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$. From (14), (15) one finds:

$$\underline{\lambda \rightarrow 0} \quad \Psi^+ \rightarrow \begin{pmatrix} e^{-iQ^-} & \lambda u^* e^{iQ^-} \\ -\lambda u e^{-iQ^-} & e^{iQ^-} \end{pmatrix}, \quad (\text{A.6a})$$

$$\underline{\lambda \rightarrow \infty} \quad \Psi^+ \rightarrow \begin{pmatrix} e^{iQ^-} & \lambda^{-1} v^* e^{-iQ^-} \\ -\lambda^{-1} v e^{iQ^-} & e^{-iQ^-} \end{pmatrix}, \quad (\text{A.6b}),$$

where $Q^- \equiv Q^-(x)$. Substituting (A.6a) into (A.5), one finds:

$$\delta\Psi_{12}^0(x) = \delta\Psi_{21}^0(x) = 0; \quad (\text{A.7a})$$

$$I_2(i\delta Q^- + \frac{1}{i\pi} \int_{\Gamma^+} \frac{\Phi_{11} d\xi}{\xi}) + \sigma_3 \text{diag}(\delta\Psi_{11}^0(x), \delta\Psi_{22}^0(x)) = 0 ; \quad (\text{A.7b})$$

$$\delta u = -\frac{1}{i\pi} \int_{\Gamma^+} \frac{\Phi_{21} d\xi}{\xi^2} + 2i\delta Q^- u ; \quad (\text{A.7c})$$

$$\delta u^* = \frac{1}{i\pi} \int_{\Gamma^+} \frac{\Phi_{12} d\xi}{\xi^2} - 2i\delta Q^- u^* . \quad (\text{A.7d})$$

Substituting (A.6b) into (A.5) and using (A.7), one obtains:

$$\delta\Psi_{11}^0(x) = -\delta\Psi_{22}^0(x) = i\delta Q^- = -\frac{1}{2i\pi} \int_{\Gamma^+} \frac{\Phi_{11} d\xi}{\xi} ,$$

and then

$$\begin{aligned} \delta u &= -\frac{1}{i\pi} \int_{\Gamma^+} \left(\frac{\Phi_{21}}{\xi^2} + u \frac{\Phi_{11}}{\xi} \right) d\xi ; \\ \delta u^* &= \frac{1}{i\pi} \int_{\Gamma^+} \left(\frac{\Phi_{12}}{\xi^2} + u^* \frac{\Phi_{11}}{\xi} \right) d\xi ; \\ \delta v &= \frac{1}{i\pi} \int_{\Gamma^+} \left(\Phi_{21} + v \frac{\Phi_{11}}{\xi} \right) d\xi ; \\ \delta v^* &= -\frac{1}{i\pi} \int_{\Gamma^+} \left(\Phi_{12} + v^* \frac{\Phi_{11}}{\xi} \right) d\xi . \end{aligned} \quad (\text{A.8})$$

Equations (A.8) are equivalent to formula (45) without the contribution from the discrete spectrum. As it is known, the discrete spectrum is introduced into the RH problem through the solution of a certain system of algebraic equations. Working with the variation of this system, occuring due to the variation of the discrete scatterind data, is much more cumbersome than simply picking up the contribution from the poles in the corresonding integrals (formula (42)), therefore we did not pursue the former way here.

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8 Caption to the Figure

Fig.1. Contours $C_0^\pm$, $C_\infty^\pm$, and $\Gamma^\pm$ in the complex λ -plane. Contour Γ is the union of Γ^+ and Γ^- .