FILTERING OUT NEAR-SURFACE UNCERTAINTIES FROM HELIOSEISMIC INVERSIONS


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Filtering out near-surface uncertainties from helioseismic inversions

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ABSTRACT
The differences between observed solar p-mode frequencies and the frequencies of a reference model can be inverted to infer the structure of the Sun using a variety of linear inversion techniques. However, it is well known that the adiabatic description, which is often employed in frequency calculations, breaks down in the outermost layers of the Sun. This and other uncertainties in the treatment of the superficial layers introduce a slowly varying frequency-dependent function into the frequency differences.

We present here a technique to pre-process the frequency differences prior to applying any of the standard inversion techniques in order to eliminate the frequency-dependent component which arises from the near-surface uncertainties, suppressing also the corresponding contributions from the kernels relating frequency differences to differences in structure. This is achieved by applying suitable linear filters to the frequency differences and to the mode kernels. We obtain the filtered kernels and demonstrate that the surface contribution can be successfully suppressed. We also present the results of the inversions performed after the data and mode kernels have been filtered.

Key words: methods: data analysis — Sun: interior — Sun: oscillations

1 INTRODUCTION
The generic inverse problem consists of determining the properties of a given physical system from sets of measured data. A basic assumption in inversion procedures is that the corresponding forward problem has been solved; that is, given a model of the relevant properties of the physical system, the corresponding theoretical values for the data are calculable. Yet under many circumstances this assumption is not satisfied: the relations between the model and the data suffer from uncertainties concerning other aspects of the system, or potential errors in the way that the data relate to the model (cf. Mosegaard & Jacobsen 1995). In such cases it is necessary to design inversion methods that are, as far as possible, insensitive to the uncertain aspects.

In helioseismology, potential uncertainties arise both from the basic structure of the solar models and from the physics of the oscillations. In inversions for solar structure, it is generally assumed that the Sun is spherically symmetric and in hydrostatic equilibrium. Departures from spherical symmetry (such as that caused by rotation) can in general be dealt with through a perturbation expansion. However, the assumption of hydrostatic equilibrium may be compromised by effects of convection, contributing to the hydrostatic balance through the turbulent pressure, as well as by small-scale magnetic fields near the solar surface effectively acting as a pressure.

The second area of uncertainty is the physics of the modes. It is frequently assumed in the helioseismic forward problem that the solar oscillations satisfy the equations of linear adiabatic oscillations (cf. Unno et al. 1989). However, while the adiabatic approximation is expected to be valid in the interior (where the thermal relaxation time scales are much larger than the oscillation periods), it certainly breaks down at the photosphere. There is no currently accepted formulation to treat the effect of nonadiabaticity on the oscillation frequencies; for, although several attempts have been made to calculate the effect, the results do not agree (e.g. Christensen-Dalsgaard & Frandsen 1983; Cox, Guzik & Kilman 1989; Pesnell 1990; Balinforth 1992; Guenther 1994).

Much of this uncertainty arises from the treatment of turbulent convection: the calculations must take into account the perturbations in the convective flux and the turbulent pressure, the latter having also potentially a direct dynam-
ichanical effect on the oscillation frequencies (e.g. Rosenthal et al. 1985).

Inversion for solar structure (e.g. Gough & Kosovichev 1988, 1990, 1993; Dziembowski, Pamyatnykh & Sienkiewicz 1995; Däppen et al. 1991; Antia & Basu 1994; Dziembowski et al. 1994) generally proceeds through a linearization of the equations of stellar oscillation around a known reference model. The differences between the structure of the Sun and the reference model are then related to the differences in the frequencies of the Sun and the model by kernels (cf. equation [2] below). Errors in the forward modelling will mean that the kernels do not fully account for the frequency differences, and thus errors may be introduced into the results of the inversions.

An important property is that the errors in the assumptions underlying our forward modelling are mostly concentrated very near the solar surface. This is certainly the case for the effects of turbulent convection and nonadiabaticity, and is likely to be true also of magnetic effects unless extremely strong fields are present in the deep solar interior. Comparison of theoretical p-mode frequencies of different models shows that the frequency differences for low- to moderate-degree modes due to changes in the surface structure are largely a function of frequency (e.g. Christensen-Dalsgaard & Berthomieu 1991; Christensen-Dalsgaard & Thompson 1995b). In ray-theoretic terms, this comes about because in the near-surface layers the ray paths of such modes are essentially independent of the degree l of the mode. It follows that the local effect of these layers on the frequencies of low- and moderate-degree modes is independent of l, and is therefore a function only of frequency. There is a degree-dependence that comes about simply because shallowly penetrating modes are more easily perturbed than deeply penetrating ones. To be precise, it may be shown that frequency changes resulting from such effects are inversely proportional to the mode inertia (e.g. Christensen-Dalsgaard & Berthomieu 1991).

A modification to the model that is localized at some radius $r^*$ introduces a perturbation to the frequencies that is proportional to

$$
\cos(2\tau^* \omega + \phi), \quad \tau^* = \frac{\int_{r^*}^R \frac{dr}{c}}{c},
$$

where $\tau^*$ is the acoustical depth of the model modification, $c$ being the adiabatic sound speed; also, $\omega$ is the angular frequency of oscillation and $\phi$ is a phase. Hence the modification contributes to the frequency differences a component which is an oscillatory function of $\omega$, with a ‘frequency’ $f = 2\tau^*$ which increases with the depth of the modification (e.g. Gough 1990; Christensen-Dalsgaard & Pérez Hernández 1992). In particular, provided they have no intrinsic frequency dependence of their own, the superficial layers are expected to introduce shifts which are slowly varying functions of frequency.

In the asymptotic description of acoustic modes the effects of the near-surface region are contained in a phase function $\alpha(\omega)$ which essentially describes the reflection of waves at the upper turning point. Even the initial asymptotic inversion to determine solar sound speed (e.g. Gough 1984; Christensen-Dalsgaard et al. 1985) included procedures to determine $\alpha$ and eliminate its influence on the data, thereby suppressing the effects of the near-surface uncertainties on the results of the inversion. In the differential asymptotic inversion (Christensen-Dalsgaard, Gough & Thompson 1989) the frequency differences between the Sun and a reference model are written in terms of two functions, $\mathcal{H}_1$ and $\mathcal{H}_2$, where $\mathcal{H}_1$ is determined by the sound-speed difference in the bulk of the Sun and $\mathcal{H}_2$ by the differences in conditions near the solar surface [see also equation (9) below]. The function $\mathcal{H}_2$ depends mainly on frequency; in particular, it contains a contribution varying slowly with frequency from the effects arising very near the surface. It is determined as part of the fit to the data, thus eliminating the near-surface contributions from the function $\mathcal{H}_1$ from which the sound-speed correction is inferred.

In the general treatment of the helioseismic inverse problem, the variational principle for the frequencies of adiabatic oscillation is used to express the frequency differences between the Sun and a model in terms of corresponding differences in structure (e.g. Gough & Thompson 1991). To this must be added the contribution from the effects of the near-surface errors. From the discussion above of the properties of these it is plausible that the general expression for the frequency differences is of the form

$$
\frac{\delta \omega_i}{\omega_i} = \int K_{l^* 2}^l(r) \frac{\delta f_1(r)}{f_1(r)} \, dr + \int K_{l^* 1}^l(r) \frac{\delta f_2(r)}{f_2(r)} \, dr + \frac{F_{\ell 0}(\omega_i)}{Q_l},
$$

(cf. Dziembowski et al. 1990). Here $\delta \omega_i$ is the difference in the frequency $\omega_i$ of the $l^{th}$ mode between the solar data and a reference model. The functions $f_1$ and $f_2$ are an appropriate pair of model parameters. The kernels $K_{l^* 2}^l$ and $K_{l^* 1}^l$ are known functions of the reference model which relate the changes in frequency to the changes in $f_1$ and $f_2$ respectively; and $Q_l$ is the inertia of the mode, normalized by the photospheric amplitude of the displacement, divided by the inertia that a radial (l = 0) mode of the same frequency would have (Christensen-Dalsgaard 1986). The term in $F_{\ell 0}$ results from the near-surface errors.

The pair $(f_1, f_2)$ can involve, for example, sound speed $c$, pressure $p$, density $\rho$ and the adiabatic exponent $\Gamma_1 = (\partial \ln p/\partial \ln \rho)$; here $p$ and $\rho$ are related through the equation of hydrostatic support. In the following, we shall as an example use $(c^2, \rho)$. If in addition the equation of state and the heavy-element abundances are assumed to be known, $\Gamma_1$ can be expressed in terms of $p$, $\rho$, and the helium abundance $Y$. This allows equation (2) to be expressed in terms of a thermodynamic variable and $Y$. Convenient pairs are $(u, Y)$, where $u = p/\rho$, and $(\rho, Y)$.

A number of techniques have been used to solve equation (2). One is to obtain a least-squares solution of the equations with a penalty function to ensure that the solution is sufficiently smooth. This is known as the regularized least squares (RLS) technique. Another such technique is optimal localized averages (OLA), which involves forming linear combination of frequency differences in such a way that the corresponding combination of kernels provides a localized average of the unknown function. Each of these techniques has developed ways to handle the surface term. In RLS the function $F_{\ell 0}(\omega)$ is obtained simultaneously with the functions $\delta f_1(r)/f_1(r)$ and $\delta f_2(r)/f_2(r)$ (cf. Dziembowski et al. 1990; Antia & Basu 1994). In OLA, the surface term has been taken care of by constraining the linear combi-
nations of data to be insensitive to the presence of slowly varying functions of frequency in the scaled frequency differences (e.g., Dapper et al. 1991; Kosovichev et al. 1992; Christensen-Dalsgaard & Thompson 1995a).

Pérez Hernández & Christensen-Dalsgaard (1994a,b) developed a filtering technique to eliminate the effects of uncertainties in the structure function in the asymptotic description of the frequencies. They successfully obtained the signature of the second helium ionization zone by suppressing the slowly varying components in \( N_2 \). The aim of this paper is to generalize such an approach to general inversion techniques. Specifically we require a method for filtering out the effects of surface uncertainties in the frequency differences before carrying out the inversion. Since such filtering unavoidably also suppresses those components of \( \delta \omega_i/\omega_i \), which arise from genuine differences in structure, i.e., \( \delta f_1/f_1 \) and \( \delta f_2/f_2 \), the corresponding contributions from the kernels \( K_{1,2} \) and \( K_{2,3} \) are also suppressed for consistency. We do this by employing a filter which removes low-frequency components from the frequency differences while simultaneously suppressing the kernels at the surface. In this paper we show how such a filter can be constructed, and illustrate the effect of filtering on helioseismic data and kernels. We also demonstrate the efficacy of the technique by inverting for the relative differences in sound speed between two solar models. It is shown that results for structural inversions improve when the surface contributions are removed from the data and the kernels.

2 SOLAR MODELS USED

We use two solar models to test the techniques that we present here: one acts as a known reference model, and the other is a test model that takes the role of the unknown Sun. Both models are constructed with OPAL opacities (Iglesias, Rogers & Wilson 1992) and the CEF equation of state (cf. Christensen-Dalsgaard & Dapper 1992). The reference model (R1) has no diffusion of helium or other elements but is otherwise a standard solar model. The test model T1 incorporates gravitational settling of helium. In addition, the surface properties of model T1 have been altered by increasing the surface opacity to twice the normal value in order to give a large surface contribution to the frequency differences between the two models. The properties of the models are summarized in Table 1.

Table 1. Properties of solar models

<table>
<thead>
<tr>
<th>Model</th>
<th>( Z )</th>
<th>( Y_e )</th>
<th>( Y_c )</th>
<th>( T_c )</th>
<th>( \rho_c )</th>
<th>( r_d/R_\odot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1(1)</td>
<td>0.02</td>
<td>0.2798</td>
<td>0.641</td>
<td>15.60</td>
<td>150.7</td>
<td>0.722</td>
</tr>
<tr>
<td>T1(2)</td>
<td>0.02</td>
<td>0.2402</td>
<td>0.652</td>
<td>15.71</td>
<td>156.6</td>
<td>0.709</td>
</tr>
</tbody>
</table>

The heavy element abundance \( Z \); the current envelope and central helium abundances \( Y_e \) and \( Y_c \); the central temperature \( T_c \) and density \( \rho_c \); and the radial location of the base of the convective envelope \( (r_d/R_\odot) \), in the models used in this paper.

The heavy element abundance \( Z \); the current envelope and central helium abundances \( Y_e \) and \( Y_c \); the central temperature \( T_c \) and density \( \rho_c \); and the radial location of the base of the convective envelope \( (r_d/R_\odot) \), in the models used in this paper.

Notes: (1)Does not incorporate gravitational settling of helium or other elements. (2)Has gravitational settling of helium but not of heavy elements; also, has higher than normal opacities in the surface layers.

\[ \sum c_i(r_0)\delta \omega_i/\omega_i \] provides a localized average of the quantity \( \delta f_1(r)/f_1(r) \) around a target radius \( r = r_0 \). It follows from equation (2) that

\[ \sum c_i(r_0)\delta \omega_i/\omega_i = \int_0^R \sum c_i(r_0)K_{1,2}^i(r)\delta f_1(r)/f_1(r)dr \]

\[ + \int_0^R \sum c_i(r_0)K_{2,3}^i(r)\delta f_2(r)/f_2(r)dr \]

\[ + \sum c_i(r_0)E_{\text{avf}}(\omega_i)/Q_i \]  

The first term in the right hand side of equation (3) is an average of the underlying difference \( \delta f_1/f_1 \) weighted by an averaging kernel

\[ K(r_0,r) \equiv \sum c_i(r_0)K_{1,2}^i(r) \]  

The second term on the RHS of equation (3) is the influence of the second function on the solution of the first, viz an integral of \( \delta f_2/\delta f_2 \) with weighting

\[ C_{2,1}(r_0,r) \equiv \sum c_i(r_0)K_{2,3}^i(r) \]  

The third term shows the influence of the surface terms.

The SOLA method allows one to specify the desired shape of the averaging kernel in terms of a target function \( T(r_0,r) \). We choose \( T(r_0,r) \) to be a Gaussian of unit area centred at \( r_0 \).

The coefficients \( c_i \) are selected in such a way as to balance the aims of making the averaging kernel resemble the target function, minimizing the contamination from \( \delta f_2/\delta f_2 \) via \( C_{2,1} \), and minimizing the effect of noise in the data. Specifically, they are chosen to minimize

\[ \int \left( \sum c_iK_{1,2}^i - T \right)^2 dr + \beta \int \left( \sum c_iK_{2,3}^i \right)^2 dr \]

\[ + \mu \sum c_iE_{\text{avf}} \]  

with the constraint that the averaging kernel be unimodular, i.e.,

\[ \sum c_i(r_0) \int_0^R K_{1,2}^i(r)dr = 1 \]
Here $E_i$ is the covariance matrix of errors in the data. The parameter $\beta$ controls the contribution of the second term to the sum given by equation (3), and $\mu$ is a trade-off parameter which controls the effect of data noise.

Additional constraints may be applied to reduce the influence of near-surface uncertainties. In particular, one can demand that
\begin{equation}
\sum \lambda (\tau_0) Q_1^{-1} \Phi_\lambda (\omega) = 0, \quad \lambda = 0, \ldots, \Lambda .
\end{equation}
where the $\Phi_\lambda$ are chosen functions, e.g. Legendre polynomials with a suitably scaled argument (cf. Dapen et al. 1991). In this work we shall not apply such additional constraints. Indeed it is our contention that the same goal of suppressing the surface uncertainties can be achieved with greater flexibility by the method of filtering that we present below. However, we shall return to the question of additional constraints in Section 6.

\section{The Filtering Procedure}

The fact that the unwanted surface effects in helioseismic data manifest themselves as slowly varying functions of frequency means that one can design a filter to remove them. In essence, this requires a high-pass filter which blocks slowly varying functions of frequency. In what follows, we describe how a general filter function can be used to construct a filter suitable for removing the slowly varying component from helioseismic scaled frequency differences.

Guided by asymptotics, we can express the frequency differences between two solar models, or the Sun and a solar model, as
\begin{equation}
S_k \frac{\delta \omega_i}{\omega_i} \simeq \mathcal{H}_1 \left( \frac{\omega_i}{L} \right) + \mathcal{H}_2 (\omega_i)
\end{equation}
(cf. Christensen-Dalsgaard, Gough & Perez Hernandez 1988), with $L = i+1/2$, where $L$ is the degree of mode $i$. Here $S_k$ is a scaling factor which asymptotically is proportional to $Q_1$, (cf. Christensen-Dalsgaard 1991), and the slowly varying component of $\mathcal{H}_2 (\omega)$ corresponds asymptotically to the function that we wish to filter out. In practice, we make a least-squares fit of the sum of two splines to the left-hand side of equation (9). Thus, the functions $\mathcal{H}_1$ and $\mathcal{H}_2$ are expressed as splines in $w = \omega / L$ and $\omega$, respectively (Christensen-Dalsgaard et al. 1989): in particular,
\begin{equation}
\mathcal{H}_2 (\omega) = \sum b_k \phi_k (\omega) ,
\end{equation}
where the $\phi_k$ are the expansion functions for the spline. The coefficients $b_k$ are then linearly related to the data through coefficients $B_k$:
\begin{equation}
b_k = \sum B_k S_k \frac{\delta \omega_i}{\omega_i} .
\end{equation}
Thus $\mathcal{H}_2 (\omega)$ can be expressed as a linear combination of the data.

Our aim is to remove from the data a slowly-varying function of frequency. We achieve this by first using a low-pass filter that will isolate the slowly-varying contribution from $F_{\text{unrpt}}$, and from the functions $tf_1$ and $tf_2$ present in the scaled frequency differences. This can then be subtracted from the original scaled data. To this end we introduce a suitable mesh $\{ \omega_n \}$ and construct a matrix $F_{\alpha \beta}$ that has the effect of a low-pass filter. Specifically, the effect of applying the filter to some function $g(\omega)$, on the mesh $\{ \omega_n \}$, is obtained as
\begin{equation}
f(\omega_n) = \sum \alpha \beta F_{\alpha \beta} g(\omega_\beta) .
\end{equation}
The filter is characterized by a stopband frequency $f_s$, such that functions of the form $g(\omega) \propto \sin (f_s \omega)$ with $f \gtrsim f_s$ are suppressed. Further details about the filters were given by Perez Hernandez & Christensen-Dalsgaard (1994a).

According to equation (12) the result of filtering $\mathcal{H}_2$ is given at the points in the mesh by
\begin{equation}
\mathcal{H}_2^\delta (\omega_n) = \sum \alpha \beta \mathcal{H}_2 (\omega_\beta) .
\end{equation}
Since we need to evaluate the effect of the filtering at all the frequency points $\omega_i$ in the observed mode set, we introduce a new spline fit to $\mathcal{H}_2^\delta$, with spline basis $\hat{\phi}_i$:
\begin{equation}
\mathcal{H}_2^\delta (\omega_i) = \sum \hat{b}_i \hat{\phi}_i (\omega) .
\end{equation}

The effect of applying $\mathcal{F}_{ij}$ corresponds to projecting onto functions of frequency alone and applying the low-pass filter. The low-pass filter $\mathcal{F}_{ij}$ was constructed with reference to the asymptotic expression (9) for the frequency differences; however, due to the close equivalence of $S_i$ and $Q_i$, it may equally well be applied to the full expression (2). By construction, it has essentially no effect on $F_{\text{unrpt}}$, provided this is indeed slowly varying as we assume. Hence, applying the filter to equation (2) and subtracting the result from the original equation yields
\begin{equation}
\sum_j G_{ij} \frac{\delta \omega_j}{\omega_j} = \int \hat{K}_{1,i}^\delta (r) \frac{\delta f_1 (r)}{f_1} dr + \int \hat{K}_{2,i}^\delta (r) \frac{\delta f_2 (r)}{f_2} dr ,
\end{equation}
where
\begin{equation}
G_{ij} = (b_{ij} - \mathcal{F}_{ij}) Q_i ,
\end{equation}
and
\begin{equation}
\hat{K}_{p,q}^\delta = \sum_j G_{ij} \hat{K}_{p,q}^\delta .
\end{equation}
Hence, we have achieved the desired consistent filtering, the filtered data on the left of equation (18) being related to the unknown functions $\delta f_1, \delta f_2$ through transformed kernels $\hat{K}_{1,i}^\delta, \hat{K}_{2,i}^\delta$. 

\begin{align*}
\end{align*}
5 RESULTS

To demonstrate the filtering technique, we use a low-pass filter \( F_{\alpha \beta} \) constructed as described by Pérez Hernández & Christensen-Dalsgaard (1994a), with a stopband frequency \( f_s = 530 \text{ s} \). From \( F_{\alpha \beta} \) we may then construct our filters \( \mathcal{F} \) and \( \mathcal{G} \). It follows from equation (1) and the value of \( f_s \) that the resulting \( \mathcal{G} \) suppresses signals arising from the near-surface region with \( r \gtrsim 0.995 R_\odot \).

In order to calculate the coefficients \( \Phi_{ij} \), we use all modes in the modest of Libbrecht, Woodard & Kaufman (1990) having degrees \( 4 \leq l \leq 100 \) and frequency \( 1 \text{ mHz} \leq \nu \leq 5 \text{ mHz} \). The original data \( \mathbf{d} \) is represented by a B-spline, with 25 equally spaced knots in log \( \nu \).

The log of the effect of the filters is represented as a linear combination of the data: the filtered data \( \tilde{d} = Gd \), where \( d_i = \log \omega_\nu \). If this linear transformation were invertible, the information contained in the processed data would be the same as that in the original data. Thus the transformed and original problems would be equivalent. If the the process of filtering is genuinely to remove information of the surface layers from the data, then the matrix \( \mathcal{G} \) must be singular.

To study the properties of the filter \( \mathcal{G}_{ij} \), we have carried out a singular value decomposition (SVD):

\[
\mathcal{G} = U \Sigma V^T,
\]

where \( \Sigma \) is a diagonal matrix containing the singular values \( \{\sigma_i\} \) of \( \mathcal{G} \). The matrices \( U \) and \( V \) have the property that both \( U^T U \) and \( V^T V = V V^T \) are equal to identity matrix. Evidently, the diagonal entries of \( \mathcal{G} \) are equivalent to the presence of at least one singular value that is equal to zero.

The original filter given by \( F_{\alpha \beta} \) is exactly singular, with one singular value equal to zero. However, the smallest singular value of \( \mathcal{G} \) is of order \( 10^{-14} \), and hence the matrix is in principle invertible. This is probably the effect of accumulation of round-off errors in the process of obtaining \( F \) from \( \mathcal{G} \). The singular value spectra of \( F_{\alpha \beta} \) and \( \mathcal{G} \) are illustrated in Fig. 1. From Fig. 1(a) we see that \( F_{\alpha \beta} \) has nine singular values which are less than 0.5, corresponding to those components of the data and kernels which are suppressed the most. This behaviour is reflected in the singular values of \( \mathcal{G}_{ij} \), where again there are nine small singular values.

We can force the filtering to suppress completely those components corresponding to the smallest singular values by setting those singular values to zero. Suppressing the nine smallest singular values in this way does not change the frequency response of the filter significantly. This can be illustrated by considering the frequency response function, defined as the normalized response to purely sinusoidal func-

![Figure 1](image1.png) Figure 1. The singular value spectrum of the basic filter \( F_{\alpha \beta} \) (panel a) and the constructed filter \( \mathcal{G} \) (panel b). The inset in each panel shows the smallest singular values plotted on a logarithmic scale for the sake of clarity.

![Figure 2](image2.png) Figure 2. The frequency response function of the original filter \( \mathcal{G} \) (continuous line) and the filter reconstructed after removing the lowest singular values (dotted line).

ations (see Pérez Hernández & Christensen-Dalsgaard 1994a for details). Fig. 2 shows the results for the original and the modified matrix \( \mathcal{G} \). Evidently the modification does not adversely affect the response of the filter.

The matrix \( V \) consists of the right singular vectors of the matrix \( \mathcal{G} \). Corresponding to the smallest singular values are the low-frequency components of \( V \) which are removed from the data when they are passed through the filter. These
Figure 3. The right singular vectors $V$ of the filter $\mathcal{G}$ which correspond to the lowest singular values. Panel (a)-(i) correspond to singular values $2.580 \times 10^{-14}$, $1.243 \times 10^{-14}$, $2.038 \times 10^{-14}$, $2.098 \times 10^{-14}$, $2.787 \times 10^{-14}$, $3.767 \times 10^{-14}$, $5.395 \times 10^{-14}$, $2.713 \times 10^{-2}$ and $0.164$ respectively. These are the singular values that are set to zero when the filter is reconstructed.

Figure 4. The original and filtered scaled frequency differences between models T1 and R1. The circles represent the original data, and the filled squares the filtered data. For $w < -1.2$ only every second point has been plotted for clarity.

are shown in Fig. 3. As is clear from the figure, for larger singular values the corresponding columns of $V$ contain components of increasing frequency.

As the filtered data are obtained as a linear combination of the original data, one cannot immediately identify a

given datum $d_i$ in $d$ with a specific mode. None the less, the filter $\mathcal{G}_{ij}$ is sufficiently diagonally dominated that it is meaningful to associate $d_i$ with the $i^{th}$ mode in the original set. This facilitates the comparison of the unfiltered and filtered data and kernels, and we shall use this convention in the following. Note, however, that such an identification is
irrelevant for the use of the filtered results in an inversion.

The effect of filtering the scaled signal frequency differences is shown in Fig. 4, where we show the filtered and original scaled frequency differences between test model T1 and reference model R1. Filtering greatly reduces the spread in the scaled frequency differences by removing the smooth frequency-dependent components. The residual frequency difference is almost purely a function of \( w = \omega/L \). The remaining frequency dependence are the higher frequency components than are not cut off by the filter, and correspond to changes at radii smaller than 0.995 \( R_\odot \).

The properties of the filtered kernels can be examined by means of the SVD of the filter. Using Eqs (20) and (21), we obtain

\[
\hat{K}_{12}^{*}(r) = \sum_{j} (U \Sigma V^T)_{ij} K_{12}^{*} + \sum_{k} U_{ik} \sigma_k K_{12}^{*}
\]

where

\[
\hat{K}_{12}^{*}(r) = \sum_{j} V_{kj} K_{12}^{*}(r).
\]

According to equation (22) all \( \hat{K}_{12}^{*} \) for which \( \sigma_k \) is zero or small will be suppressed in the filtered kernel \( \hat{K}_{12}^{*} \). In Fig. 5 we show \( \hat{K}_{12,\rho} \) for the kernels of sound speed for the lowest singular values. Clearly the lowest singular values correspond to the near-surface components of the kernels. Thus the filter does indeed suppress the kernels at the surface. The efficiency of the filter in so doing can be seen from Fig. 6, showing the original and filtered kernels as a function of frequency at a given radius. The original mode kernels show an oscillatory behaviour which becomes more slowly varying with increasing \( r \). The filtered kernels have smaller amplitude at these radii, showing that filtering has suppressed the sensitivity near-surface structural changes. Suppression is almost complete for \( r > 0.995 R_\odot \), as intended. Fig. 7 shows some of the filtered and unfiltered kernels as a function of radius for a few modes. In each case the suppression of the modes at the surface is very clear.

Thus it is indeed possible to remove the slowly varying function of frequency from the scaled frequency differences, while simultaneously modifying the kernels in a self-consistent manner. With the filtered data and kernels, we can proceed with the inversion.

We invert the frequency differences between two models to determine the relative differences in sound speed between them. We have done the inversion both with and without filtering. We scale the target width at each target radius with the sound speed at that radius, relative to the width at a typical target radius, the width being selected to prevent ringing in the averaging kernels obtained (cf. Pijpers & Thompson 1994).

In Fig. 8 we show the inferred solution for relative sound speed differences between models T1 and R1, both with and without filtering. Evidently there is a definite improvement in the inversion when we use filtering.

\[
\begin{align*}
\text{Kernels for } & \delta c^2/c^2 \\
\text{Kernels for } & \delta \rho/\rho \\
\text{(a) Unfiltered} & \\
\text{(b) Filtered} & \\
\end{align*}
\]

Figure 7. The surface structure of the kernels for a few modes. In each panel, the continuous line is the unfiltered kernel while the dotted line is the filtered one.

\[
\begin{align*}
\text{Figure 8. The unfiltered (a) and filtered (b) sound-speed inversion results for model T1 when inverted using R1 as the reference model. In each panel the continuous line is the exact model difference. The points mark the target radii. The vertical error bars indicate the error in inversion due to the errors in the data. The horizontal error bars are half the distance between the quartile points of the averaging kernel at that radius, and is an indication of the resolution of the inversion.} \\
\end{align*}
\]

6 EQUVALENT FILTERING IN OTHER FORMULATIONS

As has been mentioned in the introduction, other methods have been developed to handle the surface term. In the SOLA method the surface terms are commonly accounted for by constraining the solution to remain unchanged in the presence of slowly varying functions of frequency in the data.
If the functions are represented by polynomials \( \Phi_\lambda \) of degree \( \lambda \), we obtain a set of constraints of the form (8). To compare such procedures with the filtering proposed here, we note that the filters project the data onto a space orthogonal to the space spanned by the surface contribution \( F_{\text{surf}} \). If a polynomial expansion of the surface term \( F_{\text{surf}} \) is adopted, the inversion is carried out under the constraint that only components orthogonal to \( F_{\text{surf}} \) contribute to the solution. It was pointed out by Gough (1995) that if the filters project the data into a space which is identical to that orthogonal to the polynomials used to represent \( F_{\text{surf}} \), then the results of the inversion done after filtering will be identical to those obtained with the constraints (8). A demonstration of this result and details of the construction of the equivalent filter can be found in the Appendix.

We note, however, that filtering the data before inversion has the advantage that the surface contribution is handled in the same manner, regardless of the inversion technique, facilitating a comparison of different methods. Furthermore, the filtering provides additional insight into the properties of the inversion, as illustrated by the behaviour of the filtered kernels. Finally, one might hope that the use of filtering offers a substantial flexibility in the design of procedures for eliminating the near-surface problems.

### 7 EFFECT OF HIGH-DEGREE MODES

To define our filters and carry out the inversions, we have only used modes with degrees \( \ell \leq 100 \). This restriction ensures that the surface uncertainties introduce a contribution to the data that is essentially just a function of frequency. From the asymptotic analysis of high-degree modes it is known that a decomposition of the form given in equation (9) is not sufficient to describe the frequency difference of these modes. For high-degree modes the surface contribution has an additional dependence on \( \ell \); this arises because the ray paths in the near-surface layers deviate appreciably from the vertical direction, in a manner that depends on the degree \( \ell \). As a result, second- and higher-order terms are needed in the asymptotic description of these modes (e.g., Brodsky & Vorontsov 1993; Gough & Vorontsov 1995). In the formulation of our filter we assumed that the frequency differences could be described by the simple expression given in equation (9). Therefore, we do not expect kernels corresponding to modes of higher degree to be suppressed at the surface (see also Antia 1995).

To demonstrate this, we have constructed a filter with reference model R1 for a mode set consisting of the BISON model set (Elsworth et al. 1994) at low degree, and containing all modes in the table of Libbrecht et al. (1999): this mode set thus extends up to \( \ell = 1600 \). In Fig. 9 we plot the amplitude at \( r = R_\odot \) of the unfiltered kernels and those filtered with the new filter, as a function of the degree \( \ell \). At lower degree the kernels are suppressed, but the amplitude of the kernels at higher degree is not suppressed much by the filtering. For \( \ell \geq 500 \), the filtered kernels are actually even larger than the unfiltered ones. Thus to handle high \( \ell \) modes one has to take into account the higher-order terms in the formulation of the filters.

### 8 CONCLUSIONS

We have shown that it is possible to pre-process helioseismic data prior to applying standard inversion techniques in order to eliminate the frequency-dependent component which arises from near-surface uncertainties. At the same time the corresponding contributions from the kernels which relate the frequency differences to near-surface differences in structure are suppressed in a self-consistent way. For this purpose we use filters which eliminate low-frequency components from functions of frequency.

The procedure has been demonstrated by applying it to artificial data computed for solar models. The filters have a sharp cut-off in frequency to ensure that only the undesired components of the data are filtered out. On passing the frequency differences through the filter, we find that the slowly varying frequency-dependent term is indeed suppressed. The residual is largely a function of \( \omega / \ell \), with a remaining small frequency dependence arising from the deeper layers of the Sun. The kernels which relate the frequency differences to differences in structure are also suppressed at the surface.

We have assumed that the near-surface contributions to the frequencies vary slowly with frequency. This assumption would be compromised by an intrinsic rapid variation with frequency of the physical effects near the surface such as might be caused, for example, by resonance between convective time scales and the oscillation periods. We have also assumed that the near-surface contributions depend on degree only as a result of the variation with penetration depth of the mode inertia. This would be true only if departures from spherical symmetry can be neglected. We note, however, that the same assumptions underlie other procedures.
currently used to deal with the near-surface uncertainties.

The results of inferences for the test models improve substantially when the data have been filtered. Although we have only used SOLA for carrying out the inferences, the regularized least-squares method might also have been used. Here, however, it should be kept in mind that the covariance matrix resulting from the filtering is no longer diagonal; thus the inversion would require use of a generalized least-squares fit, resulting in a considerable increase in computing time over the normal least-squares inversion.

It is remarkable that the properties of solar oscillations allow investigations of the deep solar interior, despite the uncertainties in the physics of the near-surface region which have very substantial effects on the frequencies. Although other techniques have successfully suppressed these effects, the present procedure offers additional flexibility in the design of inversion algorithms and insight into their properties.

Although the main thrust of this paper has been to eliminate the influence of the near-surface layers, we note finally that this region is of substantial interest in its own right. Observations of high-degree modes by, e.g., Bachmann et al. (1995), and by the imminent SOI/MDI instrument on the SOHO satellite, are providing new information about these layers, which have been subject also to several recent theoretical investigations (e.g., Patera et al. 1993; Monteiro, Chrissten-Dalsgaard & Thompson 1995; Rosenthal et al. 1995). Thus we can hope both to learn about the external layers and to minimize their influence on the investigations of the deeper interior.

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APPENDIX: CONSTRUCTING EQUIVALENT FILTERS

The SOLA inversion with additional surface constraints (e.g. Däppen et al. 1990) is equivalent to a SOLA without additional constraints applied to data filtered in a suitable way. Here we demonstrate this equivalence and construct explicitly the equivalent filters. We also derive a similar result for the Regularized Least Squares method.

Consider the constrained optimization problem of finding a vector \( c \) (of length \( M \)) to minimize

\[
\frac{1}{2} c^T A c - 2 c^T t ,
\]

subject to constraints

\[
\begin{align*}
& c^T f^{(j)} = 0, \quad j = 1, \ldots, N_f \quad (A1) \\
& c^T h^{(k)} = \gamma^{(k)} , \quad k = 1, \ldots, N_h . \quad (A2)
\end{align*}
\]

The solution to this problem satisfies

\[
Ac = t + \sum_j \lambda_j f^{(j)} + \sum_k \mu_k h^{(k)} .
\]

(A3)

where the \( \lambda \)'s and \( \mu \)'s are Lagrange multipliers, the values of which are determined by the constraints. Assuming that matrix \( A \) is invertible, it follows that

\[
c = A^{-1} \left( t + \sum_j \lambda_j f^{(j)} + \sum_k \mu_k h^{(k)} \right) .
\]

(A4)

The SOLA inversion with additional constraints is a problem of this form: the surface constraints are of the first type, and the unimodularity constraint is of the second type.

Suppose now that \( G \) is an \((M - N_f) \times M\) matrix of full rank such that

\[
G f^{(j)} = 0, \quad j = 1, \ldots, N_f .
\]

(A5)

Define a new optimization problem of minimizing

\[
\frac{1}{2} c^T \hat{A} c - 2 c^T \hat{t}
\]

subject to

\[
\frac{1}{2} c^T \hat{h}^{(k)} = \gamma^{(k)} , \quad k = 1, \ldots, N_h ,
\]

(A7)

where \( \hat{A} \), \( \hat{t} \) and \( \hat{h}^{(k)} \) are defined by

\[
\hat{A} = GA^T, \quad \hat{t} = Gt, \quad \hat{h}^{(k)} = Gh^{(k)} .
\]

(A8)

By analogy with (A3), the solution to the new problem satisfies

\[
\hat{A} \hat{c} = \hat{t} + \sum_k \hat{\mu}_k \hat{h}^{(k)} .
\]

(A9)

Now the SOLA inversion solution is given in terms of the optimization solution vector \( c \) and the helioseismic data vector \( d \) by

\[
c^T d ,
\]

(A10)

If, in parallel to the transformations (A8), we construct transformed data

\[
d = Gd ,
\]

(A11)

then the condition that the original and transformed problems give the same solution is \( \hat{c}^T d = c^T d \), i.e.,

\[
G^T \hat{c} = c .
\]

(A12)

It is straightforward to show, by substituting from (A8) and (A12) in the left-hand side of equation (A9), that if \( \hat{c} \) satisfies (A12) then (A9) is indeed satisfied. Moreover, the conditions (A7) are also satisfied since

\[
\gamma^{(k)} = c^T h^{(k)} = \hat{c}^T G h^{(k)} = \hat{c}^T \hat{h}^{(k)} .
\]

(A13)

Hence any quantity of the form (A10) – and this includes not only the SOLA solution but also SOLA averaging kernels – where the quantity \( d \) transforms as (A11), will be the same whether one solves the original problem (A1) or the transformed problem (A6). The transformed problem is precisely of a form corresponding to our filtered SOLA. Indeed it is straightforward to show that the matrices in the SOLA have the required transformation properties.

Given the assumption that \( G \), of size \((M - N_f) \times M\), is of full rank the solution \( \hat{c} \) is unique. However, one might choose instead to work in terms of a rank-deficient \( G \), for example of size \( M \times M \). Then the solution \( \hat{c} \) will not be unique, but \( G^T \hat{c} \) and hence also \( G^T d \) will still be unique.

To find explicitly a suitable equivalent filter for our SOLA inversion with constraints (8), it simply remains to find a suitable matrix \( G \). Constraints (8) can be written

\[
c^T F = 0 ,
\]

(A14)

where

\[
F_j = \frac{\Phi_j(\omega_e)}{Q} , \quad j = 1 \ldots A .
\]

(A15)

In terms of the above analysis, the vectors \( f^{(j)} \) are the columns of matrix \( F \), which are assumed to be linearly independent. An SVD of \( F \) yields \( F = U F , \Sigma V^T \), whence it follows that

\[
G = I - U \Sigma U^T
\]

(A16)

has the property that \( GF = 0 \) and hence, as desired, \( G F^{(j)} = 0 \). Thus equation (A16) provides a suitable equivalent filter.

It can further be shown that this is an equivalent filter not only for the SOLA inversion, but also for an appropriately formulated Regularized Least Squares (RLS) inversion (e.g. Craig & Brown 1986). In the RLS inversion with surface constraints, one seeks a solution vector \( y \) and coefficients \( \eta_j \) to minimize

\[
(d - d_m)^T E^{-1} (d - d_m) + \lambda^2 \| Ly \|^2 ,
\]

(A17)

where

\[
d_m = B y + \sum_j \eta_j f^{(j)} .
\]

(A18)

Here the error covariance matrix \( E \), matrices \( B \) and \( L \), and parameter \( \lambda^2 \) are all fixed. The transformed problem is to minimize

\[
(d - d_m)^T E^{-1} (d - d_m) + \lambda^2 \| Ly \|^2 .
\]

(A19)

where

\[
d_m = B y .
\]

(A20)
The transformation properties are
\[ \hat{E} = GEG^T, \quad \hat{B} = GB, \quad \hat{d} = Gd. \] (A21)

If \( \hat{E} \) is singular, \( \hat{E}^{-1} \) in equation (A19) must be replaced by a generalized inverse \( \hat{E}' \): if \( \hat{E} \) is decomposed as \( \hat{E} = U_E \text{diag}(\sigma_1, \ldots, \sigma_{M-N}, 0, \ldots, 0) U_E^T \) then \( \hat{E}' = U_E \text{diag}(\sigma_1^{-1}, \ldots, \sigma_{M-N}^{-1}, 0, \ldots, 0) U_E^T \).

It can be shown that the solutions \( y \) and \( \hat{y} \) are identical, and hence that the RLS inversion with surface terms included is equivalent to a suitably filtered RLS inversion without surface terms.

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