The Mean Square Radius of Deuteron in the Bethe-Salpeter Formalism

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Abstract

We investigate the mean square radius of deuteron in the Bethe-Salpeter formalism with keeping Lorentz covariance. All possible relativistic effects are considered within the additive current concerning the constituent nucleons. We argue the possibilities to explain the discrepancy in the mean square radius of deuteron between the experiment and the non-relativistic nuclear potential models. It is found that the relativistic corrections cannot account for the discrepancy, since the relevant relativistic corrections cancel out each other. For the relativistic corrections to the non-relativistic theory, it is necessary to have at least a consistent treatment to both the equation and the electromagnetic current of the deuteron as bound system. Furthermore, we estimate the contributions of the off-shell effects for photon-nucleon interaction vertex. These off-shell effects may be candidates to explain the discrepancy.

1. Introduction

The mean square (MS) radius of the deuteron \( r_n^2 \) is a characteristic quantity derived from the deuteron wave function. Hence, the calculation of the MS radius may be a good check of the nucleon-nucleon potential models. In the non-relativistic (NR) nuclear theory, \( r_n^2 \) is given [1] as

\[
r_n^2 = \frac{1}{4} \int \left[ u(r)^2 + w(r)^2 \right] dr ,
\]

(1.1)

where \( u(r) \) and \( w(r) \) are, respectively, the usual radial S- and D-wave functions of the deuteron. In the typical NR potential models, the theoretical value of \( r_n^2 \) is given as \( r_n^2 = (1.95 - 1.98 \text{fm})^2 \) [2,3] (The value changes a little with the model used). On the other hand, the value of \( r_n^2 \) is experimentally derived from the observation of the MS charge radius \( r_{ch}^2 \), for there exists the following relation[4] between \( r_n^2 \) and \( r_{ch}^2 \) with the proton charge radius \( r_p^2 = 0.862(12) \text{fm} \) [5] and the MS neutron charge radius \( r_n^2 = -0.1192(18) \text{fm}^2 \) [6],

\[
r_n^2 = r_{ch}^2 + r_p^2 + r_n^2 + \frac{3}{4m^2} ,
\]

(1.2)

where the final term represents the relativistic correction for Zitterbewegung and we have assumed that the isospin symmetry takes the mass \( m \) for both the proton and neutron. For extremely low momentum transfer, it is possible to obtain the value of \( r_{n,exp}^2 \) from the experimental data with good confidence by measuring the ratio of the cross section of the elastic electron-deuteron scattering to that of electron-proton scattering. One of the most widely accepted value is given as \( r_{n,exp}^2 = (1.9546(47) \text{fm})^2 \) [7-9]. Thus the discrepancy between the NR potential models and the experiment is

\[
\Delta r_n^2 = r_{n,exp}^2 - r_n^2 = -0.08 \text{fm}^2 .
\]

(1.3)

It has been reported [9] that the NR potential models are unable to explain this discrepancy simultaneously with the other low energy parameters of the proton-neutron system. The main efforts to explain this discrepancy have been done along two directions, taking into account the relativistic effects or the effects of meson-exchange current (MEC) [10]. In order to make any meaningful quantitative prediction on electromagnetic processes it is indispensable to have an effective form of conserved current for deuteron system, and in doing so its covariant description is necessary. In the previous work [11], we have investigated the magnetic and quadrupole moments of deuteron in the BS formalism. We have shown that the well-known difficulty concerning the magnetic and quadrupole moments in the NR nuclear potential models may be
solved through a nonperturbative relativistic effect, “normalization effect”. That is, if we assume an admixture of negative energy amplitudes with magnitude of a few hundredths of positive ones, we can fit both the experimental values of magnetic and quadrupole moments at the same time with a single value of the D-wave state probability.

In this paper we investigate all possible relativistic effects to the MS radius of deuteron in the BS formalism with a special attention on keeping Lorentz covariance. We discuss the possibilities to explain the discrepancy in Eq (1.3). Here, it should be noted that some important parts out of the MEC effects such as “pair” current in the NR nuclear theories are automatically included in our covariant treatment.

This paper is organized as follows. In the next section, we review the description of deuteron in the BS formalism so far as required for our problem. In section III, we present the qualitative analysis on the MS radius of deuteron and argue the possibilities to explain the discrepancy. In section IV, the details of the relativistic corrections to the MS radius of deuteron are studied by the semi-quantitative analysis. The off-shell effects in the photon-nucleon interaction vertex are also discussed. The concluding remarks will be given in the final section.

II. Bethe-Salpeter formalism

In the bound-state BS equation for the deuteron, the BS amplitude $\chi_n(p,P)$ satisfy[12]

$$S^\nu_{\alpha\beta}(p_\tau)=\sum_{\alpha\beta} S^{\nu
\alpha\beta}(p_\tau) \chi_n(p,P)=\int d^2p' G(p-p')\chi_n(p',P),$$

(2.1)

where $G(p-p')$ is the meson-nucleon interaction kernel and the inverse of the nucleon propagator (Dirac operator) for the nucleon momentum $p_\alpha$ is given as

$$S^{\nu\alpha}(p_\tau)=\left[\gamma^\nu p_\tau+\alpha-mi\tau\right].\quad(\tau=1,2),$$

(2.2)

where $p_\tau=(p_\tau+p_\beta)_\alpha$ and $p_\tau=\frac{1}{2}(p_\tau-p_\beta)_\alpha$ denote the total four momentum and the relative one, respectively. $M_\alpha$ is the mass of a deuteron and $P_{\alpha\beta}=-M_\beta^\alpha$: $\alpha$ represents the other quantum numbers, such as angular momentum, of the deuteron state. It is well-known that the BS equation can be reduced to the Schrödinger equation in the non-relativistic limit as $\nu \ll c$.

In general, the BS amplitude has sixteen components because the constituent nucleons are the Dirac particles. Here we carry out the partial wave decomposition of the BS amplitude in the rest frame of deuteron in terms of the product of the free Dirac spinor $u^{\nu\alpha}_{\beta}(p_\tau)$ of constituent nucleon as follows:

$$\chi(p,P^{(0)}) = \sum_{\alpha\beta} \Phi^{\nu\alpha}_{\beta}(p,p_\tau) u^{\nu\alpha}_{\beta}(p) \otimes u^{\nu\alpha}_{\beta}(-p).$$

(2.3)

where $P^{(0)}=(0,iM_\alpha)$, $p=(p_1,p_2)$, and $s=(s_1,s_2)$; the index $s_1=(+,-)$ and $s_2=(-,+)$. The component amplitude $\Phi^{\nu\alpha}_{\beta}(p,p_\tau)$ is represented by a sum of the eight partial wave component amplitudes with the spectroscopic number $\alpha_\nu=\pi l_\nu p_\nu$.

$$^{1}S_{\nu}^{\tau\nu}, \ 2I_{\nu}^{\tau\nu}, \ 1D_{\nu}^{\tau\nu}, \ 2P_{\nu}^{\tau\nu}, \ 3P_{\nu}^{\tau\nu}, \ 1P_{\nu}^{\tau\nu}, \ 2P_{\nu}^{\tau\nu}, \ 3P_{\nu}^{\tau\nu}, \ 1P_{\nu}^{\tau\nu},$$

(2.4)

where $S, L$ and $J$ denote the total spin, orbital angular momentum and the total angular momentum, respectively. Here the reduction of the numbers from the initial sixteen for $\Phi$ is due to parity conservation. It should be noted that only the first two amplitudes with $^{1}S_{\nu}^{\tau\nu}$ and $^{2}D_{\nu}^{\tau\nu}$ states correspond to the usual states in the NR potential models. According to our previous work[11], the positive energy amplitudes for the ordinary one boson exchange (OBE) interaction picture are expected to be damping for the region of $p$ and $p_\tau$

$$\Phi^{1\nu\alpha\beta}(p,p_\tau) \to 0 \quad \text{for} \quad |p| \gg \sqrt{m(m-M)}, \quad p_\tau \gg (m-M),$$

(2.5a)

and for the negative energy amplitude

$$\Phi^{2\nu\alpha\beta}(p,p_\tau) \to 0 \quad \text{for} \quad |p| \geq \mu_\nu, \quad p_\tau \gg \mu_\nu,$$

(2.5b)

where $\rho= (-,-),(+,+) \text{ or } (-,+).$ Here $\mu_\nu$ stands for the mass of the exchange boson in the meson-nucleon interactions and $M=M_\alpha/2$.

We assume that the electromagnetic (EM) interaction of deuteron is given as a sum of the individual ones of constituent nucleons. Due to the invariance of the BS equation for the exchange of constituent nucleon one and nucleon two as the isoscalar nature of deuteron, the conserved EM current of deuteron is given as

$$J^{\nu\alpha\beta}(p,P) = -ieN_e \int d^4p' \chi_n(p',P') \Gamma^{(0)}(q) \gamma^{\nu\alpha}_{\beta}(p-P') \chi_n(p,P),$$

(2.6)

where $p' = p + q/2$, $P' = P + q$, $N_e = 1/(2\pi)^4 \sqrt{2p_\tau p_\beta}$, and $\chi$ is the conjugate of $\chi$. It should be noted that the effective EM current in Eq (2.6) satisfies the current conservation as $q \cdot J^{\nu\alpha\beta}(P,P) = 0$. The photon-nucleon interaction vertex $\Gamma^{(0)}(q)$ is given by the on-shell form

$$\Gamma^{(0)}(q) = \gamma^{\nu\alpha\beta}_\gamma F_1(q^2) - \frac{K_\gamma}{2m} \gamma^{\nu\alpha\beta}_\gamma F_2(q^2),$$

(2.7)

where $\sigma_{\nu\alpha\beta} = (\gamma_{\nu},\gamma_{\alpha},\gamma_{\beta})/2i$ and $F_1(q^2)$ and $F_2(q^2)$ represents the isoscalar Dirac (Pauli)
form factor of nucleons with \( F_1(0) = F_2(0) = 1 \) and \( \kappa = \kappa_p + \kappa_n \) (\( \frac{2m_p}{2m} = \frac{1}{2} \kappa_p \) and \( \frac{2m_n}{2m} = \frac{1}{2} \kappa_n \) stand for the proton and neutron magnetic moments, respectively).

We need the BS amplitude for deuteron in the general moving frame in order to calculate the matrix element of the deuteron current in Eq. (2.6). For this purpose, we describe the transformation property of the BS amplitude under the Lorentz transformation \( \Lambda_{\alpha\beta} \), namely

\[
\chi_{\mu}(P, P') = S^{(\alpha)}(\Lambda)S^{(\beta)}(\Lambda)\chi_{\alpha}(\Lambda^\mu P, \Lambda^\nu P).
\]  

(2.8)

\[
S^{(\alpha)}(\Lambda) = \left[ \frac{E + M_{\alpha}}{E + M_{\beta}} \right]^{\alpha\beta},
\]  

(2.9)

where \( \alpha, \beta \) is the Pauli spin matrix of the ith nucleon and \( E = \sqrt{P^2 + M^2_{\alpha}} \).

The BS amplitude should be normalized so that the total charge of deuteron, given by the EM current Eq. (2.6), becomes correctly \( e_s = e \). By using formula \( e = \lim_{t \to 0} J^{(\alpha)}(P^\mu, P) \) and Eq. (2.3), after some calculations, we can write the normalization condition of BS amplitude as

\[
P_{\mu} - P_{\nu} = 1, \quad P_{\mu} = (C_{\mu}^1 + C_{\mu}^2)^1, \quad P_{\nu} = (C_{\nu}^1 + C_{\nu}^2)^1 = 2|C_{\nu}^1|^2 + 2|C_{\nu}^2|^2,
\]

(2.10)

with the definition of "pseudoprobability"

\[
|C_{\mu}\rangle = iN_{\mu} \int d^3p \phi_{\mu}(p, \rho_0) \left[ \frac{P_{\mu} + P_{\nu}}{2} \right]^2 F_{\mu - \nu} - M_{\mu},
\]  

(2.11)

where \( \phi_{\mu}(p, \rho_0) \) represents the radial BS amplitude and we have used the simplified notation \( |C_{\mu}\rangle = |C_{\mu}\rangle \) representing the pseudoprobability of the \( ^1S_{\mu^\prime} \) state of deuteron and in the same manner \( |C_{\mu}\rangle = |C_{\mu}\rangle \) representing \( |C_{\mu}\rangle \) and \( |C_{\mu}\rangle = |C_{\mu}\rangle \) and \( |C_{\mu}\rangle = |C_{\mu}\rangle \). Thus we see that the inclusion of negative energy states into the amplitude necessarily reinforces the contribution of the positive energy states. In our previous work[11], we have shown by using the ordinary OBE interaction kernel that the pseudoprobability for \( ^1S_{\nu^2} \) partial states is dominant over the other negative energy ones as \( P_{\nu^2} = (C_{\nu}^1)^2 = 0.015 \). Therefore, for the MS radius, it is expected that the contributions of the negative energy states, except for \( ^1S_{\nu^2} \) partial state, are much smaller than the discrepancy. In the following analysis, we are exclusively concerned with the contribution of the \( ^1S_{\nu^2} \) partial state.

III. The qualitative analysis of the MS radius of deuteron

In this section we shall make a general analysis on the MS radius of deuteron, taking into account the qualitative behaviors of the component amplitude in Eq. (2.5).

The MS charge radius \( r_{\mu}^s \) in the BS formalism is given by the following formula[13]

\[
r_{\mu}^s = \frac{1}{e} \lim_{q \to \infty} \frac{1}{q^2} J^{(\alpha)}(q/2, -q/2),
\]

(3.1)

where \( J^{(\alpha)}(q/2, -q/2) \) is the charge density in the Breit frame deriving from the EM current in Eq. (2.6). In Eq. (3.1), we take the polarization of deuteron as \( \alpha' = \alpha = 1 \) (with the third axis as a direction of quantization), reflecting our physical situation. The Breit frame is defined by \( P_{\alpha}^{(\alpha)} = (q/2, i\rho_0) \) and \( P_{\alpha}^{(\alpha)} = (-q/2, i\rho_0) \). By substituting Eq. (2.8) into Eq. (2.6), the EM current in the Breit frame is represented in terms of the BS amplitude in the rest frame as follows:

\[
J_{\mu}(P^{(\alpha)}, P^{(\beta)}) = J_{\alpha}(q/2, -q/2) \chi(p, P^{(\beta)}),
\]

(3.2)

with \( \chi(p, P^{(\beta)}) = S^{(\alpha)}(\Lambda)S^{(\beta)}(\Lambda)\Gamma(\alpha) q^{(\alpha)} S^{(\beta)}(\Lambda)S^{(\alpha)}(\Lambda) \) and the relative momentum \( p_{\mu} \) is related by the boost transformation \( p_{\mu} = N_{\mu} p + \frac{1}{2} A_{\mu q} \). Furthermore, by using the decomposition of the BS amplitude in Eq. (2.3), we obtain the final expression of the EM current of deuteron as

\[
J_{\mu}(q/2, -q/2) = -iN_{\mu} \sum_{\alpha, \beta} \int d^3p \tilde{\Phi}_{\mu}(p, \rho_0) \tilde{\Phi}_{\mu}(p, \rho_0) \Phi_{\alpha}(p, \rho_0) S_{\alpha}^{(\alpha)}(p_{\mu})^{-1},
\]

(3.3)

with

\[
\tilde{\Phi}_{\mu}(p, \rho_0) = \tilde{\Phi}_{\mu}(p, \rho_0) \Phi_{\alpha}(p, \rho_0) S_{\alpha}^{(\alpha)}(p_{\mu})^{-1},
\]

(3.4)

where \( S_{\alpha}^{(\alpha)}(p_{\mu})^{-1} = p_{\mu} E_{\alpha} - M_{\alpha} + p_0 \) and \( E_{\alpha} = \sqrt{p_{\mu}^2 + m^2} \). In the most of the conventional relativistic models the process of resorting to the Breit frame is missing and the effective current in the rest frame, aside from the conservation of the space momentum, has been directly applied for the static charge and current densities.

By substituting Eq. (3.3) into Eq. (3.1), the MS charge radius \( r_{\mu}^s \) in the BS formalism is given as

\[
r_{\mu}^s = (r_{\mu}^s)^{\alpha}_{\nu^2} + (r_{\mu}^s)^{\alpha}_{\nu^2} + (r_{\mu}^s)^{\alpha}_{\nu^2}
\]

(3.5)

where

\[
(r_{\mu}^s)^{\alpha}_{\nu^2} = iN_{\alpha} \sum_{\rho_0} \int d^3p \left[ \frac{1}{e} \lim_{q \to \infty} \frac{1}{q^2} \tilde{\Phi}_{\mu}(p, \rho_0) \right] \Phi_{\alpha}(p, \rho_0) S_{\alpha}^{(\alpha)}(p_{\mu})^{-1},
\]

(3.6a)
\[
\langle r^2_{\text{ms}} \rangle = 2i N_s \sum_{p, p', r} \int d^4 p' \lim_{q \to 0} \left[ \frac{\delta}{\delta q} \hat{\Phi}^\dagger(p', p) \right] \left[ \frac{\delta}{\delta r} \Phi(p, p_s) S_{\text{ms}}(p_s)^{-1} \right] \Phi^\dagger(p, p_s) S_{\text{ms}}(p_s)^{-1},
\]
(3.6b)

\[
\langle r^2 \rangle = -i N_s \sum_{p, p'} \int d^4 p \Phi^\dagger(p, p_s) \lim_{q \to 0} \left[ \frac{\delta}{\delta q} \right] \left[ \frac{\delta}{\delta r} \Phi(p, p_s) S_{\text{ms}}(p_s)^{-1} \right] \Phi(p, p_s) S_{\text{ms}}(p_s)^{-1}
\]
(3.6c)

\[
+ \lim_{q \to 0} \frac{\delta^2}{\delta q^2} N_s.
\]

Here, \( \langle r^2_{\text{ms}} \rangle \), \( \langle r^2 \rangle \) and \( \langle r^2 \rangle_{\text{ms}} \) include various relativistic effects in our covariant treatment of \( r^2_{\text{ms}} \) and \( r^2 \) corresponding to the MS radius in the NR formula Eq. (1.1) and \( r^2_{\text{ms}} - r^2 \), respectively. We obtain the following form for \( \langle r^2 \rangle \) as

\[
\langle r^2 \rangle = \langle r^2 \rangle_{\text{ms}} + \langle r^2 \rangle_{\text{nr}}.
\]
(3.7)

where

\[
\langle r^2 \rangle_{\text{nr}} = -i N_s \sum_{p, p'} \int d^4 p \Phi^\dagger(p, p_s) \Phi(p, p_s) S_{\text{ms}}(p_s)^{-1},
\]
(3.8a)

\[
\langle r^2 \rangle_{\text{nr}} = -i N_s \sum_{p, p'} \int d^4 p \Phi^\dagger(p, p_s) \Phi(p, p_s) S_{\text{ms}}(p_s)^{-1},
\]
(3.8b)

\[
\hat{r} = \frac{1}{M_s^2} \left[ -2 M p_0 + \frac{\mu^2}{M^2} \frac{\delta^2}{\delta q^2} - 3 M p - q^2 \right] + \frac{\delta}{\delta p} \frac{\delta}{\delta q} - \frac{\delta}{\delta q} \frac{\delta}{\delta p},
\]
(3.8c)

by using the formulae

\[
\lim_{q \to 0} \frac{\delta}{\delta q} = \frac{M - p_0}{M}, \lim_{q \to 0} \frac{\delta^2}{\delta q^2} = \frac{p - p_0}{M}, \lim_{q \to 0} \frac{\delta^2}{\delta q^2} = \frac{p - p_0}{M}, \lim_{q \to 0} \frac{\delta}{\delta q} = \frac{M - p_0}{M}.
\]
(3.9)

By carrying out the angular integral and by transforming the spherical Bessel function \( j_r(p r) \) as

\[
\Phi^\dagger(p, p_s) = \int_0^{\infty} d k j_r(k r) e^{i k p_s},
\]
(3.10)

we obtain the following Fourier conjugate spatiotemporal expression of \( \langle r^2 \rangle_{\text{nr}} \) in Eq (3.8a) for \( 1S^0 \) and \( 1D^0 \) states as

\[
\langle r^2 \rangle_{\text{nr}} = -i N_s \sum_{p, p'} \int d^4 p \Phi^\dagger(p, p_s) \Phi(p, p_s) S_{\text{ms}}(p_s)^{-1} \left( \frac{p_0 + p - m - M}{2} \right).
\]
(3.11)

where we use the non-relativistic approximation to \( E_p \to m \) and \( p_0 \to 0 \) for \( S_{\text{ms}}(p_s)^{-1} \) and the isoscalar nature of deuteron. It is clear that the formula (3.11) with \( p = (p, +) \) corresponds to the non-relativistic one in Eq (1.1). For the contribution from the \( 1S^0 \) state, it is expected to be

\[
\langle r^2 \rangle_{\text{nr}} = -\langle r^2 \rangle P = -\langle r^2 \rangle P R^2 - \left( \frac{m_s}{m_p} \right)^2 P R^2 = -10^{-2} P R^2.
\]
(3.12)

where \( m_s \) and \( m_p \) are mass of the pion and rho-meson, respectively. Thus the contribution for the negative energy state becomes negligible to the discrepancy. As a result, \( \langle r^2 \rangle_{\text{nr}} \) may be estimated as

\[
\langle r^2 \rangle_{\text{nr}} = \langle r^2 \rangle_{\text{nr}} + \langle r^2 \rangle_{\text{nr}} - \left( P - 10^{-2} P - 10^{-2} P \right) - \left( 1 + P - 10^{-2} P \right) - \left( 1 + P \right).
\]
(3.13)

Thus, if we have an admixture of the negative energy amplitudes with pseudoprobability \( P \) into the positive ones, the MS radius \( r^2_{\text{ms}} \) in Eq (1.1) is modified through the normalization condition in Eq (2.10). This effect is called the normalization effect. It is essential for the normalization effect that the contributions of the negative energy states themselves are negligible to the extra contribution through the normalization condition.

The second term \( \langle r^2 \rangle_{\text{nr}} \) in Eq (3.7) and \( \langle r^2 \rangle_{\text{nr}} \) in Eq (3.5) represent the Lorentz deformation effects, such as the relativistic effects for the relative motion of constituent nucleons, the Lorentz contraction and retardation, depending on the detail of the BS amplitude. These terms may be canceled out each other (See the discussion to be given in section IV), and then it may be numerically negligible to the order of discrepancy, although quite interesting theoretically. Thus we neglect these terms.

For the term of \( \langle r^2 \rangle \), see the appendix A, we get the following formula for the case of \( 1S^0 \) and \( 1D^0 \) states as

\[
\langle r^2 \rangle = \langle r^2 \rangle + \frac{3}{4 m^2} P^2 + \frac{3}{8 m^2} P^2 + \frac{3}{8 m^2} P^2 - \frac{m^2}{M} (P + P).
\]
(3.14)

It may be worthwhile to note that the relativistic correction for the Zitterbewegung is deduced from this term. By substituting Eq (3.13) and Eq (3.14) into Eq (3.5), we obtain the final result of the charge radius of deuteron in the BS formalism corresponding to Eq (1.2) as

\[
\langle r^2 \rangle_{\text{ns}} = \langle r^2 \rangle + \langle r^2 \rangle + \frac{3}{4 m^2} P^2 + \frac{3}{8 m^2} P^2 + \frac{3}{8 m^2} P^2 - \frac{m^2}{M} (P + P).
\]
(3.15)

This formula is identical with the usual one in Eq (1.2), by putting on \( P = 1 (P = 0) \) and then
\[ \delta r_{\text{ms}} = \frac{3}{8m^2} (1 + 2x) \frac{m - M}{M} \approx 10^{-3} \frac{1}{m^2}, \] which is negligible to the discrepancy. However, from our previous analysis[11] as \( P = |C_2|^2 = 0.015 \), there is the contribution out of the normalization effect, and then we numerically obtain the final result as follows:

\[ (r^2)_{\text{ms}} = (196 \text{fm})^2 \quad \Rightarrow \quad (r^2)_{\text{ms}} = (197 \text{fm})^2, \] where \( (r^2)_{\text{ms}} \) and \( (r^2)_{\text{bs}} \) are the redefined MS radii in the NR nuclear theory and the BS formalism, respectively. From Eq (3.15), these quantities have the relation that

\[ (r^2)_{\text{ms}} = (r^2)_{\text{bs}} + \delta r_{\text{ms}}. \] (3.17)

Summarizing the results of the above analysis in this section, we conclude that the most of relativistic corrections, except for the normalization effect, to the NR MS radius are numerically too small to explain the discrepancy. For the pseudoprobability \( P = |C_2|^2 = 0.015 \), the order of magnitude of extra contribution from the normalization effect is favor, but its correction has the wrong sign. It seems that the situation in the NR potential models does not make a change for the better.

IV. The possibilities to explain the discrepancy

A. Semi-quantitative estimate to the relativistic corrections

In the previous section we do not take into account the contributions of both \( (r^2)_{\text{ms}} \) and \( (r^2)_{\text{bs}} \).

In order to estimate their contributions, we need the concrete expression of BS amplitude depending on the interaction kernel. It is difficult, however, to analytically solve the BS equation, even in the case of the OBE interaction kernel. In this subsection we shall make a semi-quantitative analysis to estimate their contributions reproducing the low energy behavior of deuteron. It is well-known that the BS equation is reduced to the Salpeter equation[14] by making the instantaneous approximation as \( p_0 - p_0 = 0 \) for the interaction kernel \( G(p - p') \).

Furthermore, if we neglect the negative energy states and make the NR approximation \( F_p = \sqrt{p^2 + m^2} = m + \frac{p^2}{2m} \), the Salpeter equation coincides with the Schrödinger equation.

From this fact, we assume, so far as an analysis of the MS radius, that the Salpeter amplitude \( \Psi^{*} \) for the positive energy states are approximately equal to the NR wave function of deuteron \( u(p), v(p) \). Furthermore, we neglect the contributions of the negative energy states, except for the normalization effect, from the analysis in the previous section. The relationship between the BS amplitude \( \Phi^{*} \) and the Salpeter amplitude \( \Psi^{*} \) for the positive energy states are given as

\[ \Phi^{*}(p, p') = \left[ S_{\text{ms}} \right]^* \Psi^{*}(p), \quad \left[ S_{\text{bs}} \right] = \frac{E - M}{\sqrt{(E - M)^2 - p^2 - \epsilon}}. \] (4.1)

By substituting Eq (4.1) into Eq (3.5) and by carrying out the integral with respect to the relative energy \( p \), we obtain the following formula as

\[ r^2 = (r^2)_{\text{ms}} + (r^2)_{\text{bs}} + \frac{3}{4m^2} \delta r_{\text{ms}} + \delta r_{\text{bs}} + \delta r_{\text{cor}}, \] (4.2)

where \( (r^2)_{\text{ms}} \) is correspondent to \( r^2 \) in Eq (1.1). \( \delta r_{\text{ms}}, \delta r_{\text{bs}}, \delta r_{\text{cor}} \) denote the relativistic corrections for \( (r^2)_{\text{ms}}, (r^2)_{\text{bs}}, (r^2)_{\text{cor}} \) and \( (r^2)_{\text{ms}} \) in Eqs (3.5) and (3.7), respectively. These explicit forms are given in appendix B. We here tentatively use the wave function in the Paris potential[3], which is widely accepted in the NR potential model, to the Salpeter amplitude \( \Psi^{*} \). By using the formulae (B.2)-(B.6) and carrying out numerical calculations, we obtain as follows in the case of \( P = |C_2|^2 = 0.015 \):

\[ \langle r^2 \rangle_{\text{ms}} = (196 \text{fm})^2 \quad \Rightarrow \quad \delta r_{\text{ms}} = -1.006 \times 10^{-4} \text{fm}, \]
\[ \delta r_{\text{cor}} = -9.0186 \times 10^{-4} \text{fm}, \quad \delta r_{\text{bs}} = -9.0186 \times 10^{-4} \text{fm}. \] (4.3)

where \( \langle r^2 \rangle_{\text{ms}} \) represents the MS radius in the Paris potential model. The relativistic corrections for \( \delta r_{\text{ms}} \) and \( \delta r_{\text{cor}} \) have favor order of magnitude to the discrepancy, but their terms apparently cancel each other due to their opposite sign. This result indicates that the calculation of the relativistic corrections to a NR theory is quite delicate. As a result, \( \delta r_{\text{ms}} + \delta r_{\text{bs}} + \delta r_{\text{cor}} = -1.054 \times 10^{-3} \text{fm} \), leading to a negligible result with right sign to the discrepancy and we obtain the similar result in the previous section:

\[ \langle r^2 \rangle_{\text{bs}} = (197 \text{fm})^2 \quad \Rightarrow \quad \langle r^2 \rangle_{\text{bs}} = (198 \text{fm})^2. \] (4.4)

In this way, we conclude that the relativistic corrections are not able to explain the discrepancy concerning the MS radius of deuteron in the NR nuclear potential models. Another possibility may be the MEC effect. However, it has been reported[9,10] that the MEC contribution has the wrong sign and is several times larger than the discrepancy, although the value depends strongly on the model used.
B. The off-shell effects of the photon-nucleon interaction vertex

In this subsection, we shall like to argue the possibility to explain the discrepancy of the MS radius, by taking into account the off-shell effect of the photon-nucleon interaction vertex for the bound system. So far we have assumed the on-shell form for the photon-nucleon interaction vertex in Eq. (2.7). This assumption is introduced mainly to avoid the complexity for the calculations. In the case of the bound system, the photon-nucleon interaction vertex has to modify the on-shell form in Eq. (2.7) as

$$\Gamma_{\mu}^{(i)}(q) \rightarrow \Gamma_{\mu}^{(i)}(p', p) = \Gamma_{\mu}^{(i)}(q) + \Lambda_{\mu}^{(i)}(p', p),$$

(4.5)

where $\Lambda_{\mu}^{(i)}(p', p)$ represent the off-shell vertex and it vanishes for the on-shell case $p'^2 = -m^2$ and $p^2 = -m^2$. It is obtained by calculating the higher order corrections of photon- and meson-nucleon interactions and by renormalizing. Generally, it is difficult to obtain its explicit form since the nuclear interaction is strong. Then, we phenomenologically assume the relevant form for $\Lambda_{\mu}^{(i)}(p', p)$ and estimate its correction to the MS radius of deuteron. Since $\Lambda_{\mu}^{(i)}(p', p)$ is the Lorentz invariant and it vanishes for the on-shell case, we are able to expect its explicit form as

$$\Lambda_{\mu}^{(i)}(p', p) = C_i(p') \gamma^{(i)} + C_{ij}(p', p) \gamma^{(j)} + C_{ij}(p', p) \gamma^{(j)} + C_{ij}(p', p) \gamma^{(j)} + C_{ij}(p', p) \gamma^{(j)} + \cdots,$$

(4.6)

where $C_i, (i = 1, 2, 3, \cdots)$ is a function of $p'^2$ and $p^2$. Furthermore, due to the conservation of deuteron current, the form in Eq. (4.6) is restricted by the following Ward-Takahashi identity:

$$q_i \Gamma_{\mu}^{(i)}(p', p) = S_{\mu}^{(i)}(p') - S_{\mu}^{(i)}(p') \gamma^5,$$

(4.7)

with

$$S_{\mu}^{(i)}(p') = \gamma_{\nu}^{(i)} p_{\nu} + m + \Sigma_{\mu}(p'),$$

(4.8)

where $\Sigma_{\mu}(p')$ represents the renormalized self-energy and it vanishes for the on-shell case. Then the meson-nucleon interaction kernel is modified and the BS equation in Eq. (2.1) is generally changed to the renormalized one. It is interesting that the third term in Eq. (4.6) automatically satisfy the current conservation without the change of the BS equation and the normalization condition. However, its correction to the MS radius is negligible to the discrepancy, since the leading $q^2$-dependence become $C_1(m - \bar{M}) q^2$.

We assume the most simple form for $\Lambda_{\mu}^{(i)}(p', p)$ as

$$\Lambda_{\mu}^{(i)}(p', p) = C(p') \gamma^{(i)} \left( \frac{d \gamma^{(i)}}{dq^2} \right).$$

(4.9)

with a constant $C$. This equation satisfies the current conservation without the change of the nucleon propagator, although the normalization condition slightly modifies the order of the binding energy. It is worthwhile to note that the Eq. (4.9) has a similar form that we have obtained by assuming the relativistic separable interaction kernel to the BS equation[15]. By using Eq. (4.9), we have

$$\langle r_{\text{in}}^2 \rangle = \langle r_{\text{na}}^2 \rangle + \frac{3}{4m^2} + \frac{3}{4m^2} + \frac{3}{2} C = -2Cm^2 - \frac{3}{4m^2},$$

(4.10)

This off-shell effect has the possibility to explain the discrepancy of MS radius, if $Cm^2 = 1$ and $C > 0$. It should be noted that the contribution of this off-shell effect in Eq. (4.9) to the magnetic and quadrupole moments is negligible. This off-shell effect may play more important role for the high-energy nuclear physics. More detailed analysis on this off-shell effect will be given separately.

V. Concluding Remarks

In this paper we have investigated the MS radius of deuteron in the BS formalism with a special attention on keeping Lorentz covariance, starting directly from the expression in the Breit frame of conserved effective deuteron current and argued the possibilities to explain the discrepancy in Eq. (1.3). In our investigation we have intended to extract the results as general as possible, being independent of dynamical details. It should be noted that some important part out of the MEC effects such as "pair" current in the NR nuclear theories are automatically included in our covariant treatment. One characteristic features of the BS formalism is the existence of the negative energy states. By neglecting the negative energy states, the BS amplitude has only two states (1S- and 1D-) which correspond to them in usual NR nuclear potential models. From this point of view, we can classify the contributions to the MS radius of deuteron as following three parts: the main part comes from the positive energy states including many kind of relativistic effects such as the relative motions of constituent nucleons and the boost effects: the second one comes from the negative energy states themselves: the third one is the normalization effect, which is non-perturbative and appears through the normalizing condition. It should be noted that the second and third contributions may be related each other (See the discussion in the
section III).

First, we have shown by the qualitative argument that the all relativistic corrections in the positive energy states, except for the Zitterbewegung term $\frac{3}{4m^3}$, to the MS radius of deuteron is relatively small compared to the discrepancy, so that we obtained the ordinary formula for the MS radius in Eq (1 2). Secondly, we have investigated the details of the relativistic corrections by carrying out the appropriate three dimensional reduction, corresponding to the instantaneous approximation, so that it reproduces the low energy behavior of deuteron. Here, we must make the three dimensional reduction after the calculation of the matrix elements concerning the MS radius. We have shown that the three relativistic corrections have the same order of the magnitude as the discrepancy, but their terms apparently cancel each other due to their opposite sign. We have found that the calculation of the relativistic corrections to the NR theory is sensitive and that it is important to have at least a consistent treatment to both the equation and the electromagnetic current of the deuteron as bound system. We have shown also, by the qualitative argument, that the contributions of the negative energy states themselves are negligible compared to the discrepancy. For the normalization effect, if we take $\rho_s = 0.015$, its contribution has a favor order of magnitude with a wrong sign. As a result, since the total relativistic corrections tend to make the discrepancy larger, we may conclude that it is impossible to explain the discrepancy concerning the MS radius of deuteron in the NR nuclear potential model including the relativistic corrections within the additive current for the constituent nucleons.

Finally, we have phenomenologically estimated the contributions of the off-shell effects for photon-nucleon interaction vertex. These off-shell effects may be candidates to explain the discrepancy.

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Appendix A

In this appendix, we describe the calculations for the term of $\langle \mathbf{r}^2 \rangle$. From Eq.(3.4), we get the formula for the case of $^1S^0$ and $^1D^0$ states as

$$\begin{align*}
\lim_{\epsilon \to 0} \frac{\delta^2}{\delta \mathbf{q}^2} \tilde{F}_i(p', \mathbf{p}, \mathbf{q}) = & - \frac{6}{\mathbf{q}^2} \delta_{\mathbf{q}^2} F_i(q)[\alpha^{(1)}]_{\mathbf{q}^2} + \frac{3}{4M_d^3} R^{p_2}_i(p) - R^{p_2}_i(p) \\
& \left[ \frac{1}{\alpha^{(1)} - \alpha^{(2)}} \right] \frac{(\alpha^{(2)} \mathbf{p})(\mathbf{p}^{(2)} \mathbf{p})}{\mathbf{p}^2} \delta_{\mathbf{q}^2} \delta_{\mathbf{p}^2}.
\end{align*}
$$

(A 1)

where $\langle \cdot \rangle_{\mathbf{q}^2}$ denotes the expectation value for the spin states and $R_i(p)$, $(i = 1 \sim 3)$ is defined by

$$R_i^p(p) = \frac{3(1 + \rho_s)}{4m(E_p + \mathbf{m})} \frac{\mathbf{p}_i}{8mE_p} \left[ 2 + \frac{m^2}{2E_p} \right] \left[ \frac{m - \mathbf{M}}{M^2} \right] \frac{1 - m}{8m(E_p + \mathbf{m})} \frac{1}{m + \mathbf{E}_p} \left( 1 - \frac{m}{E_p} \right)^2 \left( 1 + \rho_s \right)
$$

$$+ \frac{1}{8mE_p} \frac{1}{(E_p + \mathbf{m})} \left[ 1 + \frac{m}{E_p} \right] \left[ 1 + \frac{m}{E_p} \right]^2 \frac{2}{E_p} \frac{\mathbf{p}_i}{2mE_p} \left( 1 + \frac{m^2}{4E_p} + \frac{m^2}{4E_p} \right)
$$

$$- \frac{\mathbf{p}_i}{E_p} \left[ 2 + \frac{m^2}{2E_p} \right] \left[ \frac{m - \mathbf{M}}{M^2} \right] \left[ \frac{1}{m + \mathbf{E}_p} \right] \left[ \frac{1}{m + \mathbf{E}_p} \right]^2 \left( 1 + \rho_s \right)
$$

$$\frac{\mathbf{p}_i}{M \mathbf{E}_p} \left[ 2 + \frac{m^2}{2E_p} \right] \left[ \frac{m - \mathbf{M}}{M^2} \right] \left[ \frac{1}{m + \mathbf{E}_p} \right] \left[ \frac{1}{m + \mathbf{E}_p} \right]^2 \left( 1 + \rho_s \right)
$$

Therefore, we obtain the following formula

$$\langle \mathbf{r}^2 \rangle = \langle \mathbf{r}^2 \rangle^{\text{on-shell}} + \langle \mathbf{r}^2 \rangle^{\text{off-shell}} - \frac{3}{4M_d^3} \lim_{\epsilon \to 0} \frac{\delta^2}{\delta \mathbf{q}^2} N_i.$$

(A 3)

with

$$\langle \mathbf{r}^2 \rangle^{\text{on-shell}} = -iN_i \sum_{p,p'} \int d^3p \tilde{\Phi}^i_{p,p'}(p, p_p) \left[ R_i^p(p) + R_i^p(p) \right] \Phi^i_{p', p_p}(p_p) S^{(2)}(p_p),$$

(A 4a)

and

$$\langle \mathbf{r}^2 \rangle^{\text{off-shell}} = -iN_i \sum_{p,p'} \int d^3p \tilde{\Phi}^i_{p,p'}(p, p_p) R_i(p) \left[ \alpha^{(1)} - \alpha^{(2)} \right] \frac{\mathbf{p}^{(2)} \mathbf{p}}{\mathbf{p}^2} \Phi^i_{p', p_p}(p_p) S^{(2)}(p_p),$$

(A 4b)

Furthermore, by using $\lim_{\epsilon \to 0} \frac{\delta^2}{\delta \mathbf{q}^2} N_i = -\frac{3}{4M_d^3}$ and $-\frac{6}{\mathbf{q}^2} \delta_{\mathbf{q}^2} F_i(q)[\alpha^{(1)}]_{\mathbf{q}^2} = \mathbf{r}^2 \mathbf{r}^2 - \frac{3}{2m^2}$, we obtain the...
formula

\[ \langle r_0^2 \rangle - r_0^2 + r_1^2 + \frac{3}{4m^2} P_2 + \frac{3\kappa}{2m^2} P_2^* + \frac{3}{8m^2} (1 + 2\kappa) \frac{M - M}{M} (P_2 + P_2^*) . \] (A.5)

which is derived by \( \langle r_1^2 \rangle = 0 \) by virtue of \( R_1(p) = 0 \) and

\[ R_1^m(p) = \frac{3}{4m^2} \left( 1 + \frac{p_3^2}{\kappa} + \frac{M - m}{M} p_2 \right), \quad R_1^m(p) = \frac{3}{4m^2} \left( 1 + \frac{p_3^2}{\kappa} + \frac{M - m}{M} p_2 \right) . \] (A.6)

Here, we neglect the terms much smaller than the discrepancy from the qualitative behavior of the BS amplitudes in Eq (2.5) and the non-relativistic approximation \( E = m \) and \( p_3 = 0 \).

Appendix B

We present the formula of the MS radius of deuterium in the instantaneous approximation as,

\[ \langle r_0^2 \rangle = \langle r_0^2 \rangle + \langle r_1^2 \rangle + \frac{3}{4m^2}\langle \delta r^2 \rangle + \delta r_{av}^2 + \delta r_{v}^2 \] (B 1)

with

\[ \langle r_0^2 \rangle = \frac{N_0}{2n} \int d^4p \left[ \frac{1}{4} \frac{\partial^2}{\partial p^2} \Psi^{*(p)} \right] \Psi^{*(p)} . \] (B 2)

\[ \delta r_{av}^2 = \frac{N_0}{2n} \int d^4p \left[ \frac{1}{4E_s(E_s - M)} \left[ \frac{p_1^2}{E_s} - \frac{p_2^2}{E_s} + \frac{p_3^2}{E_s} \right] \right] \Psi^{*(p)} . \] (B 3)

\[ \delta r_{v}^2 = \frac{N_0}{2n} \int d^4p \left[ \frac{3}{2M} \left( 1 - \frac{E_s - M}{2M} \right) \right] \left[ \frac{1}{4} \frac{\partial^2}{\partial p^2} \Psi^{*(p)} \right] \Psi^{*(p)} . \] (B 4)

\[ \delta r_{v}^2 = \frac{N_0}{2n} \int d^4p \left[ \frac{1}{2M} \left( 1 - \frac{E_s - M}{2M} \right) \right] \left[ \frac{1}{4} \frac{\partial^2}{\partial p^2} \Psi^{*(p)} \right] \Psi^{*(p)} . \] (B 5)

\[ \delta r_{v}^2 = \frac{N_0}{2n} \int d^4p \left[ \frac{3}{2M} \left( 1 - \frac{E_s - M}{2M} \right) \right] \left[ \frac{1}{4} \frac{\partial^2}{\partial p^2} \Psi^{*(p)} \right] \Psi^{*(p)} . \] (B 6)

\[ \delta r_{v}^2 = \frac{N_0}{2n} \int d^4p \left[ \frac{3}{2M} \left( 1 - \frac{E_s - M}{2M} \right) \right] \left[ \frac{1}{4} \frac{\partial^2}{\partial p^2} \Psi^{*(p)} \right] \Psi^{*(p)} . \] (B 7)

where \( \Psi^{*(p)} \) represents the S- and D-wave states, and \( \Psi^{*(p)} \) represents the radial Salpeter amplitude for D-wave state corresponding to the \( w(p) \) in the NR treatment. The Eq (B.2) is coincided with the Fourier conjugate expression in the Eq (1.1) by identifying \( \Psi^{*(p)} \) with the NR wave function.

References

