DYNAMICS OF DIMENSIONS IN FACTOR SPACE COSMOLOGY

U. Bleyer\textsuperscript{1}\textsuperscript{§*}, M. Mohazzab\textsuperscript{2}\textsuperscript{††}, M. Rainer\textsuperscript{3} §

\textsuperscript{§} Gravitationsprojekt/Projektgruppe Kosmologie
Institut für Mathematik, Universität Potsdam
P.O.Box 601553, D-14415 Potsdam, Germany

* Urania Berlin e.V.
An der Urania 17, D-10787 Berlin, Germany

† Institute for Studies in
Theoretical Physics and Mathematics
P.O.Box 5746, Tehran 19395, Iran

‡ Physics Dept. Alzahra University
Tehran 19834, Iran

Abstract

We consider multidimensional cosmological models with a generalized space-time manifold \( M = R \times M_1 \times \cdots \times M_n \), composed from a finite number of factor spaces \( M_i, i = 1, \ldots, n \).

While usually each factor space \( M_i \) is considered to be some Riemannian space of dimension \( d_i \in \mathbb{N} \), here it is, more generally, a fractal space, the dimension of which is a smooth function of time \( d_i(t) \in \mathbb{R} \). Hence, besides the scale factor exponents \( \beta_i = \ln a_i \) and their derivatives, we consider also the dimensions \( d_i \) of the factor spaces as classical dynamical variables.

The classical equation of motions and the corresponding Wheeler-de Witt equation are set up generally, and the qualitative behaviour of the system is discussed for some specific model with 2 factor spaces.

\textsuperscript{1}E-mail: ubleyer@aip.de
\textsuperscript{2}E-mail: masoud@irarn.bitnet
\textsuperscript{3}E-mail: mrainer@aip.de

Financially supported by the DAAD and DFG grant Bl 365/1-1

1
1 Introduction

The observed correlation function of galaxy clusters and the fluctuation of microwave background radiation seem to have fractal structure [1], [2]. These kinds of observations suggest to attribute a fractal structure to the universe. When the building blocks of space-time or some of its subspaces have a fractal structure, its dimension may have a noninteger value.

The assumption that space has a continuous dimension, was first proposed in [3] where a specific \((d + 1)\)-dimensional cosmological model with isotropic and homogenous \(d\)-dimensional spacelike slices was proposed. Its starting point is an extended Einstein-Hilbert Lagrangian in arbitrary \(d\) space dimension together with a natural constraint between dimension and scale factor of the universe. The constraint arises when a cellular structure is attributed to space.

Suppose the dimension of a space \(M_0\) is \(d_0\) and its size \(a_0\). This space can be constructed from a finite number \(N\) of \(d_0\)-dimensional cells \(e_0\). Now suppose we have a space \(M\) of dimension \(d > d_0\). In order to build such a space from the same number \(N\) of cells, these should have an extra dimension \(d - d_0\). Let us take the extra \((d - d_0)\)-cell to be of small size \(\ell\) corresponding to some fundamental length (e.g. the Planck scale). Then we get the \(d\)-dimensional volume of \(M\) as \(\text{vol}_d(M) = N \text{vol}_{d_0}(e_0) \ell^{d-d_0}\), and analogously the volume of the \(d_0\)-space \(M_0\) as \(\text{vol}_{d_0}(M_0) = N \text{vol}_{d_0}(e_0)\). Since the volume of a \(d\)-dimensional space of size \(a\) is proportional to \(a^d\), the constraint

\[
\left(\frac{a}{\ell}\right)^d = \left(\frac{a_0}{\ell}\right)^{d_0}
\]

arises. The model [3] predicts that the universe quickly becomes a FRW universe during its expansion. Actually this universe may oscillate between a lower scale (the fundamental length) and an upper scale (the size of the universe now). During its expansion, the dimension of the universe decreases to the present observed value while during its contraction the dimension of the universe increases to a finite number.

This model solves the horizon problem, since there is no starting time for the evolution of the universe, and therefore during several contractions and expansions all points become correlated. Furthermore there is no big bang singularity in the model.
On the other hand recent works [4, 5] on multidimensional cosmology, generalizing the Kaluza-Klein idea, use the possibility to assign further (however constant) dimensions $d_i$ of additional internal factor spaces $M_i$ to the universe in its early stage of evolution. Instead of a dynamical reduction of the spacial dimension (like in [3]) here the scales of the internal factor spaces contract.

Multidimensional geometric models are an interesting class to study in cosmology, because on one hand, they are rich enough to model features of phenomenological interest, on the other hand they provide a well defined minisuperspace. The latter is a convenient starting point for covariant and conformally equivariant quantization, with the energy constraint yielding the Wheeler-de Witt (WdW) equation.

Here, we generalize the above works by admitting the factor spaces to have a fractal dimension which is a smooth function of time. As an example we study the case of two factor spaces, one flat and the other compact, where the latter has a constant dimension. In fact, the contribution from the scale factor of the compact space with constant dimension is formally equivalent to some matter field like the perfect fluid of [3]. Actually, we find it more conclusive (as compared to the standard approach) to have all matter created from the geometry of space-time.

In Sec. 2 we give the setup of the canonical formalism for a multidimensional cosmology with Riemannian and, more specifically, constant curvature factor spaces. After a proper reformulation of the Lagrangian and Hamiltonian on the minisuperspace, the canonical quantization can be applied.

Sec. 3 deals with conformally equivariant quantization on a minisuperspace. The first quantization of the energy constraint is performed in a generally covariant and conformally equivariant manner. Hence, there is no factor ordering problem.

Sec. 4 then considers the Lagrangian variation with dynamical dimensions, where a constraint might be taken into account by a Lagrange multiplier. However the resulting equations of motion are in general too difficult to be solved analytically.

Therefore in Sec. 5 we consider another, very specific, Lagrangian model and derive its equation of motion.

3
In Sec. 6 then the qualitative behaviour of this specific system is discussed.

Sec. 7 refers to the WdW equation for this system and Sec. 8 finally resumes the results.

2 Riemannian factor space cosmology

A convenient reduction of the superspace of geometries is at hand for the class of multidimensional geometries. Usually, with \( d_i \in \mathbb{N} \), such a geometry is described by a manifold

\[
M = \mathbb{R} \times M_1 \times \ldots \times M_n, \\
D := \dim M = 1 + d_1 + \ldots + d_n \\
g \equiv ds^2 = -e^{2\gamma dt} \otimes dt + \sum_{i=1}^{n} a_i^2 ds_i^2, \tag{2.1}
\]

where \( a_i = e^{\beta_i} \) is the scale factor of the factor space \( M_i \) of dimension \( d_i \in \mathbb{N} \). Here we choose

\[
ds_i^2 = g^{(i)}_{kl} dx^k_{(i)} \otimes dx^l_{(i)}
\]

such that \( ds_i^2 \) is a regular bounded measure on \( M_i \), with a finite standard volume

\[
\text{vol}_i := \int_{M_i} ds_i < \infty.
\]

The scale factor exponents \( \beta^i = \ln a_i, i = 1, \ldots, n \), provide a set of coordinates for the \( n \)-dimensional minisuperspace \( \mathcal{M} \) over \( M \). We subject the minisuperspace coordinates \( \beta^1, \ldots, \beta^n \) to the principle of general covariance w.r.t. minisuperspace coordinate transformations.

Like in [4, 5], we restrict here the \( M_i \) to be Einstein spaces of constant curvature. Then the Ricci scalar curvature of \( M \) is

\[
R = e^{-2\gamma} \left\{ \left( \sum_{i=1}^{n} (d_i \dot{\beta}^i) \right)^2 + \sum_{i=1}^{n} d_i [(\dot{\beta}^i)^2 - 2\gamma \dddot{\beta}^i + 2\dot{\beta}^i] \right\} + \sum_{i=1}^{n} R^{(i)} e^{-2\beta^i}. \tag{2.2}
\]

The action is usually taken in the standard form

\[
S = S_{EH} + S_{GH} + S_M, \tag{2.3}
\]
where

\[ S_{EH} = \frac{1}{2\kappa} \int_M \sqrt{|g|} R \, dt \]

is the Einstein-Hilbert action, \( S_{GH} \) is the Gibbons-Hawking boundary term, and \( S_M \) some matter term.

Here we choose the boundary conditions such that the terms with \( \gamma, \beta \) from (2.2) and \( S_{GH} \) cancel out. Since we always have the possibility to introduce one more dilatonic scale factor from the geometry instead of some scalar matter field, here we set \( \delta S_M = 0 \) without restriction.

Then the variational principle of (2.3) is equivalent to a Lagrangian variational principle over the minisuperspace \( M \), given in coordinates \( \beta^i \).

\[ S = \int L \, dt, \]

\[ L = \frac{1}{2\mu} \exp \left\{ -\gamma + \sum_{i=1}^{n} d_i \beta^i \right\} \left\{ \sum_{i=1}^{n} d_i (\beta^i)^2 - \left[ \sum_{i=1}^{n} d_i \beta^i \right]^2 \right\} - V(\beta^i) \]

(2.4)

with

\[ V(\beta^i) = \mu \exp \left\{ \gamma + \sum_{i=1}^{n} d_i \beta^i \right\} \left[ -\frac{1}{2} \sum_{i=1}^{n} R^{(i)} e^{-2\beta^i} \right] \]

where

\[ \mu := \kappa^{-1} \prod_{i=1}^{n} \text{vol}_i. \]

(2.5)

Let us define a metric on \( M \), given in coordinates \( \beta^i, i = 1, \ldots, n \). We set

\[ G_{kl} := d_k \delta_{kl} - d_k d_l \]

(2.6)

\( k, l = 1, \ldots, n \), thus defining the tensor components \( G_{ij} \) of the minisuperspace metric

\[ G = G_{ij} d \beta^i \otimes d \beta^j. \]

(2.7)

Then with a lapse function \( N \), we obtain the Lagrangian

\[ L = \frac{\mu}{2N^2} G_{ij} \dot{\beta}^i \dot{\beta}^j - V(\beta^i) \]

(2.8)
with the energy constraint

$$\frac{\mu}{2N^2} G_{ij} \beta^i \dot{\beta}^j + V(\beta^i) = 0. \quad (2.9)$$

A convenient gauge for $N$ is the harmonic one [5, 6] given by

$$N^2 := \exp \left\{ \gamma - \sum_{i=1}^n d_i \beta^i \right\} \equiv 1. \quad (2.10)$$

Nevertheless, here we do not want to restrict to a specific gauge. Unlike in [4, 5, 6], we will prefer to implement a relation like $C := \gamma - \sum_{i=1}^n d_i \beta^i = 0$ as constraint on the configuration variables $\beta^i$, rather than as a gauge for $\gamma$. Note that for a set of $m$ constraints $C_k, \ k = 1, \ldots, m$ we have to amend the potential $V(\beta^i)$ by $- \sum_{k=1}^m \lambda_k C_k(\beta^i)$, yielding

$$L = \frac{\mu}{2N^2} G_{ij} \dot{\beta}^i \dot{\beta}^j - V(\beta^i) + \sum_{k=1}^m \lambda_k C_k(\beta^i), \quad (2.11)$$

with Lagrange multipliers $\lambda_k$. Let us recall that, although in general the variation of (2.11) w.r.t. $\beta^i$ and $\lambda_k$ does not commute with the implementation of the constraints, at least the set of solutions for (2.8) with $C_k(\beta^i) = 0$ resolved and substituted before the variation, is a subset of the solutions of the variation of (2.11) including the Lagrange multipliers.

Let us consider now the minisuperspace $\mathcal{M}$ Its signature of is Lorentzian for $n > 1$ (see [5]). After diagonalization of (2.6) by a minisuperspace coordinate transformation $\beta^i \rightarrow \alpha^i \ (i = 1, \ldots, n)$, there is just one new coordinate, say $\alpha^1$, in the direction of which the corresponding eigenvalue of $G$ is negative. With a further (sign preserving) coordinate rescaling, $G$ is equivalent to the Minkowski metric [6]. Hence $\mathcal{M}$ is flat. Note that, unlike conformal flatness, flatness is not an invariant property under conformal transformation on $\mathcal{M}$.

In [6] it was pointed out that while $\beta^i \rightarrow \alpha^i$ is only a coordinate transformation on $\mathcal{M}$, it transforms a multidimensional geometry (2.1) with scale exponents $\beta^i$ to another geometry which is of the same multidimensional type (2.1). This has the same dimensions $d_i$ and first fundamental forms $ds_i^2$, but new scale exponents $\alpha^i$ of the factor spaces $M_i$. We can always
perform the diagonalization of (2.6) such that $\alpha^1$ and hence $M_1$ belongs to the unique negative eigenvalue of $G$. This $M_1$ is identified as "external" space. The scale factors of the "internal" spaces $M_2, \ldots, M_n$ contribute only positive eigenvalues to the metric of $\mathcal{M}$. $\alpha^1$ assumes in $\mathcal{M}$ the role played by time in usual geometry and quantum mechanics. In this way the "external" space is distinguished against the "internal" ones, since its scale factor provides a natural "time" coordinate on $\mathcal{M}$. Note however that the "minisuperspace time" $\alpha^1$ can be considered as a time equivalent to $t$ in the underlying multidimensional geometry $g$ only if the space $M_i$ with $\alpha^1$ is strictly expanding w.r.t. time $t$. Then the Lorentzian structure of $\mathcal{M}$ provides a natural "arrow of time" [7].

3 Canonical minisuperspace quantization

Canonical quantization essentially consists in replacing the constraint equation (2.9) by the WdW equation [8]

$$\left(-\frac{1}{2}[\Delta - \xi_e R] + V\right)\Psi = 0$$

(3.1)

for a wave function $\Psi$.

We set in the following

$$N =: e^{-2f}$$

(3.2)

and admit $f \in C^\infty(\mathcal{M})$ to be an arbitrary smooth function on $\mathcal{M}$.

In the time gauge given by $f$ the Lagrangian is

$$L^f := \frac{\mu}{2} \dot{G}^{ij}(\beta) \dot{\beta}^i \dot{\beta}^j - V^f(\beta)$$

(3.3)

and the energy constraint is

$$E^f := \frac{\mu}{2} \ddot{G}^{ij}(\beta) \dot{\beta}^i \dot{\beta}^j + V^f(\beta) = 0,$$

(3.4)

where

$$\ddot{G} = e^{2f} G \text{ and } V^f = e^{-2f} V.$$
With the canonical momenta
\[ \pi_i = \frac{\partial L^f}{\partial \dot{x}_i} = \mu^f G_{ij} \dot{x}_j \]
(3.5)
this is equivalent to a Hamiltonian system given by
\[ H^f = \frac{1}{2\mu} (G)^{ij} \pi_i \pi_j + V^f \]
(3.6)
and the energy constraint
\[ H^f = 0. \] (3.7)
The inverse of the minisuperspace metric is given by \( (G)^{-1} = e^{-2f} G^{-1} \), where for the system with Eq. (2.6) the components of \( G^{-1} \) are
\[ G^{ij} = \frac{\delta_{ij}}{d_i} + \frac{1}{1 - \sum_{i=1}^n d_i}. \] (3.8)
At the quantum level \( H^f \) has to be replaced by an operator \( \hat{H^f} \), acting in analogy to (3.7) as
\[ \hat{H}^f \Psi^f = 0 \] (3.9)
on wavefunctions, which are in a conformal representation of weight \( b \) given as
\[ \Psi^f = e^{bf} \Psi. \] (3.10)
Conformally equivariant quantization of \( H^f \) from (3.6) yields
\[ \hat{H}^f = e^{-2f} e^{bf} \hat{H} e^{-bf} \]
\[ \hat{H}^f = -\frac{1}{2\mu} \left[ \Delta^f - \xi_c R^f \right] + V^f, \] (3.11)
on wave functions \( \Psi^f = e^{bf} \Psi \), where
\[ b = -(n - 2)/2, \] (3.12)
\[ \xi_c = \frac{n - 2}{4(n - 1)}, \] (3.13)
\[ \Delta^f = G^{ij} \nabla^f_i \nabla^f_j \]  
(3.14)

and both, \( R^f \) and the covariant derivative \( \nabla^f \), are determined by the connection \( \Gamma^f \) corresponding to the metric \( G^f \).

The WdW equation (3.9) is conformally equivariant if and only if Eq. (3.9) for any \( f \) is equivalent to

\[ \dot{H} \Psi = 0 \]  
(3.15)

where

\[ \dot{H} = \dot{H}^f |_{f=0} \text{ and } \Psi = \Psi^f |_{f=0} \]

are the Hamilton operator and the wave function in the gauge \( f = 0 \).

4 Variation with dynamical dimensions

We now pick up the Lagrangian (2.11) of Riemannian factor space cosmology and consider it as Lagrangian with dynamical dimensions \( d_i \) of some fractal factor spaces. Hence the dynamical configuration variables are both \( \beta^i \) and \( d_i \) for \( i = 1, \ldots, n \).

In order to implement constraints \( C_i = 0 \) on the dynamics of dimensions, we follow [3]. Their constraint is given as follows: Suppose each factor space is constructed from a number \( P \) of \( d \)-cells, each of which is a product of a \( d_0 \)-cell of macroscopic scale and a \( (d - d_0) \)-cell of microscopic scale. From these cells we can build a \( d \) dimensional space. For finite \( P \) and a vanishing measure on the \( (d - d_0) \)-cells, the \( d \)-volume of the resulting space is zero. However if the \( d - d_0 \)-cells have a length scale \( \ell > 0 \) and nonvanishing \( (d - d_0) \)-volume then the volume of each cell is \( v_d = v_{d_0} \ell^{d-d_0} \). Now if we write the scale of the factor space as \( \ell e^{\beta} \), we will have \( \ell^d e^{d \beta} = P v_d \) or \( e^{d \beta} = e^{d_0 \beta_0} \).

Since \( d_0 \) and \( \beta_0 \) are constant initial data for the dynamics (take the present day values), we obtain the constraint \( d \beta = c \) with constant \( c \).

There are many alternatives for generalizing the above considerations to the case of multidimensional cosmology, e.g.:

1) \( \sum_{i=1}^n \beta^i d_i = \gamma \)
2) Constant \( \sum_{i=1}^m \beta^i d_i = c_m \) for some \( m < n \).
3) Constant \( \beta^i d_i = c \) for \( 1 \leq i \leq m \leq n \).
Slightly more general than case (1) is the following constraint:

$$\frac{\mu}{2} e^{-\gamma + \sum_{i=1}^{n} d_i \beta_i} = C,$$

(4.1)

with a further function $\mu$, and $C$ independent of the dynamical variables.

This constraint might also be reinterpreted as a generalization of a harmonic time gauge of constant $\mu$ and $\gamma$ as in [6]. With the harmonic time gauge many interesting cosmological models (see also [4, 5]) have been constructed.

Note that for constant curvature factor considered here we have $R^{(i)} = K d_i (d_i - 1)$.

Taking the constraint (4.1), with Lagrange multiplier $\lambda$, the Lagrangian (2.11) is

$$L = \frac{\mu}{2} \exp \{-\gamma + \sum_{i=1}^{n} d_i \beta_i \} \left[ G_{ij} \dot{\beta}^i \dot{\beta}^j + K e^{2\gamma} \sum_{i=1}^{n} d_i (d_i - 1) e^{-2\beta_i} \right] + \lambda \left( e^{-\gamma + \sum_{i=1}^{n} d_i \beta_i} - \frac{2C}{\mu} \right).$$

(4.2)

We assume that

$$\text{vol}_i = m_i (d_i) \bar{l}_i^{d_i},$$

(4.3)

where $\bar{l}_i$ is some characteristic length of $M_i$, and $m_i$ is a function of $d_i$. For dimension $d$ the coupling $\kappa$ is

$$\kappa = \ell^{d-1},$$

(4.4)

for some fundamental length $\ell$.

Then with Eq. (2.5) and dimensionless $l_i := \frac{\bar{l}_i}{\ell}$ we obtain

$$\mu := \ell \prod_{i=1}^{n} m_i (d_i) l_i^{d_i}.$$  

(4.5)

Variation w.r.t. $\lambda$ just reproduces the constraint. Varying w.r.t. $d_k$ we obtain

$$0 = \frac{\partial L}{\partial d_k}.$$
\begin{align*}
&= C \{ \beta^i \dot{\beta}^j \left[ \beta G_{ij} \right] + \left( \frac{\partial m_k}{\partial k} + \ln l_k + \beta^k \right) G_{ij} \} + \\
&\quad K e^{2\gamma} \sum_{i=1}^{n} d_i (d_i - 1) e^{-2\beta^i} \left( \frac{\partial m_k}{\partial k} + \ln l_k + \beta^k \right) + (2d_k - 1) e^{-2\beta^k} \} + \\
&\quad \lambda\{ \beta^k e^{-\gamma + \sum_{i=1}^{n} d_i \beta^i} + \frac{2C}{\mu} (\ln l_k + \frac{\partial m_k}{\partial k}) \}. \tag{4.6}
\end{align*}

The variation w.r.t. \( \beta^k \) yields

\begin{align*}
0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{\beta}^k} - \frac{\partial L}{\partial \beta^k} \\
&= C \{ \dot{G}_{kj} \dot{\beta}^j + G_{kj} \dot{\beta}^j \} + \dot{C} G_{kj} \dot{\beta}^j + \\
&\quad K e^{2\gamma} 2C d_k (d_k - 1) e^{-2\beta^k} - \lambda d^k e^{-\gamma + \sum_{i=1}^{n} d_i \beta^i}. \tag{4.7}
\end{align*}

For a general space \( M_i \), the functions \( m_i \) are too complicated, and the equations above can hardly be solved analytically. Therefore we have taken the restriction to spaces of constant curvature, and in the next section we consider a more specific model.

## 5 Some specific Lagrangian model

Let us now consider more specific cases of the Lagrangian with \( n \) factor spaces \( M_i \) of dimension \( d_i \). In what follows we specify the Lagrangian for the cases that all factor spaces are of constant curvature. Then, a space \( M_i \) of positive curvature is a \( d_i \)-dimensional sphere. For radius \( r_i \), its volume is

\begin{equation}
\text{vol}_i = \frac{2^{d_i} \pi^{d_i/2}}{(d_i - 1) \Gamma(d_i/2)} r^{d_i}, \tag{5.1}
\end{equation}

where \( \Gamma \) is the factorial function. Hence here

\begin{equation}
m_i(d_i) = \frac{1}{(d_i - 1) \Gamma(d_i/2)} \quad \text{and} \quad l_i = 2\sqrt{\pi} r_i. \tag{5.2}
\end{equation}
In the case of an open factor space, we can regularize the measure \( ds_i \) such that the volume is \( \int ds_i = \text{vol}_i < \infty \). This could be done e.g. by a conformal map reducing the radial extension \( \infty \rightarrow l_i \). For flat \( M_i \) in Eq. (4.3) \( m_i \) is constant.

In the following we assume:
a) One of the factor spaces, say \( M_n \), is compact with constant \( R^{(n)} \), all the other factor spaces \( M_1, \ldots, M_{n-1} \) are flat.
b) The dimension \( d_n \) of this space is constant, all other dimensions are variable.
c) Here we choose \( \gamma = \beta_0 d_1 \), where \( \beta_0 = \beta_1(t_0) \) is the present value of the scale exponent of the external space \( M_1 \).

With Eqs. (4.5), (4.4) and (5.2), normalizing all volumes with \( \tilde{l}_i = \ell \quad \forall i \), choosing the constants \( m_1, \ldots, m_{n-1} \) such that

\[
\prod_{i=1}^{n-1} m_i = \frac{2}{\ell} (d_n - 1) \Gamma \left( \frac{d_n}{2} \right),
\]

the Lagrangian (2.8) is

\[
L = e^{(\beta_1 - \beta_0) d_1 + \sum_{i=2}^n \beta^i d_i} \left\{ G_{ij} \dot{\beta}^i \dot{\beta}^j + R^{(n)} e^{2(\beta_0 d_1 - \beta^n)} \right\}
\]  (5.3)

This is the generalization of [3] to multidimensional vacuum cosmology, here with a potential from the curvature the compact factor space rather than the potential there.

In the following, we consider the alternative (3) of the previous section, writing the constraints as

\[
\beta^i d_i = c_i
\]  (5.4)

for \( 1 \leq i \leq m \).

While in the last section, in order to set up the equation of motion, we had to variate the Lagrangian with respect to \( \beta^i, d_i \), and the Lagrange multiplier taking the constraint into account, here we prefer to resolve the constraints first, thus reducing the number of variables.

For simplicity we restrict to the case of \( n = 2, m = 1, k = 2 \), assuming that \( M_1 \) is a flat space of variable dimension \( d_1 \) subject to a constraint (5.4), and \( M_2 \) a compact factor space with constant \( d_2 \).
Then the Lagrangian (5.3) simplifies to

\[
L = e^{(\beta_1 - \beta_0) d_1 + \beta_2 d_2} \left\{ \left[ -\left( \dot{\beta}_1 d_1 + \dot{\beta}_2 d_2 \right)^2 + \left( \ddot{\beta}_1 \right)^2 d_1 + \left( \ddot{\beta}_2 \right)^2 d_2 \right] + R^{(2)} e^{2(\beta_0 d_1 - \beta_2)} \right\}
\]  

(5.5)

with

\[
d_1 = \frac{c}{\beta_1}
\]  

(5.6)

where \( c \) is a constant. Its physical value can be found from the observational value of the size of the universe [3].

In general, when the constraints are implemented before the variation of the Lagrangian, we obtain a subset of the full space of solutions. Hence, unlike Sec. 4, here we resolve the different constraints in the very beginning.

Then, besides (5.6), the equations of motion will finally be

\[
\ddot{\beta}_1 + \frac{\beta_1 d_2}{c - \beta_1} \ddot{\beta}_2 + \frac{1}{2} \left( \ddot{\beta}_1 \right)^2 \left( \frac{\beta_0 c}{(\beta_1)^2} - \frac{2c - \beta_1^2}{c(c - \beta_1)} \right) + \ddot{\beta}_1 \ddot{\beta}_2 d_2 - \\
\frac{1}{2} \ddot{\beta}_2 \left( \frac{d_2 (d_2 - 1) \beta_0}{c - \beta_1} + \frac{d_2^2 \beta_1}{c - \beta_1} \right) + \frac{\beta_1 \beta_0}{2(c - \beta_1)} R^{(2)} e^{2(\beta_0 \frac{\beta_1}{2} - \beta_2)} = 0
\]  

(5.7)

and

\[
\ddot{\beta}_2 + \frac{c}{\beta_1 (d_2 - 1)} \ddot{\beta}_1 - \frac{1}{d_2 - 1} \left( \ddot{\beta}_2 \right)^2 + \left( \ddot{\beta}_1 \right)^2 \frac{c}{(\beta_1)^2 (d_2 - 1)} + \left( \frac{b c}{(\beta_1)^2} + d_2 \right) \frac{c}{\beta_1 (d_2 - 1)}
\]

\[
+ \ddot{\beta}_2 \left( \frac{1}{d_2 - 1} - \ddot{\beta}_1 \right) \frac{c}{(\beta_1)^2 (d_2 - 1)} - \left( 1 - \frac{d_2}{2} \right) R^{(2)} e^{2(\beta_0 \frac{\beta_1}{2} - \beta_2)} = 0
\]  

(5.8)

6 Qualitative behaviour of the system

From the Lagrangian (5.5) the Hamiltonian can be written

\[
H = e^{-\beta_0 d_1 + \beta_2 d_2} G_{ij} \dot{\beta}_i \dot{\beta}_j - R^{(2)} e^{\beta_0 d_1 + (d_2 - 2) \beta_2}
\]  

(6.1)
As the Lagrangian of this model is not an explicit function of time, the Hamiltonian \( H \) is a constant of motion. When the system has a solution \( E := -\frac{H}{c^2} \) is a positive constant.

Let us now first consider the case of constant \( \beta^2 := q \). At the early universe, it is \( d_1 \gg d_0 := d_1(t_0) \), we get the following equation

\[
(\dot{\beta}^1)^2 + \frac{1}{c^2}(\beta^1)^2 e^{2\beta_0 d_1 - q d_2} R^{(2)} = \frac{E}{c^2}(\beta^1)^2 e^{\beta_0 d_1 - q d_2}
\]

Eq. (6.2) is a 1-dimensional mechanical system of constant energy \( E \). The minimum value for \( \beta^1 \) (where \( \dot{\beta}^1 = 0 \)) is

\[
\beta^1_{\text{min}} = \frac{c\beta_0}{\ln \left( \frac{E}{R^{(2)} t} \right) - q(d_2 - 2)}.
\]

Therefore a maximum value for the dimension of the early universe can be derived

\[
d_{1,\text{max}} = \frac{\ln \left( \frac{E}{R^{(2)} t} \right) - q(d_2 - 2)}{\beta_0}
\]

On the other hand from (6.2), one finds that

\[
\dot{\beta}^1 > 0 \quad \forall t
\]

Therefore \( \beta^1_{\text{min}} \) is a local minimum. As a result the universe of this model contracts to a minimum and expands again. This behaviour is similar to the previous work [3]. The behaviour of this universe at larger \( \beta^1 \) will be determined by the factor \( (\beta^1)^2 e^{\frac{\beta^1}{\beta^1 t}} \) in equation (6.2). Therefore with \( k = \pm \sqrt{\frac{E}{R^{(2)} t} e^{-q(d_2 - 2)}} - 1 \), \( \beta^1 \) will behave as \( e^{kt} \). Therefore this model predicts the late expansion of the external space to be inflationary.

In the case \( \beta^2 \neq 0 \), for the early universe \( \beta^1_{\text{min}} \) is shifted to a larger value. Thus the dynamics of the internal factor space \( M_2 \) prevents the external factor space \( M_1 \) (our universe) from reaching a singularity in the very early universe. Another virtue of the second factor space to control the inflation. The graceful exit of the factor space \( M_1 \) can only be obtained as an effect of \( \beta^2(t) \).

Complementary to the first case above, we have to examine also the situation for constant \( \beta^1 \). There, the behaviour of \( \beta^2 \) can be solved easily from (6.2). For \( d_2 = 2 \) we find \( \beta^2 \) proportional to \( \ln(t) \).
Therefore the scale factor of the second space decreases as $1/t$. As a result the general behaviour of the continuous dimensional space is an exponential expansion and the second compact constant dimensional space contracts slowly. An exact analysis of the behaviour, needs the simulation of (5.7) and (5.8).

7 WdW equation for the model

Recall that the Lagrangian in an arbitrary gauge, with $N = e^{-2f}$, is given as

$$L^f := NL = \frac{1}{N} G_{ij} \dot{\beta}^i \dot{\beta}^j - NV(\beta^i)$$

$$= \frac{1}{N} [G_{ij} \dot{\beta}^i \dot{\beta}^j - N^2 V(\beta^i)]. \quad (7.1)$$

With the Lagrangian (5.5) and the gauge $f = \frac{1}{4}(\beta_0 d_i - \sum_i \beta^i d_i)$ we get

$$L^f = e^{\frac{i}{2}(\beta^i - \beta_0) d_i + \beta^2 d_2} [G_{ij} \dot{\beta}^i \dot{\beta}^j + R^{(2)} e^{2(\beta_0 d_i - \beta^2)}]. \quad (7.2)$$

Now we consider the corresponding Hamiltonian

$$H^f := NH = \frac{1}{N} [G_{ij} \dot{\beta}^i \dot{\beta}^j + N^2 V(\beta^i)],$$

and change from the gauge $f = -\frac{1}{4}(\beta_0 d_i - \sum_i \beta^i d_i)$ to $f = 0$. Then, in the new gauge the Hamiltonian is

$$H^0 = G^{ij} \dot{\beta}^i \dot{\beta}^j + V(\beta^i)$$

$$= \frac{1}{4} G^{ij} \pi_i \pi_j + V(\beta^i) \quad (7.3)$$

where $V(\beta^i) = -R^{(2)} e^{2(\beta_0 d_i - \beta^2)} N^{-2} = -R^{(2)} e^{\sum_i \beta^i d_i + \beta_0 d_i - 2\beta^2}$. Canonical quantization in this gauge yields

$$\hat{H}^0 = -\frac{1}{4} \Delta + \hat{V},$$

$$\Delta = G^{ij} \nabla_i \nabla_j \frac{1}{\sqrt{-\det G}} \frac{\partial}{\partial \beta_i} (\sqrt{-\det G} G^{ij} \frac{\partial}{\partial \beta^j}).$$

15
Recall that here \( n = 2 \), hence Eq. (3.11) applies simply with \( \xi_e = 0 \) and \( b = 0 \). With \( d_1 = \frac{c}{\beta^1} \) and constant \( d_2 \) the inverse of \( G_{ij} \) is

\[
(G^{ij}) = \begin{bmatrix}
\frac{\beta^1}{c} + \frac{c}{c(1-d_2)-\beta^1} & \frac{c}{c(1-d_2)-\beta^1} \\
\frac{c}{c(1-d_2)-\beta^1} & \frac{1}{d_2} + \frac{c}{c(1-d_2)-\beta^1}
\end{bmatrix},
\]

and \( \sqrt{-\det G} = \frac{1}{\beta^1 \sqrt{c(d_2 - 1)d_2 \beta^1 + c^2 d_2}} \).

Since with \( n = 2 \) the conformal weight (3.12) is \( b = 0 \), a solution \( \Psi = \Psi^0 \) of the WdW equation \( \hat{H}^0 \Psi = 0 \) in the gauge \( f = 0 \) is also a solution of the the WdW equation \( \hat{H}^f \Psi^f = e^{-2f} \hat{H} \Psi = 0 \) in the original gauge \( f = \frac{1}{4}((\beta_2 - \beta^1)d_1 - \beta^2 d_2) \).

Actually here we have minisuperspace curvature \( R[G] = 0 \) and \( \det G < 0 \) for \( \beta_1 > 0 \) and \( d_2 > 1 \). Hence the minisuperspace \( M \) is the 2-dimensional flat Minkowski space, like in the analogous case of constant dimensions.

In order to get a feeling for the qualitative structure of the WdW equation, let us write it explicitly along a line of constant \( \beta^2 = q \). There, at the limit of large \( d_1 \) it is

\[
\left\{ \left( \frac{\partial}{\partial \beta^1} \right)^2 + \frac{c}{(\beta^1)^2} \frac{\partial}{\partial \beta^1} + 4K d_2(d_2 - 1)^2 e^{c-q(d_2-2)} e^{\beta_2 c} \right\} \Psi_{\beta^2=q}(\beta^1) = 0 \quad (7.6)
\]

Orthogonally to this, the WdW equation along some line of constant \( \beta^1 = \frac{c}{\beta^1} = p \) is

\[
\left\{ \frac{d_1 - 1}{d_2(d_1 d_2 - d_1 + 1)} \left( \frac{\partial}{\partial \beta^2} \right)^2 - 4K d_2(d_2 - 1)e^{c+\beta_0 d_1} e^{\beta^2(d_2-2)} \right\} \Psi_{\beta^1=p}(\beta^2) = 0.
\]

Finally we want to compare our previous minisuperspace to the case with 2 constraints \( d_1 \beta^i = c_i \), for both, \( i = 1 \) and \( i = 2 \). There

\[
(G^{ij}) = (\beta_1 \beta_2 - \beta_1 c_2 - \beta_2 c_1)^{-1} \begin{bmatrix}
(\beta^2 - c_2) \frac{\beta^1}{c_1} & \beta^1 \beta^2 \\
\beta^1 \beta^2 & (\beta^1 - c_1) \frac{\beta^2}{c_2}
\end{bmatrix},
\]

and \( \sqrt{-\det G} = \frac{1}{\beta^1 \beta^2} \sqrt{c_1 c_2 (\beta_2 c_1 + c^1 \beta^2 - \beta^1 \beta^2)} \). Here, for \( c_{1/2} > 0 \) and \( d_{1/2} > 1 \), the minisuperspace \( M \) is always Lorentzian. However, unlike the
previous example, it is no longer homogeneous, since

$$R[G] = -\frac{1}{2}\beta_1 \beta_2/(c_2 \beta_1 + c_1 \beta^2 - \beta_1 \beta_2)^2.$$  \hfill (7.9)

For $\beta_1 \to \infty$ or $\beta_2 \to \infty$ the curvature decays $R[G] \to 0$ and $\mathcal{M}$ becomes the usual homogeneous Minkowskian space. For $\beta_1 \to 0$ or $\beta_2 \to 0$ there appears a singularity in the minisuperspace curvature, $R[G] \to \infty$. However, taking the conformal quantization scheme seriously, we should be aware that the minisuperspace curvature itself is not an invariant property of the quantum system because it may be changed by conformal transformations. More specifically, in our case the minisuperspace $\mathcal{M}$ is 2-dimensional, hence there exists a gauge $f$ such that $\mathcal{M}$ is flat and therefore also homogeneous. So for this model the inhomogeneity (7.9) is only a property for the specific gauge $f = 0$. According to [6], in canonical minisuperspace quantization $\beta^1$ and $\beta^2$ are just coordinates for $\mathcal{M}$. Hence the singularity of (7.9) is the analogue of a classical coordinate singularity.

8 Conclusion

We have investigated the effect of dynamical dimensions of fractal factor spaces on the evolution of a multidimensional cosmological model. In a mathematically closed approach, we could have set up the differential geometry of fractal spaces in the very beginning. This can essentially be done using a definition of generalized manifolds by simplicial complexes (as exemplified e.g. in [9]). However, for brevity, here we rather preferred to derive the Lagrangian as for constant dimensions, and then to consider the dimensions as variables of just this Lagrangian.

More specifically, we discussed a multidimensional cosmological model with two factor spaces, one of them flat with dynamical dimension, the other compact with constant curvature and constant dimension. In fact the latter behaves like a matter field. By qualitative analysis, the behaviour of the system shows that the universe, i.e. the factor space $\mathcal{M}_1$, contracts, passes through a state of minimum size (maximum dimension), and expands.

For a static internal space (compare [4]), i.e. for constant $\beta^2 = q$, in the late expansion, for large $\beta^1$, the universe inflates double exponentially.
Therefore, the dimension decreases very fast to its minimum value. Actually, the scale factor $\beta^2$ of the compact constant dimensional space $M_2$ controls the behaviour of the expansion of universe. Eventually the dynamics of $\beta^2$ might be the only way to obtain a graceful exit to the effective model [3]. Again this model, as in the case of [3], has no big bang singularity. However, in order to find the complete behaviour of this model, a more sophisticated analysis would be required.

In the case of a static internal space $M_2$, near the maximum of $d_1$, i.e. minimum of $\beta^1$, the present model is effectively represented by a model [4, 6] with only constant dimensions. There the minimum value of $\beta^1$ could be related to the quantum creation of the real Lorentzian space-time from an Euclidian region. So, the present model is compatible with a quantum creation of our universe.

We have further derived the WdW equation for the minisuperspace of this model. Like in previous works on multidimensional cosmology, here the metric describes a Minkowskian minisuperspace. For the slightly more general case of both factor spaces subject to the same type of constraint (5.4), in the gauge $f = 0$, we find an inhomogeneity and singularities in the minisuperspace curvature $R$. However, in the case of only 2 factor spaces, the conformal class of $G$ is in any case the flat Minkowskian one.

Acknowledgement
Support by DFG grant Bl 365/1-1 and Schm 911/6-2 is gratefully acknowledged. M. M. thanks for hospitality at the Projektgruppe Kosmologie of Universität Potsdam.

References


