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TO HEAVY-ION OPTICAL MODEL POTENTIAL
AT INTERMEDIATE ENERGIES

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International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
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**AN INVERSION SOLUTION
TO HEAVY-ION OPTICAL MODEL POTENTIAL
AT INTERMEDIATE ENERGIES¹**

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ABSTRACT

In this paper, we show by solving the inversion problem at high energies that the fundamental McIntyre parametrization of the S-matrix, for heavy ion collisions, will correspond to a Woods-Saxon type optical model potential. The parameters of such a Woods Saxon's potential are directly related to the corresponding parameters of the McIntyre parametrization. The inversion solution results were tested for the available experimental data and were found to be in good agreement.

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1-introduction

Several attempts have been made lately to describe elastic scattering processes between heavy ions in terms of the optical limit to the Glauber's model ¹). This simple model could provide a good description of nucleous-nucleous scattering over a large energy range since its only inputs are the experimental nucleon-nucleon amplitudes and the rms radii of the nuclei involved

The optical limit can be obtained from the basic assumptions of Glauber theory which implies that $E \gg U$ and $\lambda \ll a$, with a as the distance on which the potential U exhibits significant variations, λ is the wavelength and E is the energy of the incident particle ²).

For the use of the Glauber approach, the simultaneous fulfillments of two conditions are required ²); namely, the validity of the eikonal approximation, in the framework of which the deviations from rectilinear motion of the incident particle is considered to be small and the validity of the adiabatic approximation where the positions of the nucleons in the nucleous are assumed to be fixed during the flight time of the incident particle through the nucleous. At high energy both conditions are satisfied simultaneously.

On the other hand, it has been shown that the use of McIntyre parametrization of the S-matrix for high-energy heavy-ion collisions gives excellent fits to the elastic scattering data ³). Actually, the McIntyre parametrization of the S-matrix proved to give good fits to the heavy-ion data with as few as three parameters ⁴).

Moreover, the optical model analysis of heavy-ion elastic scattering experiments in the intermediate energy range indicates that the data are sensitive to the real part of the nucleous-nucleous interaction for distances smaller than the strong absorption radius ⁵).

Hence it seems reasonable to conjecture that a classical inversion procedure would enable us to get the interaction potential from the scattering data down to the region inside the strong absorption radius.

Thus based on the Mc Intyre parametrization of the S-matrix and taking Glauber's high-energy approximation into account, we solve for the heavy-ion optical model potential adopting a certain inversion procedure.

In section 2 we give details of our inversion solution. Then in the next section we use Mc Intyre parametrization of the S-matrix to obtain the phase shift parameters which are used to construct Woods-Saxon type optical model potential.

Our calculations and results are given in section 4 and where a comparison of our potential and the optical model potentials, which give best fits to the data in the case of $^{12}\text{C} - ^{12}\text{C}$ elastic scattering at $E_{\text{lab}} = 1016 \text{ MeV}$ ^{6,7}, is made.

In section 5 we give a concluding discussion. It is also to be noted that an important related appendix is encountered at the end of this paper.

2-The Solution of the Inversion Problem

Within the framework of Glauber's eikonal approximation ⁸, we can relate the elastic scattering S-matrix element $S(b)$ directly to the corresponding optical model potential $U(r)$ in the following way :-

$$S(b) = \exp(i\chi(b)) \quad (1)$$

where b is the corresponding impact parameter and $\chi(b)$ is the phase shift given by :-

$$\chi(b) = -\frac{k}{E} \int_b^\infty \frac{rU(r)dr}{\sqrt{r^2 - b^2}} \quad (2)$$

where U is the potential.

Further, writing $\chi(b)$ in the form $\chi(b) = \chi_R(b) + i\chi_I(b)$ and the corresponding optical potential as $U(r) = V(r) + iW(r)$ ($V(r)$ and $W(r)$ are real) we get :-

$$\chi(b) = \chi_R(b) + i\chi_I(b) = -\frac{k}{E} \int_b^\infty \frac{r[V(r) + iW(r)]dr}{\sqrt{r^2 - b^2}} \quad (3)$$

or we have :-

$$\chi(b) = \chi_R(b) + i\chi_I(b) = -\frac{k}{E} \int_b^\infty \frac{rV(r)dr}{\sqrt{r^2 - b^2}} - \frac{ik}{E} \int_b^\infty \frac{rW(r)dr}{\sqrt{r^2 - b^2}} \quad (4)$$

Comparing we obtain :-

$$\chi_R(b) = -\frac{k}{E} \int_b^\infty \frac{rV(r)dr}{\sqrt{r^2 - b^2}} \quad (5)$$

and

$$\chi_I(b) = -\frac{k}{E} \int_b^\infty \frac{rW(r)dr}{\sqrt{r^2 - b^2}} \quad (6)$$

Eqs (5) and (6) are of Abel's type and the inverse solution to these equations are given by ⁹ :-

$$V(r) = \frac{2E}{k\pi} \frac{1}{r} \frac{d}{dr} \int_r^\infty \frac{\chi_R(b)bdb}{\sqrt{b^2 - r^2}} \quad (7)$$

and

$$W(r) = \frac{2E}{k\pi} \frac{1}{r} \frac{d}{dr} \int_r^\infty \frac{\chi_I(b)bdb}{\sqrt{b^2 - r^2}} \quad (8)$$

Eqs. (7) and (8) give the potential in its general form.

To proceed further, we assume that our heavy-ion scattering regime to be a quasi-classical one so that at high energies the above inversion solution may be adopted.

3.The Use of Mc Intyre Parametrization

The Mc Intyre parametrization³⁾ for the elastic particle wave matrix element S_ℓ is normally given by⁴⁾:-

$$S_\ell = |S_\ell| \exp(2i\delta_\ell) \quad (9)$$

where

$$|S_\ell| = \left[1 + \exp\left(\frac{\ell_0 - \ell}{a}\right) \right]^{-1} \quad (10)$$

and

$$\delta_\ell = \frac{\mu}{1 + \exp\left(\frac{\ell - \ell_0}{a}\right)} \quad (11)$$

As can be seen from these two formulae the grazing angular momentum ℓ_0 and its related width a can be, in general, different; therefore one may be dealing with either three or five parameters for the S-matrix. Moreover, the grazing angular momentum ℓ_0 and the corresponding width a are related to the reduced radius r_0 and diffusivity d through the following semi-classical relationship:

$$\ell_0 + \frac{1}{2} = kR_0 \quad (12)$$

and

$$a = kd \quad (13)$$

with

$$R_0 = r_0 \left(A_P^{1/3} + A_T^{1/3} \right) \quad (14)$$

where A_P and A_T are the mass numbers for the incident and target nuclei respectively.

In the above relations Coulomb effects are neglected; this is because we are dealing with a high energy scattering process and because we are mainly interested in the nuclear potential.

Adopting the Mc Intyre parametrization, writing the S-matrix in the form of eq.(9) and taking into account eq.(1), we obtain:-

$$i\chi(b) = \ln|S_\ell| + 2i\delta_\ell \quad (15)$$

from eqs.(10), (11) and (15), we have:-

$$i\chi_R(b) - \chi_I(b) = -\ln \left[1 + \exp\left(\frac{\ell_0 - \ell}{a}\right) \right] + \frac{2i\mu}{1 + \exp\left(\frac{(\ell - \ell_0)}{a}\right)} \quad (16)$$

Comparing the real and imaginary parts in eq.(16), we obtain:-

$$\chi_R(b) = \frac{2\mu}{1 + \exp\left(\frac{(\ell - \ell_0)}{a}\right)} \quad (17)$$

and

$$\chi_i(b) = \ln \left[1 + \exp \left(\frac{\ell_0 - \ell}{a} \right) \right] \quad (18)$$

Using eqs (12) and (13), we get :-

$$\chi_R(b) = \frac{2\mu}{1 + \exp \left(\frac{b - b_0}{d} \right)} \quad (19)$$

and

$$\chi_i(b) = \ln \left[1 + \exp \left(\frac{b_0 - b}{d} \right) \right] \quad (20)$$

where b is the impact parameter normally given by $kb = \ell + \frac{1}{2}$ and $b_0 = R_0$,

Analytical calculations of the optical model potentials, given by eqs (7) and (8), with the above phases is practically very difficult, this is because the potentials cannot be well extracted at $r = 0$. Therefore, we shall for the moment approximate the phases $\chi_R(b)$ and $\chi_i(b)$ by the formulae :

$$\chi_R(b) = 2\mu \sum_{n=1}^N c_n \exp \left(\frac{-nb^2}{\alpha^2} \right) \quad (21)$$

and

$$\chi_i(b) = \sum_{n=1}^N b_n \exp \left(\frac{-nb^2}{\beta^2} \right) \quad (22)$$

These formulae can be shown to reproduce χ_R and χ_i to a high degree of accuracy with a restricted number of terms and allow us to calculate the potential analytically (10).

To show that this is the situation we present such a fit, for the nuclear density in the case of ^{40}Ca , in fig(1). We note that twelve terms were taken into account

Now, inserting eqs. (21) and (22) into eqs.(7) and (8) we get :

$$V(r) = \frac{2E}{\pi k} \frac{2\mu}{r} \sum_{n=1}^N c_n \frac{d}{dr} \left[\int_r^\infty \frac{\exp \left(\frac{-nb^2}{\alpha^2} \right) b db}{\sqrt{b^2 - r^2}} \right] \quad (23)$$

and

$$W(r) = \frac{2E}{\pi k} \frac{1}{r} \sum_{n=1}^N b_n \frac{d}{dr} \left[\int_r^\infty \frac{\exp \left(\frac{-nb^2}{\beta^2} \right) b db}{\sqrt{b^2 - r^2}} \right] \quad (24)$$

Thus, we have

$$V(r) = - \frac{4\mu E}{\pi k \alpha} \sum_{n=1}^N c_n \sqrt{\pi n} \exp \left(\frac{-nr^2}{\alpha^2} \right) \quad (25)$$

and

$$W(r) = - \frac{2E}{\pi k \beta} \sum_{n=1}^N b_n \sqrt{\pi n} \exp \left(\frac{-nr^2}{\beta^2} \right) \quad (26)$$

It may, now, be apparent that the use of eqs (25) and (26) greatly simplified the evaluation of our optical model potential and thus ended our inversion procedure. Yet,

in what follows and utilizing the above relations we develop a method which allows for

$V(r)$ and $W(r)$ to be written in the familiar Woods-Saxon forms

To see that, let us rewrite $V(r)$ and $W(r)$ in the forms :

$$V(r) = -\left(\frac{4\mu E}{\pi k \alpha}\right) f(R, \Delta, r) \quad (27)$$

and

$$W(r) = -\left(\frac{2E}{\pi k \beta}\right) f(R, \Delta, r) \quad (28)$$

where

$$f(x_0, a_0, x) = \frac{1}{1 + \exp\left(\frac{x - x_0}{a_0}\right)} \quad (29)$$

With the above parametrization and as is shown in appendix A, the following relations, between the parameters of the corresponding phase shifts, will hold :-

For the real part :

$$\frac{I_4(R, \Delta)}{I_2(R, \Delta)} = \frac{3}{2} \frac{I_3(b_0, d)}{I_1(b_0, d)} \quad (30)$$

and

$$\frac{I_6(R, \Delta)}{I_4(R, \Delta)} = \frac{5}{4} \frac{I_5(b_0, d)}{I_3(b_0, d)} \quad (31)$$

For the imaginary part :

$$\frac{I_4(R, \Delta)}{I_2(R, \Delta)} = \frac{3}{4} \frac{I_4(b_0, d)}{I_2(b_0, d)} \quad (32)$$

and

$$\frac{I_6(R, \Delta)}{I_4(R, \Delta)} = \frac{5}{6} \frac{I_6(b_0, d)}{I_4(b_0, d)} \quad (33)$$

where $I_i(x_0, a_0)$ is an integral given by :-

$$I_i(x_0, a_0) = \int_0^{\infty} \frac{x^i dx}{1 + \exp\left(\frac{x - x_0}{a_0}\right)} \quad (34)$$

and which can be evaluated analytically ¹¹⁾ (see appendix A).

Now, in a real application to any system we can evaluate the phase shift parameters (b 's and d 's) from a fitting procedure and hence the right-hand side of eqs (30)-(33).

Then, in each case we get two non-linear simultaneous equations which we can solve by using a certain iteration procedure such as Newton-Raphson method

Once we get the R 's and Δ 's, then there remains only the task of evaluating the parameters α and β in eqs (27) and (28). For that purpose we insert eqs (27) and (28) into eqs (5) and (6) to get :-

$$\chi_R(b) = \frac{4\mu}{\pi \alpha} \int_0^{\infty} \frac{r dr}{\sqrt{r^2 - b^2} \left(1 + \exp\left(\frac{r - R}{\Delta}\right) \right)} \quad (35)$$

and

$$\chi_i(b) = \frac{2}{\pi\beta} \int_b^{\infty} \frac{rdr}{\sqrt{r^2 - b^2} \left(1 + \exp\left(\frac{r-R}{\Delta}\right) \right)} \quad (36)$$

Taking into account eqs (19) and (20) and putting $b=0$ in the resulting expressions, we find that :-

$$\frac{1}{1 + \exp(-b_0/d)} = \frac{2}{\pi\alpha} I_0(R', \Delta') \quad (37)$$

and

$$\ln[1 + \exp(b_0/d)] = \frac{2}{\pi\beta} I_0(R, \Delta) \quad (38)$$

Accordingly we have :-

$$\alpha = \frac{2}{\pi} [1 + \exp(-b_0/d)] I_0'(R', \Delta') \quad (39)$$

and

$$\beta = \frac{2}{\pi} \frac{I_0(R, \Delta)}{\ln[1 + \exp(b_0/d)]} \quad (40)$$

where $I_0'(R', \Delta')$ and $I_0(R, \Delta)$ are given by :-

$$I_0'(R', \Delta') = R' + \Delta' \ln[1 + \exp(-R'/\Delta')] \quad (41)$$

and

$$I_0(R, \Delta) = R + \Delta \ln[1 + \exp(-R/\Delta)] \quad (42)$$

Now, eqs (27) and (28) can be rewritten in form :

$$V(r) = \frac{-V_0}{1 + \exp\left(\frac{r-R}{\Delta}\right)} \quad (43)$$

with

$$V_0 = \left(\frac{4\mu E}{\pi k \alpha} \right) \quad (44)$$

and

$$W(r) = \frac{-W_0}{1 + \exp\left(\frac{r-R}{\Delta}\right)} \quad (45)$$

with

$$W_0 = \left(\frac{2E}{\pi k \beta} \right) \quad (46)$$

where V_0 and W_0 represent the depths of the optical model potential

Thus, we have succeeded, through our inversion procedure, in obtaining Woods-Saxon's type optical model potentials with parameters that can be determined directly from the Mc Intyre parametrization of the phase shift.

4. Application to $^{12}\text{C}-^{12}\text{C}$ System at $E_{\text{lab}}=1016 \text{ Mev}$

We shall apply, here, the preceding formalism to the symmetric system $^{12}\text{C}-^{12}\text{C}$ and as it can be seen, the present method yields a five-parameters fit to any data. These parameters are μ, R, R', Δ' and Δ , and for $^{12}\text{C}-^{12}\text{C}$ system these are obtained on the basis of the phase-shift analysis carried out by Mermaz ⁴⁾. In doing that Coulomb effects on the parameters are to be subtracted.

The reduced radius r_0 and diffusivity d are related to the grazing angular momentum and angular momentum width through the following semi-classical relationship ⁴⁾

$$\ell_0 = kR_0 \left(1 - \frac{2\eta}{kR_0} \right)^2 \quad (47)$$

and

$$a = kd \left(1 - \frac{\eta}{kR_0} \right) \left(1 - \frac{2\eta}{kR_0} \right)^2 \quad (48)$$

Using the data from Mermaz ⁴⁾ for ℓ_0, ℓ'_0, a and a' and calculating the other related parameters (such as $\eta, k, \dots$ etc.), we get the phase-shift parameters as :-
 $b_0 = 4.98187 \text{ fm}$, $b'_0 = 3.07595 \text{ fm}$, $d = 0.59826 \text{ fm}$ and $d' = 0.92254 \text{ fm}$

Note that Coulomb effects have been subtracted off in evaluating the parameters cited above.

Now with these parameters at hand and with reference to appendix A, we can evaluate $I_i(b'_0, d')$ ($i=1,3,5$) and $I_i(b_0, d)$ ($i=2,4,6$) and use them with eqs (30)-(33) to write down two sets of non-linear simultaneous equation for $\{R', \Delta'\}$ and $\{R, \Delta\}$ respectively.

Using an iteration procedure such as the modified Newton-raphson method we can solve for R, Δ, R' and Δ' . Their values are :-

$$R = 3.811417, \Delta = 0.6671, R' = 3.3558 \text{ and } \Delta' = 0.87$$

Moreover we get from eqs (39) and (40) that $\alpha = 2.22451$ and $\beta = 0.29175$.

Accordingly we obtain :-

$$V_0 = 112.276 \text{ Mev and } W_0 = 91.745 \text{ Mev.}$$

V and W can be calculated now and the results are given in Figs (2) and (3) with a comparison made with the best fit Woods-Saxon geometry adopted for the same system ^{6,7)}.

From these figure we clearly see that our procedure is in good agreement with the best fit Woods-Saxon potential as far as the real part is concerned, while it gives a much deeper potential for the imaginary part.

On the other hand, if we parametrize the absorption coefficient by $\eta(b) \equiv |S_r|^2$ rather than $|S_r|$ as is taken by many authors ⁵⁾.

Then we have

$$i\chi_R - \chi_I(b) = -\frac{1}{2} \ln \left[1 + \exp \left(\frac{\ell_0 - \ell}{a} \right) \right] + \frac{2i\mu}{1 + \exp \left(\frac{(\ell - \ell'_0)}{a'} \right)} \quad (49)$$

So that, we have

$$\chi_R = \frac{2m}{1 + \exp \left(\frac{(\ell - \ell'_0)}{a'} \right)} \quad (50)$$

and

$$\chi_I(b) = \frac{1}{2} \ln \left[1 + \exp \left(\frac{\ell_0 - \ell}{a} \right) \right] \quad (51)$$

Thus, the only change we obtain, due to this new parametrization, is that, eq.(20) is multiplied by a factor of a half so that all related equations must be multiplied by the same factor. The final expected result is that the depth of the imaginary potential W_0

will be reduced by a factor of a half and hence becomes much closer to the best fit potential. This is illustrated in fig(4) and where the curves have the same meanings as before.

5. Concluding Discussion

We have seen, through our inversion procedure, that we were able to relate the Mc Intyre parametrization of the S-matrix to a Woods-Saxon type optical model potential

The assumption that the ions have straight-line trajectories through the scattering processes made it possible to use the semi-classical approach and thus to make a correspondence between the high-energy approximation and the partial wave expansion for the scattering amplitude.

It is to be emphasized that Coulomb effects contribute negligibly ($\pm 1\%$) to the various parameters of the Mc Intyre parametrization in our case of the $^{12}\text{C}-^{12}\text{C}$ scattering.

As to the result obtained for the depth of the imaginary part of the optical potential, it can be understood on the lights of the parametrization procedure we have followed. The multiplication factor present in W_0 can be reduced if we parametrize the absorption coefficient as $\eta(b) \equiv |S_r|^2$ rather than $|S_r|$ given by eq (1), this was shown to be the case and the depth of the imaginary potential got reduced by a factor of a half and became much closer to the best fit potential.

We should point out, also, that many relations between the I_V 's other than those obtained in eqs (30)-(33) are satisfied with our values of the potential parameters, for instance another set of equations between the I_V 's may be :-

$$\frac{I_8(R, \Delta)}{I_6(R, \Delta)} = \frac{7}{6} \frac{I_7(b_0, d)}{I_5(b_0, d)} \quad (52)$$

and

$$\frac{I_8(R, \Delta)}{I_6(R, \Delta)} = \frac{7}{8} \frac{I_8(b_0, d)}{I_6(b_0, d)} \quad (53)$$

These two equations are exactly satisfied by the same values obtained previously for the optical potential parameters; more details are given in appendix A

Finally, it is to be noted that in principle, coulomb effects can be incorporated in the calculations; but the case will then be much more involved ¹².

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APPENDIX A

In this appendix we given detailed calculations of the parameters R , Δ , R' and Δ' in terms of the impact parameters (b_0 and b'_0) and diffusivities (d and d')

Comparing eqs (19) and (20) with eqs (21) and (22), we have -

$$\frac{1}{1 + \exp\left(\frac{b - b'_0}{d}\right)} = \sum_n c_n \exp\left(\frac{-nb^2}{\alpha^2}\right) \quad (A.1)$$

and

$$\ln\left[1 + \exp\left(\frac{b_0 - b}{d}\right)\right] = \sum_n b_n \exp\left(\frac{-nb^2}{\beta^2}\right) \quad (A.2)$$

Now, introducing the integral :-

$$I_v(x_0, a_0) = \int_0^{\infty} \frac{x^v dx}{1 + \exp\left(\frac{x - x_0}{a_0}\right)} \quad (A \ 3)$$

and

and using eqs.(A .1) and (A .2) we get .

$$I_v(b_0, d') = \sum_n c_n \int_0^{\infty} b^v \exp\left(\frac{-nb^2}{\alpha^2}\right) db \quad (A \ 4)$$

and

$$I_v(b_0, d) = \frac{2d}{\beta^2} \sum_n b_n n \int_0^{\infty} b^{v-1} \exp\left(\frac{-nb^2}{\alpha^2}\right) db \quad (A \ 5)$$

Accordingly we have:-

For $I_v(b_0, d')$:-

$$I_1(b_0, d') = \frac{\alpha^2}{2} \sum_n c_n / n \ , \quad (A \ 6)$$

$$I_3(b_0, d) = \frac{\alpha^4}{2} \sum_n c_n / n^2 \ , \quad (A \ 7)$$

$$I_5(b_0, d') = \alpha^6 \sum_n c_n / n^3 \ , \quad (A \ 8)$$

$$I_7(b_0, d') = 3\alpha^8 \sum_n c_n / n^4 \ , \quad (A \ 9)$$

$$I_9(b_0, d') = 12\alpha^{10} \sum_n c_n / n^5 \quad (A \ 10)$$

$$I_{11}(b_0, d') = 60\alpha^{12} \sum_n c_n / n^6 \quad (A \ 11)$$

Also, for $I_v(b_0, d)$, we find :-

$$I_2(b_0, d) = d \beta^2 \sum_n b_n / n \ , \quad (A \ 12)$$

$$I_4(b_0, d) = 2d \beta^4 \sum_n b_n / n^2 \ , \quad (A \ 13)$$

$$I_6(b_0, d) = 6d \beta^6 \sum_n b_n / n^3 \ , \quad (A \ 14)$$

$$I_8(b_0, d) = 24d \beta^8 \sum_n b_n / n^4 \ , \quad (A \ 15)$$

$$I_{10}(b_0, d) = 120d \beta^{10} \sum_n b_n / n^5 \quad (A \ 16)$$

and

$$I_{12}(b_0, d) = 720d \beta^{12} \sum_n b_n / n^6 \quad (\text{A } 17)$$

Thus, from Eqs(A . 6)-(A . 11), we obtain : -

$$\frac{I_3(b_0, d)}{I_1(b_0, d)} = \alpha^2 \frac{\sum_n c_n / n^2}{\sum_n c_n / n} \quad , \quad (\text{A } 18)$$

$$\frac{I_5(b_0, d)}{I_3(b_0, d)} = 2\alpha^2 \frac{\sum_n c_n / n^3}{\sum_n c_n / n^2} \quad , \quad (\text{A } 19)$$

$$\frac{I_7(b_0, d)}{I_5(b_0, d)} = 3\alpha^2 \frac{\sum_n c_n / n^4}{\sum_n c_n / n^3} \quad , \quad (\text{A } 20)$$

$$\frac{I_9(b_0, d)}{I_7(b_0, d)} = 4\alpha^2 \frac{\sum_n c_n / n^5}{\sum_n c_n / n^4} \quad (\text{A } 21)$$

and

$$\frac{I_{11}(b_0, d)}{I_9(b_0, d)} = 5\alpha^2 \frac{\sum_n c_n / n^6}{\sum_n c_n / n^5} \quad (\text{A } 22)$$

Similarly, from Eqs.(A . 12)-(A . 17), we get : -

$$\frac{I_4(b_0, d)}{I_2(b_0, d)} = 2\beta^2 \frac{\sum_n b_n / n^2}{\sum_n b_n / n} \quad , \quad (\text{A } 23)$$

$$\frac{I_6(b_0, d)}{I_4(b_0, d)} = 3\beta^2 \frac{\sum_n b_n / n^3}{\sum_n b_n / n^2} \quad , \quad (\text{A } 24)$$

$$\frac{I_8(b_0, d)}{I_6(b_0, d)} = 4\beta^2 \frac{\sum_n b_n / n^4}{\sum_n b_n / n^3} \quad , \quad (\text{A } 25)$$

$$\frac{I_{10}(b_0, d)}{I_8(b_0, d)} = 5\beta^2 \frac{\sum_n b_n / n^5}{\sum_n b_n / n^4} \quad , \quad (\text{A } 26)$$

and

$$\frac{I_{12}(b_0, d)}{I_{10}(b_0, d)} = 6\beta^2 \frac{\sum_n b_n / n^6}{\sum_n b_n / n^5} \quad (\text{A } 27)$$

On the other hand, from eqs (25)-(26) and eqs (27)-(28), we get -

$$\frac{1}{1 + \exp((r - R) / \Delta)} = \sum_n c_n \sqrt{n\pi} \exp\left(\frac{-nr^2}{\alpha^2}\right) \quad (\text{A } 28)$$

and

$$\frac{1}{1 + \exp((r - R) / \Delta)} = \sum_n b_n \sqrt{n\pi} \exp\left(\frac{-nr^2}{\beta^2}\right) \quad (\text{A } 29)$$

Again, from these relations, we obtain :-

$$\frac{I_4(R, \Delta)}{I_2(R, \Delta)} = \frac{3}{2} \alpha^2 \frac{\sum_n c_n / n^2}{\sum_n c_n / n} \quad , \quad (\text{A } 30)$$

$$\frac{I_6(R, \Delta)}{I_4(R, \Delta)} = \frac{5}{2} \alpha^2 \frac{\sum_n c_n / n^3}{\sum_n c_n / n^2} \quad , \quad (\text{A } 31)$$

$$\frac{I_8(R, \Delta)}{I_6(R, \Delta)} = \frac{7}{2} \alpha^2 \frac{\sum_n c_n / n^4}{\sum_n c_n / n^3} \quad , \quad (\text{A } 32)$$

$$\frac{I_{10}(R, \Delta)}{I_8(R, \Delta)} = \frac{9}{2} \alpha^2 \frac{\sum_n c_n / n^5}{\sum_n c_n / n^4} \quad (\text{A } 33)$$

and

$$\frac{I_{12}(R, \Delta)}{I_{10}(R, \Delta)} = \frac{11}{2} \alpha^2 \frac{\sum_n c_n / n^6}{\sum_n c_n / n^5} \quad (\text{A } 34)$$

We, also, get :-

$$\frac{I_4(R, \Delta)}{I_2(R, \Delta)} = \frac{3}{2} \beta^2 \frac{\sum_n b_n / n^2}{\sum_n b_n / n} \quad , \quad (\text{A } 35)$$

Comparing eqs.(A . 18)-(A . 22) with eqs (A . 30)-(A . 34),we find . -

$$\frac{I_6(R, \Delta)}{I_4(R, \Delta)} = \frac{5}{2} \beta^2 \frac{\sum_n b_n / n^3}{\sum_n b_n / n^2} \quad , \quad (A . 36)$$

$$\frac{I_4(R, \Delta)}{I_2(R, \Delta)} = \frac{3}{2} \frac{I_3(b_0, d)}{I_1(b_0, d)} \quad , \quad (A . 40)$$

$$\frac{I_8(R, \Delta)}{I_6(R, \Delta)} = \frac{7}{2} \beta^2 \frac{\sum_n b_n / n^4}{\sum_n b_n / n^3} \quad , \quad (A . 37)$$

$$\frac{I_6(R, \Delta)}{I_4(R, \Delta)} = \frac{5}{4} \frac{I_5(b_0, d)}{I_3(b_0, d)} \quad , \quad (A . 41)$$

$$\frac{I_{10}(R, \Delta)}{I_8(R, \Delta)} = \frac{9}{2} \beta^2 \frac{\sum_n b_n / n^5}{\sum_n b_n / n^4} \quad (A . 38)$$

$$\frac{I_8(R, \Delta)}{I_6(R, \Delta)} = \frac{7}{6} \frac{I_7(b_0, d)}{I_5(b_0, d)} \quad , \quad (A . 42)$$

and

$$\frac{I_{10}(R, \Delta)}{I_8(R, \Delta)} = \frac{9}{8} \frac{I_9(b_0, d)}{I_7(b_0, d)} \quad (A . 43)$$

and

$$\frac{I_{12}(R, \Delta)}{I_{10}(R, \Delta)} = \frac{11}{2} \beta^2 \frac{\sum_n b_n / n^6}{\sum_n b_n / n^5} \quad (A . 39)$$

$$\frac{I_{12}(R, \Delta)}{I_{10}(R, \Delta)} = \frac{11}{10} \frac{I_{11}(b_0, d)}{I_9(b_0, d)} \quad (A . 44)$$

Again, comparing eqs.(A . 23)-(A . 27) with eqs (A . 35)-(A . 39), we find :-

$$\frac{I_4(R, \Delta)}{I_2(R, \Delta)} = \frac{3}{4} \frac{I_4(b_0, d)}{I_2(b_0, d)} \quad , \quad (A . 45)$$

$$\frac{I_6(R, \Delta)}{I_4(R, \Delta)} = \frac{5}{6} \frac{I_6(b_0, d)}{I_4(b_0, d)} \quad , \quad (A . 46)$$

$$\frac{I_8(R, \Delta)}{I_6(R, \Delta)} = \frac{7}{8} \frac{I_8(b_0, d)}{I_6(b_0, d)} \quad , \quad (A . 47)$$

$$\frac{I_{10}(R, \Delta)}{I_8(R, \Delta)} = \frac{9}{10} \frac{I_{10}(b_0, d)}{I_8(b_0, d)} \quad (A . 48)$$

and

$$\frac{I_{12}(R, \Delta)}{I_{10}(R, \Delta)} = \frac{11}{12} \frac{I_{12}(b_0, d)}{I_{10}(b_0, d)} \quad (A . 49)$$

Further, from the analytic relation for $I_\nu(x_0 - a_0)$, we have ^[11] :-

$$I_\nu(x_0 - a_0) = \frac{x_0^{\nu+1}}{\nu+1} + \sum_{m=0}^{\infty} \left[1 - (-1)^m \right] \frac{\nu! x_0^{\nu-m}}{(v-m)!} a_0^{m+1} (1-2^{-m}) \xi(m+1) \\ - \sum_{k=1}^{\infty} (-1)^{\nu+k} \exp\left(-\frac{kx_0}{a_0}\right) \nu! \frac{a_0^{\nu+1}}{k^{\nu+1}} \quad (A . 50)$$

where $\xi(z)$ is the Rieman Zeta function.

Eq (A . 50) can be simplified somewhat ^[11] ; since the infinite sum in the last term can be neglected. For the case of $\nu = 2$, with typical values of $x_0 = 2.0608 \text{ fm}$ and $a_0 = 0.513 \text{ fm}$ for ¹⁶O , the last term, if neglected would only introduce an error in I_2 which amounts to $\sim 0.02\%$.

From eq (A . 50), we find that:-

$$I_0(x_0, a_0) = x_0 - \sum_{k=1}^{\infty} (-1)^k \exp\left(-\frac{kx_0}{a_0}\right) \frac{a_0}{k} \quad (A . 51)$$

or

$$I_0(x_0, a_0) = x_0 + a_0 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \left[\exp\left(-\frac{x_0}{a_0}\right) \right]^k \quad (A . 52)$$

but

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \quad (\text{A } 53)$$

Accordingly we get :-

$$I_0(x_0, a_0) = x_0 + a_0 \ln \left[1 + \exp \left(-\frac{x_0}{a_0} \right) \right] \quad (\text{A } 54)$$

which can be written in the form

$$I_0(x_0, a_0) = a_0 \ln \left[1 + \exp \left(\frac{x_0}{a_0} \right) \right] \quad (\text{A } 55)$$

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Figure Captions

- Figure 1 A plot of the nuclear density $\rho(r)$ for ^{40}Ca (solid curve) with Woods-Saxon parameters taken as $\rho_0 = 4 \times 10^{-3} \text{ fm}^{-3}$, $R = 1.07 \text{ fm}$ and $a=0.545 \text{ fm}$, the dashed curve corresponds to the approximate density with 12 terms taken from the expansion of the forms (21) and (22).
- Figure 2 The real part of the optical model potential for $^{12}\text{C}-^{12}\text{C}$ scattering at $E_{\text{lab}}=1016 \text{ Mev}$. The solid curve corresponds to the inversion solution; while the dashed and dotted curves refer to the experimental data from references 6 and 7 respectively.
- Figure 3 The imaginary part of the optical model potential for $^{12}\text{C}-^{12}\text{C}$ scattering at $E_{\text{lab}}=1016 \text{ Mev}$. The solid curve corresponds to the inversion solution; while the dashed and dotted curves refer to the experimental data from references 6 and 7 respectively.
- Figure 4 The imaginary part of the optical model potential for $^{12}\text{C}-^{12}\text{C}$ scattering at $E_{\text{lab}}=1016 \text{ Mev}$. The solid curve corresponds to the inversion solution; (second parametrization); while the dashed and dotted curves refer to the experimental data from references 6 and 7 respectively.

Fig.1

Fig.2

Fig.3

Fig.4