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**THE BURGERS EQUATION WITH A
NON-GAUSSIAN RANDOM FORCE**

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1 Introduction

In recent years there has been a growing interest for Burgers equation with a randomly disturbed external force. Yakhot and She [YS], Orłowski and Sobczyk [OS], Bertini et. al. [BCJ-L] and Dermoune [D2] have studied the Burgers equation driven additively by a noisy force. An analysis of a singular Gaussian noise as an additive force has been presented by Holden et. al. [HLØUZ1+2]. On the other hand, Konno [K], de Lillo [L] and Benth et. al. [BDPS] have studied Burgers equation with a multiplicative noise.

It is of interest to study Burgers equation driven additively by noisy forces of non-Gaussian type, for instance Poisson noises. The equation under consideration here, is

$$u_t + \lambda u \cdot u_x = \nu u_{xx} + F(t, x, \omega)$$

The force F will be a general time-space noise.

In Holden et. al. [HLØUZ1+2], the above equation was interpreted in the Wick sense in order to cope with singular noises. We will in this paper use their version of Burgers equation with additive noise. I. e., our problem takes the form

$$\Phi_t + \lambda \Phi \diamond \Phi_x = \nu \Phi_{xx} + F(t, x, \omega) \quad (1)$$

with the given initial condition

$$\Phi(0, x) = \Phi_0(x) \quad (2)$$

In this equation, λ, ν are constants, with $\lambda, \nu > 0$. $F(t, x)$ is a singular random force, which, in our setting, can be of non-Gaussian type. Especially, we can with

our analysis treat Poisson forces. The initial data is allowed to be a generalized stochastic process. The equation is considered for $(t, x) \in (0, T] \times \mathbb{R}$. (Our assumptions will be made precise in section 3.)

As already mentioned, the problem (1) and (2) in the Gaussian setting has been considered by Holden et. al. in [HLØUZ1+2]. I.e. they work with the probability space $(\mathcal{S}', \mathcal{B}, \mu)$ where μ is a Gaussian measure. In their paper, the problem of solving (1) and (2) was transformed into a deterministic problem by the Hermite-Transform. This Hermite-transformed Burgers equation was translated by a Cole-Hopf transform into a problem of solving a heat equation with potential. For general potentials in the Gaussian distribution space $(\mathcal{S})^{-1}$ (see [ADKS], [KLS]), they solve the heat problem explicitly with a Feynman-Kac formula. Invoking the Wick Calculus from the article by Kondratiev et. al. [KLS], they produce a solution to the original Burgers equation represented by a Wick version of the Cole-Hopf transform. For a related study including the case of Poisson noise, see Dermoune [D1+3].

We are going to use the same scheme as Holden et. al. [HLØUZ1+2] for solving (1). However, in order to handle non-Gaussian forces, we will invoke the non-Gaussian infinite dimensional analysis constructed in Albeverio et. al. [ADKS] and generalized in Kondratiev et. al. [KSW]. The particular case of an infinite dimensional analysis based on Poisson noise is treated by Privault [Pr1+2] and Us [U]. In the non-Gaussian framework we must use the $\mathcal{S}_{\mathbf{p}}$ -Transform instead of the Hermite-Transform. For the corresponding deterministic heat problem, we represent the solution through a Dyson series, which is practical for showing necessary continuity and analyticity properties.

In section 2 we give the necessary background for the non-Gaussian infinite dimensional analysis found in Albeverio et. al. [ADKS]. The conditions and interpretation of problem (1) and (2) are stated in section 3, where we also show the connection to a heat equation with potential. We study the Dyson series representation of the solution of the heat problem in section 4, and show that the series and its derivatives have the correct continuity and analyticity properties. In section 5 we conclude the solution of (1) and (2) based on the results from section 4. Finally, in section 6 we look at several examples of forces, with emphasis on the Poisson case.

2 Some Mathematical Preliminaries

Our presentation of parts of the non-Gaussian White Noise Analysis is based upon the paper of Albeverio et. al. [ADKS]. For a complete account with proofs of the below results, the reader should confer [ADKS] and the references therein. A generalization of this framework including the Poisson case, can be found in Kondratiev et. al. [KSW]. We will, however, base our consideration of the Poisson case in section 6 on the paper by Us [U].

Let $\mathcal{H}$ be a real separable Hilbert space, and define $\mathcal{N}$ to be a separable

nuclear space densely topologically embedded in $\mathcal{H}$. $\mathcal{N}$ will be the projective limit of Hilbert spaces $\mathcal{H}_p, p \in \mathbb{N}_0$. Denote the dual of $\mathcal{N}$, $\mathcal{N}'$, which is the inductive limit of $\mathcal{N}'_p$. $\mathcal{N}'_p$ is here the dual of $\mathcal{N}_p$. Let μ be a Borel probability measure on $\mathcal{N}'$. We assume that its characteristic functional

$$\mathcal{X}(\phi) = \int_{\mathcal{N}'} e^{i\langle \phi, \omega \rangle} d\mu(\omega)$$

is continuous in $\phi = 0$ in the topology of $\mathcal{N}'_p$ for some $p \in \mathbb{N}$. In this paper we will consider so-called *smooth double analytic measures*. We define this class of measures:

A function

$$f : \mathcal{N}' \rightarrow Y$$

where Y is a topological vector space, is called $\mathcal{N}$ -differentiable if there exists a linear continuous mapping

$$f'(\omega) : \mathcal{N} \rightarrow Y$$

such that

$$\nabla_\phi f(\omega) = f'(\omega)\phi$$

The differential operator ∇_ϕ is defined as

$$\nabla_\phi f(\omega) = \frac{d}{dh} f(\omega + h\phi)|_{h=0}$$

We say that μ admits an integration by parts along $\mathcal{N}$ if for $\phi \in \mathcal{N}$ there exists a $\beta(\phi; \cdot) \in L^1(\mathcal{N}', \mu)$ such that for all measurable and bounded $\mathcal{N}$ -differentiable functions $f : \mathcal{N}' \rightarrow \mathbb{R}$

$$\int_{\mathcal{N}'} \nabla_\phi f(\omega) d\mu(\omega) = - \int_{\mathcal{N}'} f(\omega) \beta(\phi; \omega) d\mu(\omega)$$

Finally, the measure μ is called *smooth* when there exists a measurable infinitely $\mathcal{N}$ -differentiable mapping $\beta : \mathcal{N}' \rightarrow \mathcal{N}'$ such that

$$\beta(\phi; \omega) = \langle \phi, \beta(\omega) \rangle$$

The function β is known as the logarithmic derivative of the measure μ .

By the continuity assumption on the characteristic functional of μ , we know from the Bochner-Minlos-Sazonov theorem that the measure μ is concentrated on $\mathcal{N}'_q$ for a $q \in \mathbb{N}$. We assume the following integrability condition on μ : There exists an $\epsilon > 0$ such that

$$\int_{\mathcal{N}'} e^{\epsilon|\omega|^{-q}} d\mu(\omega) < \infty \quad (3)$$

This condition is called *the first condition of analyticity*.

∇_ϕ acts on the variable θ . By the first condition of analyticity, (3), together with the kernel theorem, it is shown in [ADKS] that there exists a polynomial mapping

$$P_n : \mathcal{N}' \rightarrow \mathcal{F}_n(\mathcal{N}')$$

such that

$$P_n(\phi^{\otimes n}; \omega) = \langle \phi^{\otimes n}, P_n(\omega) \rangle$$

This can be extended such that for $\phi^{(n)} \in \mathcal{F}_n(\mathcal{V})$ and $\omega \in \mathcal{N}'$, we have

$$P(\phi^{(n)}; \omega) = \langle \phi^{(n)}, P_n(\omega) \rangle$$

Our **P**-system is :

$$\mathbf{P} = \left\{ P(\phi^{(n)}; \omega) = \langle \phi^{(n)}, P_n(\omega) \rangle : \phi^{(n)} \in \mathcal{F}_n(\mathcal{N}), n \in \mathbb{N}_0, \omega \in \mathcal{N}' \right\} \quad (5)$$

This set is total in $L^2(\mathcal{N}', \mu)$. As already mentioned in the introduction, these polynomials have many similar properties to the Hermite polynomials. The reader is referred to Alberverio et. al. [ADKS] for more details. We continue with the construction of the biorthogonal system of the **P**-system:

A direct calculation shows that the operator ∇_ϕ on $L^2(\mathcal{N}', \mu)$ has an adjoint ∇_ϕ^* represented by

$$\nabla_\phi^* = -\nabla_\phi - \beta(\phi; \omega)$$

Define the system of functions

$$Q(\phi_1, \dots, \phi_n; \omega) = (\nabla_{\phi_1}^* \cdots \nabla_{\phi_n}^* \mathbf{1})(\omega)$$

where $\phi_1, \dots, \phi_n \in \mathcal{N}$ and $\omega \in \mathcal{N}'$. The second condition of analyticity, (4), and the kernel theorem (see [ADKS]) give the existence of a family of functions

$$Q_n : \mathcal{N}' \rightarrow \mathcal{F}_n(\mathcal{N}')$$

such that

$$Q(\phi^{\otimes n}; \omega) = \langle \phi^{\otimes n}, Q_n(\omega) \rangle$$

For $\phi^{(n)} \in \mathcal{F}_n(\mathcal{N})$ we can extend this to

$$Q(\phi^{(n)}; \omega) = \langle \phi^{(n)}, Q_n(\omega) \rangle$$

The **Q**-system is

$$\mathbf{Q} = \left\{ Q(\phi^{(n)}; \omega) = \langle \phi^{(n)}, Q_n(\omega) \rangle : \phi^{(n)} \in \mathcal{F}_n(\mathcal{N}), n \in \mathbb{N}_0, \omega \in \mathcal{N}' \right\} \quad (6)$$

The biorthogonality relation between the **P**- and the **Q**-system is the following: For $\phi^{(n)} \in \mathcal{F}_n(\mathcal{N})$ and $\psi^{(m)} \in \mathcal{F}_m(\mathcal{N})$:

$$\int_{\mathcal{N}'} P(\phi^{(n)}; \omega) Q(\psi^{(m)}; \omega) d\mu(\omega) = \delta_{nm} n! \langle \phi^{(n)}, \psi^{(m)} \rangle$$

We introduce the space of smooth random variables, also called the Kondratiev test function space: Let $\mathcal{P}(\mathcal{N}')$ be the space of polynomials, i.e. the elements in $L^2(\mathcal{N}', \mu)$ on the form

$$\phi(x) = \sum_{n=1}^{\infty} \langle \phi^{(n)}, P_n(\omega) \rangle$$

$\phi^{(n)} \in \mathcal{F}_n(\mathcal{N})$. For $p, q \in \mathbb{N}$, define $(\mathcal{N})_{p,q}^1$ to be the completion of $\mathcal{P}(\mathcal{N}')$ in the norm

$$\|\phi\|_{p,q,1}^2 = \sum_{n=1}^{\infty} (n!)^2 2^{nq} |\phi^{(n)}|_p^2$$

where $|\phi^{(n)}|_p$ denotes the norm in $\mathcal{N}_{p,q}^{\otimes n}$. Our space of test functions, denoted $(\mathcal{N})^1$, is defined as the projective limit of $(\mathcal{N})_{p,q}^1$. This space is a nuclear Fréchet space which is densely topologically embedded into $L^2(\mathcal{N}', \mu)$.

The space of distributions is defined as the inductive limit of $(\mathcal{N})_{-p,-q}^{-1}$, where $(\mathcal{N})_{-p,-q}^{-1}$ is the dual of $(\mathcal{N})_{p,q}^1$. This space will be denoted $(\mathcal{N})^{-1}$ and is called the *space of Kondratiev distributions*. We have the inclusions

$$(\mathcal{N})^1 \subset L^2(\mathcal{N}', \mu) \subset (\mathcal{N})^{-1}$$

The space of distributions can be characterized as follows:

Proposition 1 (prop. 5. [ADKS]) $\Phi \in (\mathcal{N})^{-1}$ if and only if it has the unique decomposition

$$\Phi = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, Q_n \rangle$$

where $\Phi^{(n)} \in \mathcal{F}_n(\mathcal{N}')$, such that for a pair $p, q \in \mathbb{N}_0$

$$\|\Phi\|_{-p,-q,-1}^2 = \sum_{n=0}^{\infty} 2^{-nq} |\Phi^{(n)}|_{-p}^2 < \infty$$

The action of $\Phi \in (\mathcal{N})^{-1}$ on a test function $\phi = \sum_{n=0}^{\infty} \langle \phi^{(n)}, P_n \rangle \in (\mathcal{N})^1$ is given by

$$\langle \langle \Phi, \phi \rangle \rangle = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \phi^{(n)} \rangle$$

In Gaussian White Noise Analysis, the $\mathcal{S}$ -transform plays an important role. It characterizes the space of Hida distributions, see Potthoff and Streit [PS], Hida et. al. [HKPS] and Kondratiev et. al. [KLPSW]. In non-Gaussian Analysis we have an analogue transform which provides us with a characterization

of our distribution space $(\mathcal{N})^{-1}$. If $\Phi \in (\mathcal{N})^{-1}$, we know that $\Phi \in (\mathcal{N})_{-p,-q}^{-1}$ for a pair $p, q \in \mathbb{N}_0$. Define the following set: For $\delta \in (0, 1)$

$$\mathcal{U}_{p,q}^\delta(0) = \{\xi \in \mathcal{N}_{\mathbb{C}} : 2^q |\xi|_p^2 < \delta\}$$

$\mathcal{U}_{p,q}^\delta(0)$ is a neighborhood of zero in $\mathcal{N}_{\mathbb{C}}$. By direct calculation of the norm, one can show that the μ -exponential $e_\mu(\xi; \cdot) \in (\mathcal{N})_{p,q}^1$ for $\xi \in \mathcal{U}_{p,q}^\delta(0)$. Hence, we can define the $S_{\mathbf{p}}$ -transform of Φ as follows: For each $\xi \in \mathcal{U}_{p,q}^\delta(0)$,

$$S_{\mathbf{p}}\Phi(\xi) = \langle \langle \Phi, e_\mu(\xi; \cdot) \rangle \rangle \quad (7)$$

This transform will play an important role in the consideration of equation (1) and (2).

In order to characterize the Kondratiev distributions, we need the following space of functions: Let $\mathcal{U}$ be an open neighborhood of zero in $\mathcal{N}_{\mathbb{C}}$. We say that

$$F : \mathcal{U} \rightarrow \mathbb{C}$$

belongs to $\mathcal{Hol}_0(\mathcal{N}_{\mathbb{C}})$ if it satisfies the following two properties:

- 1:** For each $\xi \in \mathcal{U}$, $\eta \in \mathcal{N}_{\mathbb{C}}$, there exists a neighborhood $V_{\xi,\eta}$ around zero in $\mathbb{C}$ such that

$$z \rightarrow F(\xi + z\eta)$$

is holomorphic.

- 2:** For each $\xi \in \mathcal{U}$ there exists an open set $\mathcal{V} \subset \mathcal{U}$ such that $F(\mathcal{V})$ is bounded.

We identify two functions F_1 and F_2 if they coincide on a neighborhood of zero. $\mathcal{Hol}_0(\mathcal{N}_{\mathbb{C}})$ is then the space of germs of functions with the above two properties.

In Albeverio et. al. [ADKS], we find the following characterization theorem:

Theorem 2 (Th. 5, [ADKS]) *If $\Phi \in (\mathcal{N})^{-1}$ then $S_{\mathbf{p}}\Phi \in \mathcal{Hol}_0(\mathcal{N}_{\mathbb{C}})$. Conversely, if $F \in \mathcal{Hol}_0(\mathcal{N}_{\mathbb{C}})$, there is a uniquely defined distribution $\Phi \in (\mathcal{N})^{-1}$ such that $S_{\mathbf{p}}\Phi = F$ on some neighborhood of zero in $\mathcal{N}_{\mathbb{C}}$.*

Since $\mathcal{Hol}_0(\mathcal{N}_{\mathbb{C}})$ is an algebra under the product, we can define the Wick product on $(\mathcal{N})^{-1}$: If $\Phi, \Psi \in (\mathcal{N})^{-1}$, then $\Phi \diamond \Psi$ is the unique $(\mathcal{N})^{-1}$ -element defined by the relation

$$S_{\mathbf{p}}(\Phi \diamond \Psi)(\xi) = S_{\mathbf{p}}\Phi(\xi) \cdot S_{\mathbf{p}}\Psi(\xi)$$

In the space $(\mathcal{N})^{-1}$, we can also talk about Wick functions, i.e. the composition of functions in the Wick sense with distributions. The following theorem is known as the Wick Calculus Theorem, and can be found in Kondratiev et. al. [KLS] in the Gaussian setting:

Theorem 3 (Th 12, [KLS]) Let $\Phi \in (\mathcal{N})^{-1}$, and assume $f : \mathbb{C} \rightarrow \mathbb{C}$ analytic in a neighborhood of the point $z_0 = \langle \Phi, \mathbf{1} \rangle \in \mathbb{C}$. Then $f^\circ(\Phi)$ defined as

$$f^\circ(\Phi) = \mathcal{S}_{\mathbf{p}}^{-1}(f(\mathcal{S}_{\mathbf{p}}\Phi))$$

is an element of $(\mathcal{N})^{-1}$.

Proof: See the proof for the Gaussian case in [KLS]. ■

Remark: We remark that the pairing

$$\langle \Phi, \mathbf{1} \rangle$$

generalizes expectation to elements in $(\mathcal{N})^{-1}$. If $\Phi \in L^2(\mathcal{N}', \mu)$, we have

$$E[\Phi] = \langle \Phi, \mathbf{1} \rangle$$

Note that

$$\mathcal{S}_{\mathbf{p}}\Phi(0) = \langle \Phi, \mathbf{1} \rangle$$

We shall need the notion of differentiation in the *weak* sense. The following notation and definitions are taken from Potthoff, [P]: Let $\mathcal{O}$ be an open set in $\mathbb{R}^2$. Consider the mapping

$$\Psi(\cdot) : \mathcal{O} \rightarrow (\mathcal{N})^{-1}$$

We say that Ψ is *weakly* differentiable in $x_0 \in \mathcal{O}$ in direction $k, k = 1, 2$, if

$$\frac{\Psi(x_0 + he_k) - \Psi(x_0)}{h}$$

exists as a weak limit in $(\mathcal{N})^{-1}$. The limit is denoted $D_k\Psi(x_0)$. We have set e_k as the unit vector in $\mathbb{R}^2$. We characterize this notion of differentiability through the $\mathcal{S}_{\mathbf{p}}$ -transform: Define the function space $C_{ib}^n(\mathcal{O}; \mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}}))$ to be the space of mappings

$$F(\cdot) : \mathcal{O} \rightarrow \mathcal{H}ol_0(\mathcal{N}C_{\mathcal{C}})$$

such that

1: For each $\xi \in \mathcal{U}$

$$x \rightarrow F(x)(\xi) \in C^n(\mathcal{O})$$

2: For each $x \in \mathcal{O}$ there exists a neighborhood of x , $\mathcal{O}_x$, such that

$$|D^\alpha F(y)(\xi)| \leq K$$

for $y \in \mathcal{O}_x, \xi \in \mathcal{U}$ and $|\alpha| \leq n$.

The index *lb* stands for *locally bounded*. In Potthoff [P] we find the following characterization lemma:

Lemma 4 *If $F \in C_{lb}^n(\mathcal{O}; \mathcal{Hol}_0(\mathcal{N}_{\mathcal{C}}))$ then $S_{\mathbf{p}}^{-1}F$ is n -times weakly continuously differentiable, i. e. $D^\alpha(S_{\mathbf{p}}^{-1}F)$ exists and is continuous for $|\alpha| \leq n$. Moreover*

$$D^\alpha(S_{\mathbf{p}}^{-1}F) = S_{\mathbf{p}}^{-1}(D^\alpha F)$$

3 Connection to the heat equation

By a solution of our problem, we shall understand a process $\Phi(t, x) \in (\mathcal{N})^{-1}$ which satisfies (1) and (2) in the weak sense. I. e., Φ_t , Φ_x and Φ_{xx} are understood as the weak partial derivatives of Φ in the sense defined in section 2.

We make the following assumptions for problem (1) and (2):

- (i): $\Phi_0(x) \in (\mathcal{N})^{-1}$ for all $x \in \mathbb{R}$.
- (ii): $\frac{\partial}{\partial x} S_{\mathbf{p}} \Phi_0(x)(\xi)$, $S_{\mathbf{p}} \Phi_0(x)(\xi)$ and $\int_{-\infty}^x S_{\mathbf{p}} \Phi_0(y)(\xi) dy$ are elements of $C_{lb}([0, T] \times \mathbb{R}; \mathcal{Hol}_0(\mathcal{N}_{\mathcal{C}}))$.
- (iii): Moreover, there exist neighborhoods $\mathcal{U}_i, i = 1, 2, 3$ around zero in $\mathcal{N}_{\mathcal{C}}$ such that:

$$\begin{aligned} \sup_{x \in \mathbb{R}, \xi \in \mathcal{U}_1} \left| \frac{\partial}{\partial x} S_{\mathbf{p}} \Phi_0(x)(\xi) \right| &< \infty \\ \sup_{x \in \mathbb{R}, \xi \in \mathcal{U}_2} |S_{\mathbf{p}} \Phi_0(x)(\xi)| &< \infty \\ \sup_{\xi \in \mathcal{U}_3} \int_{\mathbb{R}} |S_{\mathbf{p}} \Phi_0(y)(\xi)| dy &< \infty \end{aligned}$$

- (iv): $F(t, x) \in (\mathcal{N})^{-1}$ for all $(t, x) \in [0, T] \times \mathbb{R}$. In addition, there exists an element $P(t, x) \in (\mathcal{N})^{-1}$ for all $(t, x) \in [0, T] \times \mathbb{R}$ where $\frac{\partial}{\partial x} P(t, x) = F(t, x)$. (Differentiation is here understood in the weak sense).

- (v): $\frac{\partial}{\partial x} S_{\mathbf{p}} F(t, x)(\xi)$, $S_{\mathbf{p}} F$ and $S_{\mathbf{p}} P$ are elements of $C_{lb}([0, T] \times \mathbb{R}; \mathcal{Hol}_0(\mathcal{N}_{\mathcal{C}}))$.

- (vi): Moreover, there exist neighborhoods $\mathcal{U}_i, i = 4, 5, 6$ around zero in $\mathcal{N}_{\mathcal{C}}$ such that:

$$\begin{aligned} \sup_{(t, x) \in [0, T] \times \mathbb{R}, \xi \in \mathcal{U}_4} \left| \frac{\partial}{\partial x} S_{\mathbf{p}} F(t, x)(\xi) \right| &< \infty \\ \sup_{(t, x) \in [0, T] \times \mathbb{R}, \xi \in \mathcal{U}_5} |S_{\mathbf{p}} F(t, x)(\xi)| &< \infty \\ \sup_{(t, x) \in [0, T] \times \mathbb{R}, \xi \in \mathcal{U}_6} |S_{\mathbf{p}} P(t, x)(\xi)| &< \infty \end{aligned}$$

(vii): The (generalized) expectations of Φ_0 and F are real, i. e.

$$\mathcal{S}_{\mathbf{P}}\Phi_0(x)(0) \text{ and } \mathcal{S}_{\mathbf{P}}F(t, x)(0) \in \mathbb{R} \text{ for all } (t, x) \in [0, T] \times \mathbb{R}$$

Note by conditions (iii) and (vi): For each $t \in [0, T]$:

$$\mathcal{S}_{\mathbf{P}}P(t, \cdot) \in C_{lb}^2(\mathbb{R}, \mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}}))$$

and

$$\int_{-\infty}^x \mathcal{S}_{\mathbf{P}}\Phi_0(y)dy \in C_{lb}^2(\mathbb{R}, \mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}}))$$

From now on we use the following notation: For a function $f(t, x; \xi)$ we denote

$$\|f\|_{\infty} := \sup_{(t, x) \in [0, T] \times \mathbb{R}, \xi \in \mathcal{U}} |f(t, x; \xi)|$$

and, for a function $g(x; \xi)$ we denote

$$\|g\|_{\infty} := \sup_{x \in \mathbb{R}, \xi \in \mathcal{U}} |g(x; \xi)|$$

We have put $\mathcal{U} := \cap_{i=1, \dots, 6} \mathcal{U}_i$, which again will be a neighborhood around zero in $\mathcal{N}_{\mathcal{C}}$.

Let us formally $\mathcal{S}_{\mathbf{P}}$ -transform equation (1): Letting

$$v(t, x; \xi) := \mathcal{S}_{\mathbf{P}}\Phi(t, x)(\xi)$$

for ξ in a neighborhood around zero of $\mathcal{N}_{\mathcal{C}}$, we obtain

$$v_t + \lambda v \cdot v_x = \nu v_{xx} + \mathcal{S}_{\mathbf{P}}F(t, x)(\xi) \quad (8)$$

(8) is a deterministic complex valued problem on $(t, x) \in (0, T] \times \mathbb{R}$. Using the Cole-Hopf transform, the solution of (8) is given through the relation

$$v(t, x; \xi) = \left(-\frac{2\nu}{\lambda}\right) \frac{u_x(t, x; \xi)}{u(t, x; \xi)} \quad (9)$$

$u(t, x; \xi)$ is the solution of the following heat problem with potential:

$$u_t = \nu u_{xx} + V(t, x; \xi)u \quad (10)$$

with initial condition

$$u(0, x; \xi) := u_0(x; \xi) = \exp\left(-\frac{\lambda}{2\nu}\right) \int_{-\infty}^x \mathcal{S}_{\mathbf{P}}\Phi_0(y)(\xi)dy \quad (11)$$

and

$$V(t, x; \xi) = \mathcal{S}_{\mathbf{P}}P(t, x)(\xi) \quad (12)$$

Our task in the next two sections is to verify all the formal steps made above, and show that v defined in (9) really is an $\mathcal{S}_{\mathbf{P}}$ -Transform of an element in the Kondratiev space.

4 A solution of the heat equation and its properties

We turn the attention to the heat problem with potential given by (10) and (11). An integral formulation of the problem is

$$u(t, x; \xi) = ((G_t * u_0(\cdot; \xi))(x) + \int_0^t (G_{t-s} * (V(s, \cdot; \xi)u(s, \cdot; \xi)))(x)ds \quad (13)$$

Here, $G_t(x)$ is the heat kernel associated to the Laplace operator $\nu\Delta$, and $*$ denotes convolution. Iterating (13), we get a Dyson representation for the solution of (10):

$$\begin{aligned} u(t, x; \xi) = & (G_t * u_0(\cdot; \xi))(x) + \\ & \sum_{n=1}^{\infty} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} G_{t-t_n} * \{V(t_n, \cdot; \xi)G_{t_n-t_{n-1}} * \{V(t_{n-1}, \cdot; \xi)G_{t_{n-1}-t_{n-2}} * \cdots \\ & \cdots * \{V(t_1, \cdot; \xi)G_{t_1} * u_0(\cdot; \xi)\} \cdots \}(x) dt_1 \cdots dt_{n-1} dt_n \end{aligned} \quad (14)$$

Our goal is to find a weak solution of (1) with $\mathcal{S}_{\mathbf{p}}$ -Transform given by $v(t, x; \xi)$. This means that v , v_t , v_x and v_{xx} all have to be elements of $C_{lb}((0, T] \times \mathbb{R}; \mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}}))$. Since v is given through the relation (9), we see that u_x and u both have to be elements of $C_{lb}((0, T] \times \mathbb{R}; \mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}}))$. In addition, $u(t, x; 0)$ have to be positive in order to be able to define $u(t, x; \xi)^{-1}$ as an element of $\mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}})$, see th. 3. Differentiation of (9) shows that v_t , v_x and v_{xx} are given as linear combinations of products of u , u_x , u_{xt} , u_{xx} and u^{-2} . Hence, we must also show that u_x , u_{xt} and u_{xx} are elements of $C_{lb}((0, T] \times \mathbb{R}; \mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}}))$. The rest of this section is devoted to work out these properties of $u(t, x; \xi)$. In the next section, we use these properties in order to conclude that we have a Kondratiev distribution which solves (1) in a weak sense.

I Properties of $u(t, x; \xi)$:

Direct estimation of the series in (14) using the assumptions (iii) and (vi) gives:

$$\begin{aligned} |u(t, x; \xi)| & \leq \|u_0\|_{\infty} + \|u_0\|_{\infty} \sum_{n=1}^{\infty} \|V\|_{\infty}^n \left(\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} dt_1 \cdots dt_n \right) \\ & = \|u_0\|_{\infty} \left(1 + \sum_{n=1}^{\infty} \frac{\|V\|_{\infty}^n t^n}{n!} \right) \leq \|u_0\|_{\infty} \exp(\|V\|_{\infty} T) \end{aligned}$$

Hence,

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}, \xi \in \mathcal{U}} |u(t, x; \xi)| \leq \|u_0\|_{\infty} \exp(\|V\|_{\infty} T)$$

From this uniform estimate, we can conclude: Since each term in the series for $u(t, x; \xi)$ is continuous with respect to (t, x) for each $\xi \in \mathcal{U}$, we have that $u(t, x; \xi)$

itself will be continuous for each $\xi \in \mathcal{U}$. Given $\xi \in \mathcal{U}$ and $\eta \in \mathcal{N}_{\mathcal{C}}$, we have by assumptions on F and Φ_0 that $V(\xi + z\eta)$ and $u_0(\xi + z\eta)$ are analytic mappings for $z \in \mathcal{V}_{\xi, \eta}$. $\mathcal{V}_{\xi, \eta}$ is a neighborhood around zero in $\mathcal{C}$. Hence, each term in the series for $u(t, x; \xi + z\eta)$ is analytic in $\mathcal{V}_{\xi, \eta}$. An application of Vitali's theorem together with Weierstrass' theorem, gives the analyticity of the mapping:

$$z \in \mathcal{V}_{\xi, \eta} \rightarrow u(t, x; \xi + z\eta) \in \mathcal{C}$$

The uniform bound of the series gives immediately that $u(t, x; \xi)$ is bounded on $\mathcal{U}$, uniformly in (t, x) . When then have

$$u \in C_{lb}([0, T] \times \mathbb{R}; \mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}}))$$

II Properties of $u_x(t, x; \xi)$:

Differentiating each term in the sequence (14), we obtain:

$$\begin{aligned} & \frac{\partial}{\partial x} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} (\cdots) dt_1 \cdots dt_n \\ &= \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \left(\frac{\partial}{\partial x} G_{t-t_n} \right) * \{V(t_n, \cdot; \xi) \cdots\} dt_1 \cdots dt_n \end{aligned}$$

Hence, each term in the series of $u(t, x; \xi)$ is continuously differentiable with respect to x . We show that $u_x(t, x; \xi)$ exists, and is given by the series:

$$\begin{aligned} u_x(t, x; \xi) &= \left(\left(\frac{\partial}{\partial x} G_t \right) * u_0(\cdot; \xi) \right)(x) + \\ & \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{t_2} \left(\frac{\partial}{\partial x} G_{t-t_n} \right) * \{V(t_n, \cdot; \xi) G_{t_n-t_{n-1}} \{V(t_{n-1}, \cdot; \xi) G_{t_{n-1}-t_{n-2}} * \cdots \\ & \cdots * \{V(t_1, \cdot; \xi) G_{t_1} * u_0(\cdot; \xi)\} \cdots\} \}(x) dt_1 \cdots dt_n \end{aligned} \quad (15)$$

It is sufficient to show that the series in (15) converges uniformly: Direct estimation in the series in (15) shows

$$|u_x(t, x; \xi)| \leq K t^{-\frac{1}{2}} \|u_0\|_{\infty} + K \|u_0\|_{\infty} \sum_{n=1}^{\infty} \|V\|_{\infty} \int_0^t \frac{t_n^{n-1}}{(n-1)!(t-t_n)^{\frac{1}{2}}} dt_n$$

where K is a positive constant. We have the following bound:

$$\int_0^t \frac{s^k}{(t-s)^{\frac{1}{2}}} ds \leq t^k \int_0^t (t-s)^{-\frac{1}{2}} ds = 2t^{k+\frac{1}{2}}$$

Hence, $u_x(t, x; \xi)$ exists, and satisfies a uniform bound: For each $\epsilon > 0$

$$\sup_{(t,x) \in [\epsilon, T] \times \mathbb{R}, \xi \in \mathcal{U}} |u_x(t, x; \xi)| < \infty$$

We can conclude as follows: For each $\xi \in \mathcal{U}$, u_x is continuous in $(t, x) \in (0, T] \times \mathbb{R}$. By the same argumentation as in I, we have the analyticity of the mapping

$$z \rightarrow u_x(t, x; \xi + z\eta)$$

on $\mathcal{V}_{\xi, \eta}$. Hence,

$$u_x \in C_{lb}((0, T] \times \mathbb{R}; \mathcal{H}ol_0(\mathcal{N}_{\mathbb{C}}))$$

III Properties of $u_{xx}(t, x; \xi)$:

Define

$$H(t_n, \dots, t_1, z) = (G_{t_n - t_{n-1}} * \{V(t_{n-1}, \cdot; \xi) \cdots * \{V(t_1, \cdot; \xi) G_{t_1} * u_0(\cdot; \xi)\} \cdots\}) (z)$$

(Note: z is here not meant to be complex!). We do the following calculation:

$$\begin{aligned} & \frac{\partial^2}{\partial x^2} \int_0^t \cdots \int_0^{t_2} (\cdots) dt_1 \dots dt_n \\ &= \frac{\partial}{\partial x} \int_0^t \cdots \int_0^{t_2} \left(\int_{\mathbb{R}} \left(\frac{\partial}{\partial x} G_{t-t_n} \right) (x-y) V(t_n, y; \xi) H(t_n, \dots, t_1, y) dy \right) dt_1 \dots dt_n \\ &= \frac{\partial}{\partial x} \int_0^t \cdots \int_0^{t_2} \left(\int_{\mathbb{R}} \left(\frac{\partial}{\partial x} G_{t-t_n} \right) (y) V(t_n, x-y; \xi) H(t_n, \dots, t_1, x-y) dy \right) dt_1 \dots dt_n \\ &= \int_0^t \cdots \int_0^{t_2} \left(\int_{\mathbb{R}} \left(\frac{\partial}{\partial x} G_{t-t_n} \right) (y) \frac{\partial V}{\partial x}(t_n, x-y; \xi) H(t_n, \dots, t_1, x-y) dy \right) dt_1 \dots dt_n \\ &+ \int_0^t \cdots \int_0^{t_2} \left(\int_{\mathbb{R}} \left(\frac{\partial}{\partial x} G_{t-t_n} \right) (y) V(t_n, x-y; \xi) \frac{\partial H}{\partial x}(t_n, \dots, t_1, x-y) dy \right) dt_1 \dots dt_n \end{aligned}$$

where we have:

$n = 1$:

$$\frac{\partial H}{\partial x}(t_1, x-y) = \int_{\mathbb{R}} \left(\frac{\partial}{\partial x} G_{t_1} \right) (x-y-z) u_0(z; \xi) dz$$

$n \geq 2$:

$$\frac{\partial H}{\partial x}(t_n, \dots, t_1, x-y) = \int_{\mathbb{R}} \left(\frac{\partial}{\partial x} G_{t_n - t_{n-1}} \right) (x-y-z) H(t_{n-1}, \dots, t_1, z) dz$$

These calculations imply that each term in the series for u_x is continuously differentiable with respect to x . We show that the series for u_{xx} is:

$$u_{xx}(t, x; \xi) = \left(\frac{\partial}{\partial x} G_t * \frac{\partial u_0}{\partial x}(\cdot; \xi) \right)(x) + \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{t_n} \left(\frac{\partial}{\partial x} G_{t-t_n} \right) * \\ \left\{ \frac{\partial V}{\partial x}(t_n, \cdot; \xi) H(t_n, \dots, t_1, \cdot) + V(t_n, \cdot; \xi) \frac{\partial H}{\partial x}(t_n, \dots, t_1, \cdot) \right\} dt_1 \dots dt_n$$

Direct estimation of the series gives:

$$|u_{xx}(t, x; \xi)| \leq C t^{-\frac{1}{2}} \left\| \frac{\partial u_0}{\partial x} \right\|_{\infty} \\ + K \|u_0\|_{\infty} \left\| \frac{\partial V}{\partial x} \right\|_{\infty} \sum_{n=1}^{\infty} \left(\|V\|_{\infty}^{n-1} \int_0^t \frac{t_n^{n-1}}{(n-1)!(t-t_n)^{\frac{1}{2}}} dt_n \right) \\ + K \|u_0\|_{\infty} \sum_{n=2}^{\infty} \left(\|V\|_{\infty}^n \int_0^t \int_0^{t_n} \frac{t_{n-1}^{n-2}}{(n-2)!(t-t_n)^{\frac{1}{2}}(t_n-t_{n-1})^{\frac{1}{2}}} dt_{n-1} dt_n \right) \\ + K \|u_0\|_{\infty} \int_0^t \frac{dt_1}{(t_1(t-t_1))^{\frac{1}{2}}}$$

By estimation of the integrals,

$$\leq C t^{-\frac{1}{2}} \left\| \frac{\partial u_0}{\partial x} \right\|_{\infty} \\ + K \|u_0\|_{\infty} \left\| \frac{\partial V}{\partial x} \right\|_{\infty} \sum_{n=1}^{\infty} \|V\|_{\infty}^{n-1} \frac{2t^{n-\frac{1}{2}}}{(n-1)!} \\ + K \|u_0\|_{\infty} \sum_{n=2}^{\infty} \|V\|_{\infty}^n \frac{4t^{n-1}}{(n-2)!} + K \|u_0\|_{\infty}$$

Hence, the series converges uniformly. We obtain that u_{xx} exists, and is for each $\epsilon > 0$ uniformly bounded

$$\sup_{(t,x) \in [\epsilon, T] \times \mathbb{R}, \xi \in \mathcal{U}} |u_{xx}(t, x; \xi)| < \infty$$

We get the usual conclusions that u_{xx} is continuous in $(t, x) \in (0, T] \times \mathbb{R}$, and that

$$u_{xx} \in C_{lb}((0, T] \times \mathbb{R}; \mathcal{H}ol_0(\mathcal{N}_{\mathcal{C}}))$$

IV Properties of $u_t(t, x; \xi)$:

From the heat equation (10) we have

$$u_t(t, x; \xi) = \nu u_{xx}(t, x; \xi) + V(t, x; \xi)u(t, x; \xi)$$

which means that u_t will inherit the properties of V, u_{xx} and u . Hence, u_t is continuous in $(t, x) \in (0, T] \times \mathbb{R}$ for each $\xi \in \mathcal{U}$, since by condition (v) in section 3 $V \in C_{lb}([0, T] \times \mathbb{R}; \mathcal{Hol}_0(\mathcal{N}_{\mathcal{C}}))$. Therefore

$$u_t \in C_{lb}((0, T] \times \mathbb{R}; \mathcal{Hol}_0(\mathcal{N}_{\mathcal{C}}))$$

V Properties of $u_{xt}(t, x; \xi)$:

Consider IV: After differentiation, we obtain

$$\begin{aligned} u_{xt}(t, x; \xi) &= \frac{\partial}{\partial x} u_t(t, x; \xi) \\ &= \nu u_{xxx}(t, x; \xi) + \mathcal{S}_{\mathbf{P}} F(t, x)(\xi) u(t, x; \xi) + V(t, x; \xi) u_x(t, x; \xi) \end{aligned}$$

Hence, to obtain the wanted properties of u_{xt} , we consider u_{xxx} . But calculating as in III, we see that u_{xxx} exists, and has for each $\epsilon > 0$ the uniform bound

$$\sup_{(t, x) \in [\epsilon, T] \times \mathbb{R}; \xi \in \mathcal{U}} |u_{xxx}(t, x; \xi)| < \infty$$

Moreover, $u_{xxx}(t, x; \xi)$ is continuous in $(t, x) \in (0, T] \times \mathbb{R}$ for each $\xi \in \mathcal{U}$. Since by condition (v) section 3, $\mathcal{S}_{\mathbf{P}} F, V \in C_{lb}([0, T] \times \mathbb{R}; \mathcal{Hol}_0(\mathcal{N}_{\mathcal{C}}))$, u_{xt} is continuous in $(t, x) \in (0, T] \times \mathbb{R}$. Moreover, we can conclude

$$u_{xt} \in C_{lb}((0, T] \times \mathbb{R}; \mathcal{Hol}_0(\mathcal{N}_{\mathcal{C}}))$$

VI Positivity of $u(t, x; 0)$:

By assumption (vii), we have that $V(t, x; 0)$ and $u_0(t, x; 0)$ are real. Moreover, by assumptions (iii) and (vi) they are uniformly bounded. Hence, the heat problem

$$\begin{aligned} u_t(t, x; 0) &= \nu u_{xx}(t, x; 0) + V(t, x; 0)u(t, x; 0) \\ u_0(x; 0) &= \exp\left(-\frac{\lambda}{2\nu}\right) \int_{-\infty}^x \mathcal{S}_{\mathbf{P}} \Phi_0(y)(0) dy \end{aligned}$$

has a unique positive (strong) solution $u(t, x; 0)$, (see Friedman [Fr]). This solution can be represented by a Feynman-Kac formula:

$$u(t, x; 0) = E \left[u_0(b_t^x; 0) \exp\left(\int_0^t V(t-s, b_s^x; 0) ds\right) \right] \quad (16)$$

b_t^x is the modified Brownian motion which starts in x at time zero. By condition (iii), we have the following inequality:

$$\sup_{x \in \mathbb{R}, \xi \in \mathcal{U}} \left| \int_{-\infty}^x \mathcal{S}_{\mathbf{p}} \Phi_0(y)(\xi) dy \right| \leq \sup_{\xi \in \mathcal{U}} \int_{\mathbb{R}} |\mathcal{S}_{\mathbf{p}} \Phi_0(y)(\xi)| dy = K$$

where K is a positive constant. Hence

$$-\frac{\lambda K}{2\nu} \leq \left(-\frac{\lambda}{2\nu}\right) \int_{-\infty}^x \mathcal{S} \Phi_0(y)(0) dy \leq \frac{\lambda K}{2\nu}$$

for all $x \in \mathbb{R}$. Taking exponentials on both sides:

$$\exp\left(-\frac{\lambda K}{2\nu}\right) \leq u_0(x; 0) \leq \exp\left(\frac{\lambda K}{2\nu}\right)$$

for all $x \in \mathbb{R}$. A similar argumentation, using assumption (vi), gives

$$\exp(-Ct) \leq \exp\left(\int_0^t V(s, x; 0) ds\right) \leq \exp(Ct)$$

for all $x \in \mathbb{R}$. C is a positive constant. Using these bounds in (16), we obtain a kind of non-degeneracy property for the solution of our heat equation:

$$u(t, x; 0) \geq E^x[\exp(-\frac{\lambda K}{2\nu} - Ct)] \geq \exp(-\frac{\lambda K}{2\nu} - CT) > 0$$

Hence, $u(t, x; 0)$ is always a distance $\exp(-\frac{\lambda K}{2\nu} - CT)$ away from zero for $(t, x) \in [0, T] \times \mathbb{R}$.

We are ready to conclude the existence of a solution of the stochastic Burgers equation (1) and (2).

5 Solution of the stochastic Burgers equation

From **I**, we know that $u(t, x; \xi)$ is an element of $\mathcal{Hol}_0(\mathcal{N}_{\mathcal{C}})$ for all (t, x) . By **VI**, $u(t, x; 0)$ is positive and bounded away from zero. Hence, for each $(t, x) \in [0, T] \times \mathbb{R}$ there is a neighborhood around $u(t, x; 0)$ on which

$$f(z) = \frac{1}{z}$$

is analytic. By the Wick Calculus theorem 3, the element

$$\mathcal{S}_{\mathbf{p}}^{-1}(u(t, x; \cdot)^{-1})$$

is well defined as an element of $(\mathcal{N})^{-1}$. Moreover, by **II** and the Wick Calculus theorem, $\mathcal{S}_{\mathbf{p}}^{-1}(u_x(t, x; \cdot)) \in (\mathcal{N})^{-1}$ and we obtain the existence of $\Phi(t, x) \in (\mathcal{N})^{-1}$ defined by

$$\mathcal{S}_{\mathbf{p}} \Phi(t, x)(\xi) = v(t, x; \xi) = \left(-\frac{2\nu}{\lambda}\right) \frac{u_x(t, x; \xi)}{u(t, x; \xi)}$$

Moreover, we obtain from **I**, **II** and lemma (4) that $\Phi(t, x)$ is weakly continuous in (t, x) .

Differentiation of v shows that v_t , v_x and v_{xx} are given as linear combinations of products of the elements u , u_t , u_x , u_{xx} , u_{xt} and $1/u^2$. Using the results in **I-VI**, we get that v_t , v_x and v_{xx} all belong to $C_{lb}((0, T] \times \mathbb{R}; \mathcal{H}ol_0(\mathcal{N}_C))$. Hence

$$\Phi_t(t, x) = S_{\mathbf{p}}^{-1}(v_t(t, x; \cdot))$$

$$\Phi_x(t, x) = S_{\mathbf{p}}^{-1}(v_x(t, x; \cdot))$$

$$\Phi_{xx}(t, x) = S_{\mathbf{p}}^{-1}(v_{xx}(t, x; \cdot))$$

are all elements of $(\mathcal{N})^{-1}$, weakly continuous in (t, x) . Therefore, the Kondratiev distribution $\Phi(t, x)$ is a weak solution of the stochastic Burgers equation (1) and (2). We conclude what we have showed:

Theorem 5 *Assume the conditions (i)-(vii) stated in section 3. With*

$$V(t, x; \xi) = S_{\mathbf{p}}P(t, x)(\xi)$$

and $u(t, x; \xi)$ given as in (14), there exists a weakly continuous solution $\Phi(t, x) \in (\mathcal{N})^{-1}$ of (1) and (2) given by

$$S_{\mathbf{p}}\Phi(t, x)(\xi) = \left(-\frac{2\nu}{\lambda}\right) \frac{u_x(t, x; \xi)}{u(t, x; \xi)}$$

for $(t, x) \in [0, T] \times \mathbb{R}$. Moreover, $\Phi(t, x)$ is weakly continuously differentiable in t and twice weakly continuously differentiable in x for $(t, x) \in (0, T] \times \mathbb{R}$.

■

6 Applications

We will in this section look at three different kind of noises: The Gaussian case treated by Holden et. al. [HLØUZ1+2], the non-Gaussian case presented in section 2 and the Poisson case.

Gaussian case:

Let our probability space be $\mathcal{S}'(\mathbb{R}^2)$, the space of tempered distributions on $\mathbb{R}^2$. $\mathcal{S}'(\mathbb{R}^2)$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. A Gaussian probability measure μ_0 can be defined through the characteristic functional:

$$\int_{\mathcal{S}'(\mathbb{R}^2)} e^{i\langle \omega, \xi \rangle} d\mu_0(\omega) = e^{-\frac{1}{2}|\xi|_2^2}$$

We note that μ_0 falls into the class of smooth double analytic measures, see Albeverio et. al. [ADKS]. Moreover, the **P**- and **Q**-system will be equal and

coincide with the Hermite polynomials. The $\mathcal{S}_{\mathbf{p}}$ -transform is the well-known $\mathcal{S}$ -transform. We denote the Kondratiev distribution space $(\mathcal{S})^{-1}$. For more detailed information about the Gaussian setting, the reader should confer Hida et. al. [HKPS] and Kondratiev et. al. [KLS].

In this setting we have a time-space white noise which can be realized as the derivative of the Brownian sheet,

$$W_{t,x} = \frac{\partial^2 B_{t,x}}{\partial t \partial x}$$

It has chaos decomposition given by

$$W_{t,x} = \langle \omega, \delta_{t,x} \rangle$$

where $\delta_{t,x}$ is the Dirac's delta function. Moreover, its $\mathcal{S}$ -transform is

$$SW_{t,x}(\xi) = \xi(t, x)$$

As noise in our model (1), we choose

$$F(t, x) = W_{t,x}$$

and we define $P(t, x)$ to be the $(\mathcal{S})^{-1}$ -element with $\mathcal{S}$ -transform

$$SP(t, x)(\xi) = \int_0^x \xi(t, y) dy$$

We show that condition (vi) in section 3 is satisfied: Let $N > 1$ and K a positive constant. Define the following neighborhoods around zero in $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^2)$:

$$\mathcal{U}_4 = \{\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^2) : \|\xi\|_{(N, (0,1))} < K\}$$

$$\mathcal{U}_5 = \{\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^2) : |\xi|_{\infty} < K\}$$

$$\mathcal{U}_6 = \{\xi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^2) : \|\xi\|_{(N, (0,0))} < K\}$$

$|\cdot|_{\infty}$ denotes the sup-norm, and

$$\|\xi\|_{(N, \alpha)} = \sup_{(t,x) \in [0,T] \times \mathbb{R}} (1 + (t^2 + x^2)^{\frac{1}{2}})^N (|\partial_{\alpha} \xi(t, x)|)$$

$\alpha = (\alpha_1, \alpha_2)$. Hence

$$\begin{aligned} \sup_{(t,x), \xi \in \mathcal{U}_4} \left| \frac{\partial}{\partial x} SF(t, x)(\xi) \right| &= \sup_{(t,x), \xi \in \mathcal{U}_4} |\xi_x(t, x)| \\ &\leq \sup_{(t,x), \xi \in \mathcal{U}_4} (1 + |x|)^{-N} (1 + (t^2 + x^2)^{\frac{1}{2}})^N |\xi_x(t, x)| \end{aligned}$$

$$\leq \sup_{\xi \in \mathcal{U}_4} \|\xi\|_{(N,(0,1))} \cdot \sup_{x \in \mathbb{R}} (1 + |x|)^{-N} \leq K$$

For $SF(t, x)(\xi)$ we immediately obtain

$$\sup_{(t,x), \xi \in \mathcal{U}_5} |SF(t, x)(\xi)| = \sup_{(t,x), \xi \in \mathcal{U}_5} |\xi(t, x)| \leq \sup_{\xi \in \mathcal{U}_5} |\xi|_\infty \leq K$$

We estimate the $\mathcal{S}$ -transform of P equivalently:

$$\sup_{(t,x), \xi \in \mathcal{U}_6} |SP(t, x)(\xi)| \leq \sup_{\xi \in \mathcal{U}_6} \|\xi\|_{(N,(0,0))} \cdot \int_{\mathbb{R}} (1 + |y|)^{-N} dy \leq K \cdot C$$

Hence, from section 4 and 5 we have a weak solution of (1) with $W_{t,x}$ as force. We remark that this kind of force is covered by the work of Holden et. al. [HLØUZ1+2].

Non-Gaussian case:

We consider the probability space $(\mathcal{N}, \mu)$ with $\mathcal{N} = S'(\mathbb{R}^2)$ as constructed in section 2. The Kondratiev distribution space will be denoted $(\mathcal{S})^{-1}$. In Albeverio et. al. [ADKS], they suggest

$$\int_0^T \Phi_t \diamond \langle Q_1, \delta_t \rangle dt$$

as stochastic integration. Hence,

$$F(t, x) = \langle Q_1, \delta_{t,x} \rangle$$

is a possible non-Gaussian realization of what we can call a time-space white noise. If we use this as a force in our Burgers equation (1), we see by results in section 4 and 5 that a weak solution exists in $(\mathcal{S})^{-1}$. This follows since

$$\mathcal{S}_P F(t, x)(\xi) = \xi(t, x)$$

In analogy to the Gaussian case, it is also of interest to study the (distributional) derivative of the canonical coordinate process

$$S_{t,x} = \langle \omega, \mathbf{1}_{[0,t]} \cdot \mathbf{1}_{[0,x]} \rangle$$

We consider this in the special situation of a perturbed Gaussian measure μ on $S'(\mathbb{R}^2)$: Fix a $\phi \in \mathcal{S}(\mathbb{R}^2)$, and consider the measure μ defined as

$$\mu = e^{\langle \cdot, \phi \rangle} \mu_0$$

where μ_0 is the Gaussian measure. With such a measure, we have an explicit representation of the Q_n 's, see [ADKS]: With ∇_ψ acting on θ , we have

$$Q(\psi; \omega) = (-1) \left(\nabla_\psi \frac{e^{\langle \omega + \theta, \phi \rangle}}{e^{\langle \omega, \phi \rangle}} \cdot e^{\langle \omega, \theta \rangle - \frac{1}{2} |\theta|_2^2} \right) |_{\theta=0}$$

$$= -\langle \omega, \psi \rangle - \langle \phi, \psi \rangle$$

Hence

$$Q_1(\omega) = -\omega - \phi$$

This implies that

$$\langle \omega, \delta_{t,x} \rangle = -\langle Q_1(\omega), \delta_{t,x} \rangle - \phi(t, x)$$

defines an element in $(S)^{-1}$. We can use

$$F(t, x) = \langle \omega, \delta_{t,x} \rangle$$

as our force in the Burgers equation (1). Since

$$S_{\mathbf{P}} F(t, x)(\xi) = -\xi(t, x) - \phi(t, x)$$

we obtain a weak solution of (1) in $(S)^{-1}$.

Poisson case:

We consider the probability space $(S'(\mathbb{R}^2), \pi)$, where π is the Poisson measure defined by the characteristic functional

$$\mathcal{X}_\pi(\phi) = \exp\left(\int_{\mathbb{R}} (e^{i\phi(t)} - 1) dt\right) \text{ for } \phi \in S(\mathbb{R}^2)$$

The Poisson case is not covered by the non-Gaussian analysis presented in section 2. However, as already mentioned, this case is included in the general framework of Kondratiev et. al. [KSW]. We base our presentation here on the explicit consideration found in Us [U]. We first give a brief account on his results. For proofs and references, see Us [U].

The **P**-system is defined like in the non-Gaussian case by the π -exponential

$$\begin{aligned} e_\pi(\theta; \omega) &= e^{\langle \omega, \theta \rangle} \left(\int_{S'(\mathbb{R}^2)} e^{\langle \omega, \theta \rangle} d\pi(\omega) \right)^{-1} \\ &= e^{\langle \omega, \theta \rangle - \int_{\mathbb{R}} (e^{\theta(t)} - 1) dt} \end{aligned}$$

The **Q**-system, on the other hand, is defined through the relation

$$\langle Q(\phi^{\otimes n}; \cdot), \psi \rangle = \int_{S'(\mathbb{R}^2)} (\nabla_\phi^n \psi)(\omega) d\pi(\omega)$$

where $\psi \in \mathcal{P}(S')$, the continuous polynomials on $S'(\mathbb{R}^2)$. The **P**- and **Q**-systems are biorthogonal.

Like in the non-Gaussian case, we can construct a triplet of spaces

$$(S)_\pi^1 \subset L^2(S'(\mathbb{R}^2), \pi) \subset (S)_\pi^{-1}$$

where the test function space $(\mathcal{S})_\pi^1$ has a decomposition in the **P**-system, and the distribution space $(\mathcal{S})_\pi^{-1}$ in the **Q**-system.

We define the $S_{\mathbf{P}}$ -transform on $(\mathcal{S})_\pi^{-1}$: For $\Phi \in (\mathcal{S})_\pi^{-1}$ and $\xi \in \mathcal{U}_{p,q}^\delta(0)$, (for the definition of $\mathcal{U}_{p,q}^\delta(0)$, see section 2.)

$$S_{\mathbf{P}}\Phi(\xi) = \langle \langle \Phi, e_\pi(\xi; \cdot) \rangle \rangle$$

The $(\mathcal{S})_\pi^{-1}$ -distributions are characterized through the $S_{\mathbf{P}}$ -transform like in the non-Gaussian case see th. 2. Note that the Wick Calculus Theorem, th. 3, also holds for $(\mathcal{S})_\pi^{-1}$.

In the Poisson setting, one can define a system of orthogonal polynomials called the Charlier polynomials. These polynomials have

$$c(\theta; \omega) = e_\pi(\ln(1 + \theta); \omega) \text{ for } \theta > -1$$

as generating function, i. e.

$$C(\psi^{\otimes n}; \omega) = \frac{d^n}{dt^n} c(t\psi; \omega)|_{t=0}$$

By the kernel theorem, one obtains the existence of a system $C_n(\omega), n \in \mathbb{N}_0$, where

$$\langle C_n(\omega), \psi^{\otimes n} \rangle = C(\psi^{\otimes n}; \omega)$$

$\{C_n\}_{n \in \mathbb{N}_0}$ is called the **J**-system.

One can introduce a distribution space in terms of decomposition into the **J**-system, denoted $(\tilde{\mathcal{S}})_\pi^{-1}$: Each $\Phi \in (\tilde{\mathcal{S}})_\pi^{-1}$ has the unique decomposition

$$\Phi = \sum_{n=0}^{\infty} \langle C_n(\omega), \Phi^{(n)} \rangle, \Phi^{(n)} \in \mathcal{F}_n(\mathcal{S}') \quad (17)$$

where

$$\|\Phi\|_{\pi, -p, -1}^2 = \sum_{n=0}^{\infty} |\Phi^{(n)}|_{-p}^2 < \infty \quad (18)$$

Every formal series on the form (17) which satisfies (18) defines a distribution in $(\tilde{\mathcal{S}})_\pi^{-1}$. It is shown in [KSW] and [U], that $(\mathcal{S})_\pi^{-1}$ coincides with $(\tilde{\mathcal{S}})_\pi^{-1}$. We denote both spaces $(\mathcal{S})_\pi^{-1}$.

We can introduce a new $\mathcal{S}$ -transform called the $S_{\mathbf{J}}$ -transform: Let $\xi \in \mathcal{U}_{p,q}^\delta(0)$ with $\xi > -1$,

$$S_{\mathbf{J}}\Phi(\xi) = \langle \langle \Phi, c(\xi; \cdot) \rangle \rangle, \Phi \in (\mathcal{S})_\pi^{-1}$$

One sees that

$$S_{\mathbf{J}}\Phi(\xi) = S_{\mathbf{P}}\Phi(\ln(1 + \xi))$$

Since we have two $\mathcal{S}$ -transforms on $(\mathcal{S})_\pi^{-1}$, we have a-priori two different Wick products. Denote them $\diamond_{\mathbf{J}}$ and $\diamond_{\mathbf{P}}$. However, they coincide, as the following calculation shows:

$$\begin{aligned} \mathcal{S}_{\mathbf{J}}(\Phi \diamond_{\mathbf{P}} \Psi)(\xi) &= \mathcal{S}_{\mathbf{P}}(\Phi \diamond_{\mathbf{P}} \Psi)(\ln(1 + \xi)) \\ &= \mathcal{S}_{\mathbf{P}}\Phi(\ln(1 + \xi)) \cdot \mathcal{S}_{\mathbf{P}}\Psi(\ln(1 + \xi)) = \mathcal{S}_{\mathbf{J}}\Phi(\xi) \cdot \mathcal{S}_{\mathbf{J}}\Psi(\xi) \\ &= \mathcal{S}_{\mathbf{J}}(\Phi \diamond_{\mathbf{J}} \Psi)(\xi) \end{aligned}$$

We denote the Wick product in $(\mathcal{S})_\pi^{-1}$ by $\diamond$.

The Charlier polynomials are in a nice relation to the *compensated* two parameter Poisson process $N_{t,x} - tx$ on $L^2(\mathcal{S}', \pi)$:

$$\begin{aligned} N_{t,x} - tx &= \langle \omega - 1, \mathbf{1}_{[0,t]} \cdot \mathbf{1}_{[0,x]} \rangle \\ &= \langle C_1(\omega), \mathbf{1}_{[0,t]} \cdot \mathbf{1}_{[0,x]} \rangle \end{aligned}$$

Hence, we can define the *compensated* two parameter Poisson noise as the $(\mathcal{S})_\pi^{-1}$ -element

$$\frac{\partial^2 N_{t,x}}{\partial t \partial x} - 1 := \langle C_1(\omega), \delta_{t,x} \rangle$$

where $\delta_{t,x}$ is Dirac's delta-function. The $\mathcal{S}_{\mathbf{J}}$ -transform of the compensated Poisson noise is

$$\mathcal{S}_{\mathbf{J}}\left(\frac{\partial^2 N_{t,x}}{\partial t \partial x} - 1\right)(\xi) = \xi(t, x)$$

Hence, we get for the $\mathcal{S}_{\mathbf{P}}$ -transform

$$\mathcal{S}_{\mathbf{P}}\left(\frac{\partial^2 N_{t,x}}{\partial t \partial x} - 1\right)(\xi) = e^{\xi(t,x)} - 1$$

We consider the Burgers equation forced by the compensated Poisson noise:

$$\Phi_t + \lambda \Phi \diamond \Phi_x = \nu \Phi_{xx} + \left(\frac{\partial^2 N_{t,x}}{\partial t \partial x} - 1\right) \quad (19)$$

I. e.

$$F(t, x) = \frac{\partial^2 N_{t,x}}{\partial t \partial x} - 1$$

Define

$$P(t, x) = \mathcal{S}_{\mathbf{P}}^{-1}\left(\int_0^x e^{\xi(t,y)} dy - x\right)$$

We prove that F and P satisfy condition (vi) in section 3: Let K_1, K_2, K_3 be positive constants, and $N > 1$: Define the following neighborhoods of zero in $\mathcal{S}_{\mathbf{C}}(\mathbb{R}^2)$:

$$\begin{aligned} \mathcal{U}_4 &= \{\xi \in \mathcal{S}_{\mathbf{C}}(\mathbb{R}^2) : \|\xi\|_{(N,(0,1))} < K_1, |\xi|_\infty < K_2\} \\ \mathcal{U}_5 &= \{\xi \in \mathcal{S}_{\mathbf{C}}(\mathbb{R}^2) : |\xi|_\infty < K_2\} \end{aligned}$$

$$\mathcal{U}_6 = \{\xi \in \mathcal{S}_{\mathcal{C}}(\mathbb{R}^2) : \|\xi\|_{(N,(0,0))} < K_3\}$$

We estimate as follows:

$$\begin{aligned} \sup_{(t,x), \xi \in \mathcal{U}_4} \left| \frac{\partial}{\partial x} \mathcal{S}_{\mathbf{P}} F(t, x)(\xi) \right| &= \sup_{(t,x), \xi \in \mathcal{U}_4} |\xi_x(t, x) e^{\xi(t, x)}| \\ &\leq \sup_{(t,x), \xi \in \mathcal{U}_4} |\xi_x(t, x)| \cdot \sup_{(t,x), \xi \in \mathcal{U}_4} |e^{\xi(t, x)}| \leq K_1 e^{K_2} \end{aligned}$$

For the force $F(t, x)$ we get

$$\sup_{(t,x), \xi \in \mathcal{U}_5} |\mathcal{S}_{\mathbf{P}} F(t, x)(\xi)| = \sup_{(t,x), \xi \in \mathcal{U}_5} |(e^{\xi(t, x)} - 1)| \leq (e^{K_2} - 1)$$

For $P(t, x)$ we use the fact that $e^{\xi(t, x)} - 1$ is in $\mathcal{S}_{\mathcal{C}}(\mathbb{R}^2)$:

$$\begin{aligned} \sup_{(t,x), \xi \in \mathcal{U}_6} |\mathcal{S}_{\mathbf{P}} P(t, x)(\xi)| &= \sup_{(t,x), \xi \in \mathcal{U}_6} \left| \int_0^x (e^{\xi(t, y)} - 1) dy \right| \\ &\leq \sup_{\xi \in \mathcal{U}_6} \|e^{\xi} - 1\|_{(N,(0,0))} \cdot \int_{\mathbb{R}} (1 + |y|)^{-N} dy \leq K_3 \cdot C \end{aligned}$$

Hence, condition (vi) is satisfied, and from section 5 we conclude that the Burgers equation (19) with compensated Poisson noise as force, has a weak solution in $(\mathcal{S})_{\pi}^{-1}$.

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