Off-Diagonal Elements of the DeWitt Expansion from the Quantum Mechanical Path Integral *

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Abstract

The DeWitt expansion of the matrix element \( M_{xy} = \langle x | \exp \left[ \frac{1}{2} (p - A)^2 + V \right] t | y \rangle \), \((p = -i\partial)\) in powers of \( t \) can be made in a number of ways. For \( x = y \) (the case of interest when doing one-loop calculations) numerous approaches have been employed to determine this expansion to very high order; when \( x \neq y \) (relevant for doing calculations beyond one-loop) there appear to be but two examples of performing the DeWitt expansion. In this paper we adapt to the latter case a covariant approach based on representing \( M_{xy} \) by a quantum mechanical path integral. We also generalize to the case of curved space, allowing us to determine the DeWitt expansion of \( \tilde{M}_{xy} = \langle x | \exp \frac{1}{2} \left[ \frac{1}{\sqrt{g}} (\partial_\mu - iA_\mu) g^{\mu\nu} \sqrt{g} (\partial_\nu - iA_\nu) \right] t | y \rangle \) by use of normal coordinates. By comparison with results for the DeWitt expansion of this matrix element obtained by the iterative solution of the diffusion equation, the relative merit of different approaches to the representation of \( \tilde{M}_{xy} \) as a quantum mechanical path integral can be assessed. Furthermore, the exact dependence of \( \tilde{M}_{xy} \) on some geometric scalars can be determined.

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I Introduction

There is a long history of computing the elements $a_n(x_0, \Delta)$ in the expansion of $M_{xy}$ in powers of $t$,

$$M_{xy} \equiv \langle x | \exp \left\{ - \frac{1}{2} (p - A)^2 + V \right\} | y \rangle$$

$$= \frac{e^{-\Delta^2/2t}}{(2\pi t)^{D/2}} \sum_{n=0}^{\infty} a_n(x_0, \Delta) t^n,$$

in the limit $\Delta = 0$ [1]. This expansion when $\Delta = 0$ is extremely useful when examining certain properties of the generating functional at one-loop order; in particular, the divergence structure of a theory at one-loop order can be discerned. Among the approaches used to evaluate $a_n(x_0, 0)$ are the perturbative solution to the heat equation [1, 2], the use of pseudo-differential operators [3], working in momentum space [4], systematically rearranging a Schwinger expansion of (1a) in powers of $A$ and $V$ into an expression of the form (1b) [5,6] and representing (1a) as a quantum mechanical path integral (QMPI) hence expanding it in powers of $t$ [7, 8]. The only places of which the authors are aware where $a_n(x_0, \Delta)$ is considered for $\Delta \neq 0$ are in [2] and [6]. These coefficients are useful in considering multi-loop processes [9], which motivates us to pursue them further. The quantum mechanical path integral has proved useful in computing Green’s functions at one-loop order [10–13] and beyond [14–16]; this suggests using this approach to examine $a_n(x_0, \Delta)$ for $\Delta \neq 0$. Although our method is not identical to that of [7], the two approaches are similar and both results agree when $\Delta = 0$.

The representation of

$$\tilde{M}_{xy} = \langle x | \exp \left\{ \frac{1}{2} \left[ \frac{1}{\sqrt{g}} (\partial_\mu - iA_\mu) g^{\mu \nu} \sqrt{g} (\partial_\nu - iA_\nu) \right] t \right\} | y \rangle$$

in terms of a quantum mechanical path integral is not uniquely specified [17–19], as discussed in [14]. We use one of the various forms of the QMPI to expand $\tilde{M}_{xy}$ and compare our results with those of [2]. Furthermore, a partial summation of the DeWitt expansion to obtain the full dependence of $\tilde{M}_{xy}$ on $R$ and $R_{\alpha \beta} \Delta^\alpha \Delta^\beta$ is possible [20].

II Expanding $M_{xy}$

It is possible to represent $M_{xy}$ as a QMPI [21],

$$M_{xy} = \int_x^y Dq(\tau) \mathcal{P} \exp \int_{0}^{t} d\tau \left\{ -\frac{\dot{q}^2(\tau)}{2} + i\dot{q}(\tau) \cdot A(q(\tau)) - V(q(\tau)) \right\}$$
where path-ordered integration is implied over trajectories with end points \(q(0) = y\) and \(q(t) = x\). We attempt to construct a power series about some point \(x_0\) which we arbitrarily choose to be the mid-point between \(x\) and \(y\). Defining the relative coordinate \(\delta\) by

\[
q(\tau) = x_0 + \delta(\tau),
\]

and imposing the Fock-Schwinger gauge condition [22],

\[
\delta(\tau) \cdot A(x_0 + \delta(\tau)) = 0,
\]

one can expand the gauge field in powers of \(\delta\),

\[
A_\mu(x_0 + \delta(\tau)) = \int_0^1 d\alpha \alpha \delta_\lambda(\tau) F_\lambda(\alpha x_0 + \alpha \delta(\tau))
\]

\[
= \sum_{N=0}^\infty \frac{1}{N!(N+2)} [\delta(\tau) \cdot D(x_0)]^N \delta^\lambda(\tau) F_\lambda(x_0).
\]

The scalar potential can be similarly expanded,

\[
V(x_0 + \delta(\tau)) = \sum_{N=0}^\infty \frac{1}{N!(N+2)} [\delta(\tau) \cdot D(x_0)]^N V(x_0).
\]

Here gauge-covariant differentiation at \(x_0\) has been denoted by \(D(x_0)\). Together, (5b) and (6) allow (1a) to be written as

\[
M_{xy} = \int_{-\Delta/2}^{\Delta/2} D\delta(\tau) \exp \left[ - \int_0^t d\tau \frac{\dot{\delta}(\tau)^2}{2} \right] \sum_{L=0}^\infty \frac{1}{L!} \left[ \sum_{N=0}^\infty \frac{1}{N!(N+2)} [\delta(\tau) \cdot D(x_0)]^N \delta^\lambda(\tau) F_\lambda(x_0) - V(x_0) \right]^L.
\]

The path integral in (7) can be evaluated by systematic functional differentiation of the standard result [21, 23, 12]

\[
\int_{-\Delta/2}^{\Delta/2} D\delta(\tau) \exp \left[ - \int_0^t d\tau \frac{\dot{\delta}(\tau)^2}{2} + \gamma(\tau) \cdot \delta(\tau) \right]
\]

\[
= \frac{e^{-\Delta^2/2t}}{(2\pi t)^{D/2}} \exp \left\{ \int_0^t d\tau \left( -\frac{1}{2} + \frac{\tau}{t} \right) \Delta \cdot \gamma(\tau) - \frac{1}{2} \int_0^t d\tau d\tau' G(\tau, \tau') \gamma(\tau) \cdot \gamma(\tau') \right\},
\]

with respect to \(\gamma_\alpha(\tau)\) and then setting \(\gamma = 0\). (Here, \(G(\tau, \tau') \equiv \frac{1}{2}[\tau - \tau'] - \frac{1}{2}(\tau + \tau') + \frac{\tau\tau'}{t}\) is the Green’s function of a free particle on the worldline.) For example, after two such derivatives, it is easily shown that

\[
\int_{-\Delta/2}^{\Delta/2} D\delta(\tau) \delta^\alpha(\tau_\alpha) \delta^\beta(\tau_\beta) \exp \left[ - \int_0^t d\tau \frac{\dot{\delta}(\tau)^2}{2} \right]
\]

\[
= \frac{e^{-g_{\alpha\beta} \Delta^\alpha \Delta^\beta / 2t}}{(2\pi t)^{D/2}} \left[ -G(\tau_\alpha, \tau_\beta) g^{\alpha\beta} + \left( -\frac{1}{2} + \frac{\tau_\alpha}{t} \right) \Delta^\alpha \left( -\frac{1}{2} + \frac{\tau_\beta}{t} \right) \Delta^\beta \right].
\]
\[ a_0(x_0, \Delta) = 1 \]
\[ a_1(x_0, \Delta) = (\Delta \cdot D)^k V, (\Delta \cdot D)^k (\Delta^\alpha D_\beta F_{\alpha \beta}^\beta), (\Delta \cdot D)^k (\Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu}) \]
\[ a_2(x_0, \Delta) = (\Delta \cdot D)^k (D^2 V), (\Delta \cdot D)^k V (\Delta \cdot D)^l V, (\Delta \cdot D)^k V (\Delta \cdot D)^l (\Delta^\alpha D_\beta F_{\alpha \beta}^\beta), (\Delta \cdot D)^k (\Delta^\alpha D_\beta F_{\alpha \beta}^\beta) (\Delta \cdot D)^l V, (\Delta \cdot D)^k (\Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu}) (\Delta \cdot D)^l V, (\Delta \cdot D)^k \left( \Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu} \right) (\Delta \cdot D)^l V, (\Delta \cdot D)^k V (\Delta \cdot D)^l (\Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu}), (\Delta \cdot D)^k (\Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu}) (\Delta \cdot D)^l F_{\alpha \beta}, (\Delta \cdot D)^k (\Delta^\alpha D_\beta F_{\alpha \beta}) (\Delta \cdot D)^l (\Delta^\alpha D_\beta F_{\alpha \beta}), (\Delta \cdot D)^k (\Delta^\alpha D_\beta F_{\alpha \beta}) (\Delta \cdot D)^l \left( \Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu} \right), (\Delta \cdot D)^k \left( \Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu} \right) (\Delta \cdot D)^l \left( \Delta^\alpha D_\beta F_{\alpha \beta} \right), (\Delta \cdot D)^k \left( \Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu} \right) (\Delta \cdot D)^l \left( \Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu} \right) \]

(All fields and covariant derivatives are evaluated at \( x_0 \).)

\[ (k, l = 0, 1, 2, \cdots) \]

| Table 1: Possible contributions to the various coefficients of the DeWitt expansion. |

From (8), it is easily seen that no term in (7) will involve factors of \( \Delta^2 \). From this observation, combined with simple power counting arguments, it is straightforward to tabulate the possible contributions to the various coefficients \( a_n(x_0, \Delta) \) (see table 1). (When a temperature dependent QMPI is considered as in [12], then the temperature provides a second dimensionful parameter that must be considered.) The coefficients of the various contributions can be easily determined by appropriately choosing \( L \) and \( N \) in (7) and then systematically applying (8). For example, if \( L = n, N = 0 \), then it is apparent that the contribution to \( a_n(x_0, \Delta) \) proportional to \( V^n(x_0) \) is \( \frac{1}{n!}(-V(x_0))^n \). By setting \( L = N = 1 \) in (7), we find after a very short calculation that the contribution to \( a_1(x_0, \Delta) \) proportional to \( \Delta^\alpha D_\beta F_{\alpha \beta} \) is \( -\frac{i}{12} \Delta^\alpha D_\beta F_{\alpha \beta} \). With \( L = 2, N = 0 \), the contribution to \( M_{xy} \) is straightforwardly computed to be

\[ M^{(2,0)}_{xy} = \frac{e^{-\Delta^2/2t}}{(2\pi t)^{D/2}} \left( -\frac{t}{24} \Delta^\alpha \Delta^\beta F_{\alpha \mu} F_{\beta \mu} - \frac{t^2}{48} F_{\alpha \beta} F_{\alpha \beta} \right), \]

giving a contribution to both \( a_1(x_0, \Delta) \) and \( a_2(x_0, \Delta) \). These results are all consistent with the flat space limit of the expressions for \( a_n(x_0, \Delta) \) giving in [2].

### III Expanding \( \tilde{M}_{xy} \)

As has been noted in the introduction and in [14], there are various representations of the matrix element

\[ \tilde{M}_{xy} = \langle x | \exp \left\{ \frac{1}{2} \left[ \frac{1}{\sqrt{g}} (\partial_\mu - iA_\mu) g^{\mu\nu} \sqrt{g} (\partial_\nu - iA_\nu) \right] t \right\} | y \rangle \] (10)
Following the developments of [26], we have done in [19]. Upon introducing a real Bosonic field it turns out that contributions from [27], goes to zero. We are dealing with a model in which no regularization is required; indeed it is usually discarded as it gives a contribution to the effective action that is proportional to the parameter \( p \) appearing there.) This representation gives a dependence of \( \tilde{M}_{xy} \) on \( R \) that coincides with that of [20, 2], but does not fully agree with the results of [2]. The full dependence of \( \tilde{M}_{xy} \) on \( R_{\alpha \beta} \Delta^\alpha \Delta^\beta \) can also be determined.

We are now faced with evaluating

\[
\tilde{M}_{xy} = \int_y^x Dq^\alpha(\tau) \sqrt{g(q(\tau))} \mathcal{P} \exp \left\{ - \int_0^t d\tau \left[ \frac{1}{2} g_{\mu \nu}(q(\tau)) \dot{q}^\mu(\tau) \dot{q}^\nu(\tau) - i \dot{q}^\alpha(\tau) A_\alpha(q(\tau)) + \frac{1}{8} R(q(\tau)) \right] \right\} .
\]

(11)

The factor \( \sqrt{g(q(\tau))} \) in the measure also occurs in the non-linear sigma model [25], but there it is usually discarded as it gives a contribution to the effective action that is proportional to \( \delta(0) \) which, when regulated using dimensional regularization [26] or operator regularization [27], goes to zero. We are dealing with a model in which no regularization is required; indeed it turns out that contributions from \( \sqrt{g} \) are essential to render the path integral in (11) well-defined. It is most convenient to incorporate the effects of \( \sqrt{g} \) by using ghosts as was done in [19]. Upon introducing a real Bosonic field \( b^\alpha(\tau) \) and a pair of Fermionic fields \( \overline{\pi}^\alpha(\tau) \) and \( c^\alpha(\tau) \), all of which vanish at the end points \( (\tau = 0, t) \), (11) can be re-expressed as

\[
\tilde{M}_{xy} = \int_y^x Dq^\alpha(\tau) \int_0^0 Db^\alpha(\tau) \int_0^0 Dc^\alpha(\tau) D\overline{\pi}^\alpha(\tau)
\]

\[
\mathcal{P} \exp \left\{ - \int_0^t d\tau \left[ g_{\mu \nu}(q(\tau)) \left( \frac{1}{2} \dot{q}^\mu(\tau) \dot{q}^\nu(\tau) + \frac{1}{2} b^\mu(\tau) b^\nu(\tau) + \overline{\pi}^\mu(\tau) c^\nu(\tau) \right) - i \dot{q}^\alpha(\tau) A_\alpha(q(\tau)) + \frac{1}{8} R(q(\tau)) \right] \right\} .
\]

(12)

A normal coordinate expansion [28] is now made about a point \( \phi(\tau) \) so that

\[
q^\alpha(\tau) = \phi^\alpha(\tau) + \pi^\alpha(\xi(\tau)).
\]

(13)

Following the developments of [26], we have

\[
R(q(\tau)) = R(\phi(\tau)) + \frac{1}{11!} R_{\alpha \beta}(\phi(\tau)) \xi^\alpha(\tau) + \frac{1}{2!} R_{\alpha \beta \gamma}(\phi(\tau)) \xi^\alpha(\tau) \xi^\beta(\tau) + \cdots ,
\]

(14a)

\[
g_{\mu \nu}(q(\tau)) = g_{\mu \nu}(\phi(\tau)) - \frac{1}{3} R^{\alpha \beta \gamma}(\phi(\tau)) \xi^\alpha(\tau) \xi^\beta(\tau) \xi^\gamma(\tau)
\]

\[
- \frac{1}{6} R_{\mu \nu \beta \gamma}(\phi(\tau)) \xi^\alpha(\tau) \xi^\beta(\tau) \xi^\gamma(\tau) + \left( -\frac{1}{20} R_{\mu \nu \beta \gamma \delta}(\phi(\tau)) + \frac{2}{45} R_{\alpha \mu \beta \sigma}(\phi(\tau)) R^{\alpha \beta \gamma \delta}(\phi(\tau)) \right) \xi^\alpha(\tau) \xi^\beta(\tau) \xi^\gamma(\tau) \xi^\delta(\tau) + \cdots ,
\]

(14b)
\[ \dot{q}^\mu(\tau) = \dot{\phi}^\mu(\tau) + D_\tau \xi^\mu(\tau) + \frac{1}{3} R^\mu_{\alpha\beta\gamma}(\phi(\tau)) \xi^\alpha(\tau) \xi^\beta(\tau) \dot{\phi}^\gamma(\tau) + \cdots, \quad (14c) \]

\[ (D_\tau \xi^\mu(\tau) \equiv \dot{\xi}^\mu(\tau) + \Gamma^\mu_{\beta\gamma}(\tau) \dot{\phi}^\beta(\tau) + \cdots) . \]

As shown in appendix A, by imposing a gauge condition analogous to (4), one finds that the corresponding normal coordinate expansion for the gauge field is (A.3)

\[ A_\mu(q(\tau)) = \frac{1}{2} F_{\alpha\mu}(\phi(\tau)) \xi^\alpha(\tau) + \frac{1}{3} F_{\alpha\mu\beta}(\phi(\tau)) \xi^\alpha(\tau) \xi^\beta(\tau) + \cdots \quad (14d) \]

\[ + \left( \frac{1}{8} F_{\alpha\mu;\beta\gamma}(\phi(\tau)) + \frac{1}{24} R^\alpha_{\beta\gamma\mu}(\phi(\tau)) F_{\alpha\sigma}(\phi(\tau)) \right) \xi^\alpha(\tau) \xi^\beta(\tau) \xi^\gamma(\tau) \]

\[ + \left( \frac{1}{30} F_{\alpha\mu;\beta\gamma\delta}(\phi(\tau)) + \frac{1}{60} R^\alpha_{\beta\gamma\mu;\delta}(\phi(\tau)) F_{\alpha\sigma}(\phi(\tau)) \right) \xi^\alpha(\tau) \xi^\beta(\tau) \xi^\gamma(\tau) \xi^\delta(\tau) + \cdots. \]

If we take \( \phi(\tau) \) to be the geodesic mid-point, \( x_0 \), between \( x \) and \( y \), then \( \dot{\phi}^\alpha(\tau) \) vanishes and the above expansions simplify a bit. Letting \( \Delta \) denote the difference between the normal coordinates of \( x \) and \( y \), equation (12) becomes

\[ \tilde{M}_{xy} = \int_{-\Delta/2}^{\Delta/2} D\xi^\alpha(\tau) \int_0^t Db^\alpha(\tau) \int_0^t Dc^\alpha(\tau) D\xi^\alpha(\tau) e^{-Rt/8} \]

\[ \times \exp \left\{ \int_0^t d\tau \left[ -g_{\mu\nu} \left( \frac{1}{2} \xi^\mu(\tau) \dot{\xi}^\nu(\tau) + \frac{1}{2} b^\mu(\tau) b^\nu(\tau) + \bar{c}^\mu(\tau) c^\nu(\tau) \right) \right] \right\} \]

\[ \times \mathcal{P} \sum_{N=0}^\infty \frac{1}{N!} \left\{ -\int_0^t d\tau \left[ \left( -\frac{1}{3} R_{\mu\nu\alpha\beta}(\tau) \xi^\alpha(\tau) \xi^\beta(\tau) \xi^\gamma(\tau) + \cdots \right) \right. \right. \]

\[ \times \left( \frac{1}{2} \dot{\xi}^\mu(\tau) \dot{\xi}^\nu(\tau) + \frac{1}{2} b^\mu(\tau) b^\nu(\tau) + \bar{c}^\mu(\tau) c^\nu(\tau) \right) \]

\[ \left. \left. -i \left( \frac{1}{2} F_{\alpha\mu}(\tau) + \frac{1}{3} F_{\alpha\mu\beta}(\tau) \xi^\beta(\tau) + \cdots \right) \xi^\alpha(\tau) \right] \right\}^N . \]

(All geometrical and gauge quantities in (15) are evaluated at \( x_0 \).)

The standard results

\[ \int_{-\Delta/2}^{\Delta/2} D\xi^\alpha(\tau) \exp \left\{ \int_0^t d\tau \left( -\frac{1}{2} g_{\mu\nu} \dot{\xi}^\mu(\tau) \dot{\xi}^\nu(\tau) + g_{\mu\nu} \xi^\mu(\tau) \gamma^\nu(\tau) \right) \right\} \quad (16a) \]

\[ = \frac{e^{-g_{\mu\nu} \Delta^\mu \Delta^\nu / 2t}}{(2\pi t)^{D/2} \sqrt{g}} \exp \left\{ \int_0^t d\tau \left( -\frac{1}{2} + \frac{\tau}{t} \right) g_{\mu\nu} \Delta^\mu \gamma^\nu(\tau) - \frac{1}{2} \int_0^t d\tau d\tau' G(\tau, \tau') g_{\mu\nu} \gamma^\mu(\tau) \gamma^\nu(\tau') \right\} , \]

\[ \int_0^t Db^\alpha(\tau) \exp \left\{ \int_0^t d\tau \left( -\frac{1}{2} g_{\mu\nu} b^\mu(\tau) b^\nu(\tau) + g_{\mu\nu} b^\mu(\tau) B^\nu(\tau) \right) \right\} \quad (16b) \]

\[ = \frac{1}{\sqrt{g}} \exp \left\{ \frac{1}{2} \int_0^t d\tau d\tau' \delta(\tau - \tau') g_{\mu\nu} B^\mu(\tau) B^\nu(\tau') \right\} \bigg|_{B(0) = B(t) = 0} . \]
and,
\[ \int_0^\tau Dc^\alpha(\tau)\bar{D}c^\alpha(\tau) \exp \left\{ \int_0^\tau d\tau \left( -g_{\mu\nu}c^\mu(\tau)c^\nu(\tau) + g_{\mu\nu}(\eta^\mu(\tau)c^\nu(\tau) + \bar{c}^\mu(\tau)\eta^\nu(\tau)) \right) \right\} \]
\[ = g \exp \left\{ \int_0^\tau d\tau d\tau' \delta(\tau - \tau') g_{\mu\nu}\bar{c}^\mu(\tau)\eta^\nu(\tau') \right\} \eta=\bar{\eta}=0 \text{ at } \tau=t, 0, \]
permit one to compute the functional integrals appearing in (15).

For example, if we restrict our attention to the contribution to (15) that is linear in \( R_{\mu\nu\beta} \), we have
\[ \tilde{M}^R_{xy} \left( \frac{1}{24} \right) = \int_{-\Delta/2}^{\Delta/2} D\xi(\tau) \int_{0}^{\tau} D\tilde{c}^\alpha(\tau) \int_{0}^{\tau} Dc^\alpha(\tau) D\bar{c}^\alpha(\tau) e^{-Rt/8} \]
\[ \exp \left\{ \int_0^\tau d\tau \left[ -g_{\mu\nu} \left( \frac{1}{2} \hat{\xi}^\mu(\tau)\hat{\xi}^\nu(\tau) + \frac{1}{2} b^\mu(\tau)b^\nu(\tau) + \bar{c}^\mu(\tau)c^\nu(\tau) \right) \right] \right\} \]
\[ - \left( \frac{1}{3} R_{\mu\alpha\beta} \right) \int_0^\tau d\tau \xi^\alpha(\tau) \xi^\beta(\tau) \left( \frac{1}{2} \hat{\xi}^\mu(\tau)\hat{\xi}^\nu(\tau) + \frac{1}{2} b^\mu(\tau)b^\nu(\tau) + \bar{c}^\mu(\tau)c^\nu(\tau) \right) \]
\[ = \frac{-e^{-g_{\mu\nu}\Delta^\mu \Delta^\nu/2t}}{(2\pi t)^{D/2}} e^{-Rt/8} \int_0^t d\tau \left\{ \left( G^\mu_{\alpha\beta} G(\tau, \tau) + \frac{1}{2} g^\mu_{\alpha\beta} G(\tau, \tau) + \frac{1}{2} g^\mu_{\alpha\beta} G(\tau, \tau) \right) \left( \frac{1}{2} g^\mu_{\alpha\beta} G(\tau, \tau) \right) \right\} \left( \frac{1}{3} R_{\mu\alpha\beta} \right). \] (17)
The explicit form of \( G(\tau, \tau') \) when substituted into (17) leaves us with
\[ \tilde{M}^R_{xy} = \frac{e^{-g_{\mu\nu}\Delta^\mu \Delta^\nu/2t}}{(2\pi t)^{D/2}} \left( -\frac{1}{6} \right) e^{-Rt/8} \int_0^t d\tau \left\{ R \left[ \frac{1}{t} \left( -\tau + \frac{\tau^2}{t} \right) \right] - \left( \frac{1}{2} \frac{\tau}{t} \right)^2 \right\} \]
\[ + R_{\alpha\beta}\Delta^\alpha \Delta^\beta \left[ \frac{1}{t^2} \left( -\tau + \frac{\tau^2}{t} \right) + \frac{1}{t} \left( \frac{1}{2} \frac{\tau}{t} \right)^2 \right] \]
\[ = \frac{e^{-g_{\mu\nu}\Delta^\mu \Delta^\nu/2t}}{(2\pi t)^{D/2}} \frac{-R_{\alpha\beta}\Delta^\alpha \Delta^\beta + R t}{24}. \] (18)
(All dependence on \( \delta(0) \) in (17) cancels out due to the compensating contributions from the ghost-fields.) It is easily seen that the entire dependence of \( \tilde{M}_{xy} \) on \( R \) and \( R_{\alpha\beta}\Delta^\alpha \Delta^\beta \) is given by
\[ \tilde{M}_{xy}^{(R,\Delta R, \Delta \Delta)} = \frac{e^{-g_{\mu\nu}\Delta^\mu \Delta^\nu/2t}}{(2\pi t)^{D/2}} \sum_{N=0}^{\infty} \frac{e^{-Rt/8}}{N!} \left[ \frac{-R_{\alpha\beta}\Delta^\alpha \Delta^\beta + R t}{24} \right]^N. \]
\[
\frac{e^{-\frac{g_{\mu\nu}\Delta^\alpha\Delta^\beta}{(2\pi t)^{D/2}}}}{(2\pi t)^{D/2}} \exp \left[ -\frac{Rt}{12} - \frac{R_{\alpha\beta}\Delta^\alpha\Delta^\beta}{24} \right].
\]

(19)

The dependence of \(\tilde{M}_{xy}\) on \(R\) in (19) agrees with that of [20, 2, 19] (once the different normalization of \(t\) is taken into account); however the dependence of \(R_{\alpha\beta}\Delta^\alpha\Delta^\beta\) is different from that in [2] while agreeing with [19].

A completely analogous calculation can be used to fix the lowest order dependence on \(R_{\alpha\beta\gamma}\Delta^\alpha\Delta^\beta\Delta^\gamma\); it is found to vanish. This does appear to be consistent with the results of [2] (where the coefficient \(R_{\alpha\beta\gamma}\) is evaluated at the end-point \(y\), instead of the mid-point \(x_0\)). Further terms in the DeWitt expansion of \(\tilde{M}_{xy}\) can be similarly determined.

The techniques used in this section may be employed to find the effective action for a particle moving in a gravitational field [29]. This involves taking \(\phi^\alpha(\tau)\) to be arbitrary in (13) rather than restricting it to be \(x_0\).

IV Discussion

In the preceding sections we have considered how the QMPI can be used to determine the elements of the DeWitt expansion for the heat kernel both on and off the diagonal. This technique is seen to be easier to use than the original approach [6] in which the heat equation was solved perturbatively. By employing the off-diagonal elements, calculations can be done to two-loop order [9]. The method works in both flat and curved space.

Because of the simplicity of the proper-time (\(\tau\)) integrands which arise when implementing this method, we expect that high-order covariant expansions could be computerized using presently available symbolic algebra packages. This approach has already been employed in the diagonal case [7] where derivatives of equation (8) reduce to sums of Wick contractions. In the off-diagonal case, surface terms (proportional to \(\Delta\)) introduce an additional combinatorical consideration which should be tractable.

Finally, it is worth noting, that although our treatment of this problem focused specifically on the case where the vector potential \(A_\mu\) was a gauge field, this is unnecessarily restrictive and the methods presented here would work equally well for a general vector coupling to a background field. For example, if one were interested in radiative corrections to Fermionic Green’s functions in flat-space QED, the following quantum mechanical operator would be of interest [30]

\[
\begin{pmatrix}
  p^2 g_{\mu\nu} - \left(1 - \frac{1}{a}\right)p_\mu p_\nu & -\frac{\gamma_\mu \psi}{T} & \bar{\psi} \gamma_\mu \\
  \frac{\gamma_\nu \psi}{T} & 0 & \hat{p} - m \\
  -\frac{\bar{\psi} \gamma_\nu}{T} & -(\hat{p} - m)^T & 0 \\
\end{pmatrix}_{(\mu,\nu)}.
\]

(20)

By factoring in the constant supermatrix

\[
\begin{pmatrix}
  g_{\nu\lambda} & 0 & 0 \\
  0 & 0 & (\hat{p} + m)^T \\
  0 & -(\hat{p} + m) & 0 \\
\end{pmatrix}_{(\nu,\lambda)}
\]

8
and choosing the $a = 1$ gauge, operator (20) becomes
\[
\begin{pmatrix}
p^2 g_{\mu\lambda} & -\bar{\psi} \gamma_\mu (\dot{\phi} + m) & -\bar{\gamma}_\mu \psi^T (\dot{\phi} + m)^T \\
(\bar{\gamma}_\lambda \psi) & p^2 + m^2 & 0 \\
-\bar{\psi} \gamma_\lambda & 0 & p^2 + m^2 \\
\end{pmatrix}
\]_{(\mu, \lambda)}.
\]

(21)

After completing the square of $p$ (and noting that $p^T = -p$ in the coordinate-space representation), the heat kernel of this operator is easily shown to have the form of equation (1a) with vector potential,
\[
A_\nu(q) = \frac{1}{2} \begin{pmatrix}
0 & \bar{\psi}(q) \gamma_\mu \gamma_\nu & -\gamma_\nu \gamma_\mu \psi(q)^T \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]_{(\mu, \lambda)},
\]

(22)

and scalar potential,
\[
V(q) = \begin{pmatrix}
0 & \bar{\psi}(q) \gamma_\mu \left(\frac{i}{2} \dot{\phi} - m\right) & \left[\left(\frac{i}{2} \dot{\phi} - m\right) \gamma_\mu \psi(q)\right]^T \\
(\gamma_\lambda \psi(q)) & m^2 & 0 \\
-\bar{\psi}(q) \gamma_\lambda & 0 & m^2 \\
\end{pmatrix}
\]_{(\mu, \lambda)}.
\]

(23)

After the appropriate Taylor expansions of these potentials is substituted into eq. (2) the method should proceed in the obvious way.

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Appendix A. Normal Coordinate Expansion of the Gauge Field

In this appendix we discuss the construction of a gauge-covariant normal coordinate expansion for the gauge potential.

By analogy with the flat-space case discussed briefly in section II and in refs. [11, 7], the appropriate gauge condition for this expansion is the synchronous gauge [31] (a curved-space generalization of the Fock-Schwinger gauge (4) [22]) which fits very well in the normal coordinate construction. In the basis of the normal coordinate system, the gauge condition is
\[
\xi^\alpha A_\alpha (\phi + \pi(\xi)) = 0.
\]

(A.1)
Either by integrating along the geodesics (which is formally identical to equation (5a)) or by using differential forms [31] one can show, in the normal coordinate system, that the synchronous gauge leads to a gauge-covariant expansion for the vector potential which looks exactly like equation (5b) with the gauge-covariant normal coordinate derivative \( D_\alpha = \frac{\partial}{\partial \xi^\alpha} + [A_\alpha, \ldots] \). The latter derivative is not covariant under reparametrization of the manifold, however using the methods of reference [26] it is straightforward to write such normal coordinate derivatives at the origin in terms of the corresponding fully-covariant derivatives, denoted by indices trailing the semicolon (;). For example, one can show that \(^1\)

\[
D_{\beta_1} F_{\beta_0 \gamma} = F_{\beta_0 \gamma ; \beta_1}
\]  
(A.2a)

\[
D_{\beta_2} D_{\beta_1} F_{\beta_0 \gamma} = F_{\beta_0 \gamma ; \beta_1 \beta_2} + \frac{1}{3} R^\delta_{\beta_1 \beta_2 \gamma} F_{\beta_0 \delta}
\]  
(A.2b)

\[
D_{\beta_3} D_{\beta_2} D_{\beta_1} F_{\beta_0 \gamma} = F_{\beta_0 \gamma ; \beta_1 \beta_2 \beta_3} + \frac{1}{2} R^\delta_{\beta_1 \beta_2 \gamma \beta_3} F_{\beta_0 \delta} + R^\delta_{\beta_1 \beta_2 \gamma} F_{\beta_0 \delta ; \beta_3}
\]  
(A.2c)

\[
D_{\beta_4} D_{\beta_3} D_{\beta_2} D_{\beta_1} F_{\beta_0 \gamma} = F_{\beta_0 \gamma ; \beta_1 \beta_2 \beta_3 \beta_4} + \frac{3}{5} R^\delta_{\beta_1 \beta_2 \gamma ; \beta_3 \beta_4} F_{\beta_0 \delta} + 2 R^\delta_{\beta_1 \beta_2 \gamma} F_{\beta_0 \delta ; \beta_3 \beta_4} + 2 R^\delta_{\beta_1 \beta_2 \gamma} F_{\beta_0 \delta ; \beta_3 \beta_4} + \frac{1}{3} R^\delta_{\beta_1 \beta_2 \gamma} R^\epsilon_{\beta_3 \beta_4 \delta} F_{\beta_0 \epsilon}
\]  
(A.2d)

where \( \approx \) indicates equality at the origin only after symmetrization of the \( \beta_i \) indices. Substitution of equations (A.2a)-(A.2d) into eq. (5b) yields the fully covariant normal coordinate expansion to fifth-order in the normal coordinates,

\[
A_\gamma (\phi + \pi (\xi)) = \frac{1}{2} \left\{ F_{\beta \gamma} \right\} \xi^\beta + \frac{1}{3} \left\{ F_{\beta_0 \gamma ; \beta_1} \right\} \xi^{\beta_0} \xi^{\beta_1} + \frac{1}{8} \left\{ F_{\beta_0 \gamma ; \beta_1 \beta_2} + \frac{1}{3} R^\delta_{\beta_1 \beta_2 \gamma} F_{\beta_0 \delta} \right\} \xi^{\beta_0} \xi^{\beta_1} \xi^{\beta_2}
\]

\[
+ \frac{1}{315} \left\{ F_{\beta_0 \gamma ; \beta_1 \beta_2 \beta_3} + \frac{1}{2} R^\delta_{\beta_1 \beta_2 \gamma ; \beta_3} F_{\beta_0 \delta} + R^\delta_{\beta_1 \beta_2 \gamma} F_{\beta_0 \delta ; \beta_3} \right\} \xi^{\beta_0} \xi^{\beta_1} \xi^{\beta_2} \xi^{\beta_3}
\]

\[
+ \frac{1}{416} \left\{ F_{\beta_0 \gamma ; \beta_1 \beta_2 \beta_3 \beta_4} + \frac{3}{5} R^\delta_{\beta_1 \beta_2 \gamma ; \beta_3 \beta_4} F_{\beta_0 \delta} + 2 R^\delta_{\beta_1 \beta_2 \gamma} F_{\beta_0 \delta ; \beta_3 \beta_4} + 2 R^\delta_{\beta_1 \beta_2 \gamma} F_{\beta_0 \delta ; \beta_3 \beta_4} + \frac{1}{3} R^\delta_{\beta_1 \beta_2 \gamma} R^\epsilon_{\beta_3 \beta_4 \delta} F_{\beta_0 \epsilon} \right\} \xi^{\beta_0} \xi^{\beta_1} \xi^{\beta_2} \xi^{\beta_3} \xi^{\beta_4}
\]

\[+ O (\xi^6) \, .\]

All coefficients in braces \( \{ \ldots \} \) are evaluated at the origin, where the basis vectors for the normal coordinate system coincide with those of the original system. Since potential on the left hand side of this equation is not a vector at the origin, its indices must refer to the normal coordinate basis. (This is also true of equations (14a)-(14c)). The results of (A.3) agree with those of [2] to order \( O (\xi^3) \).

\(^1\)The authors suspect that the fourth derivative of a rank-two tensor implied in reference [26] is not entirely correct. The corresponding coefficients presented here for the field strength, equation (A.2d), have been verified independently.
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