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On Triple Focusing Dipole Magnets

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Abstract

We present a general, paraxial study of triple focusing (i.e., stigmatic and non-dispersive) index-free dipole magnets. The transcendental equations which describe such magnets lead to a second-degree polynomial equation which we solve analytically. The two solutions of this equation correspond to magnets having either one or no intermediate focal points in the vertical direction. The first-order optical properties of the physical solutions are studied.

1 Introduction

After the focusing properties of inclined entrance and exit faces of dipole magnets were discovered, the possibility of designing double focusing (DF, i.e., stigmatic) deflecting magnets became apparent [1, 2].

In general, these magnets also disperse the beam; that is, particles with different momenta entering on a common trajectory follow different trajectories after leaving the magnet. It is obvious that the dispersion increases inside a magnet up to a certain deflection angle of the particles, and then decreases. Therefore, it is possible to design triple focusing (TF, i.e., stigmatic and non-dispersive) dipole magnets.

Several examples of such magnets have already appeared in the literature [3, 4, 5, 6]. In each case, the choice of a particular configuration has led to a simple solution of the equations that define TF magnets. The magnet described in [3, 4] has a deflection angle of 270° and a parallel incident beam¹. In [6], a solution is obtained in the particular case of symmetric magnets. In [5], a solution is obtained through a numerical computation. No complete study has been presented yet, as “*the transcendental equations are quite unwieldy*” [5].

More recently, interest in TF magnets has been renewed for their use in bunch compression [7, 8], since the time of flight through the magnet depends on the momentum of the incident particle.

TF magnets are also useful when the path length of the particles must be minimized. The present work has, in fact, been triggered by the need for non-dispersive and stigmatic injection of an electron beam into a gas vessel for the plasma wave acceleration experiment at Ecole Polytechnique [9]. In this experiment, the use of a combination of three magnets would have led to a long path in the gas and thus to an intolerable scattering of the injected electrons before they reach the plasma.

In this paper, we present the first general study of triple focusing, dipole (zero index) single magnets. We use Enge’s paraxial formalism in the approximation of a small air gap² with the same notation as in [10, 11].

After a short presentation of this formalism, we write down the system of three equations which defines a TF magnet. In section 3, we solve this system in five variables analytically; elimination of two of the variables leads to a second-degree polynomial equation. We then derive the condition of existence of physical solutions. In section 4, we study the asymptotic behaviour of the solutions in several limiting cases. We discuss the optical properties of the magnets in section 5, where we also present solutions for

¹In this paper, angles are mixedly given in degrees and radians depending on the context.

²The air gap is small compared to the length of the particle trajectory inside the magnetic field.

special cases. In section 6, we address the practical question of the Triple Focusing use of a magnet with given pole angle.

2 Paraxial description of TF magnets

A dipole magnet is usually described by the following parameters [10]: the straight drift length A (B) before (after) the magnet, the radius of curvature R of the particles, the angle α (β) of the central ray with respect to the normal on the entrance (exit) face of the dipole, and the deflection angle φ (fig. 1). All quantities of dimension length can be normalized to the radius of curvature, in which case they are denoted by a lower case letter, e.g. $a = A/R$. The normalized drift lengths are called *arms* here. Hence, the set $(a, b, \varphi, \alpha, \beta)$ defines a dipole magnet.

This parametrisation might seem counter-intuitive. In practice, one would rather position a source with a given beam direction in front of a magnet with given dipole angle ω , thus fixing a and α . The deflection and exit angles φ and β would then be determined by ω , a , and α and the position of the exit face of the magnet. On the contrary, it appears here as if one designs the magnet around a given central trajectory.

Let (O, x, y, z) be the moving frame of the central ray, in which (Ox) is the tangent, (Oy) the normal, (Oz) the bi-normal, and s is the curvilinear coordinate (fig. 2). The plane (Oxy) is called the median or horizontal plane of the magnet, and the direction of the magnetic field (Oz) is called the vertical direction. The angles are written $y' = dy/dx$, and the momentum deviation $\Delta p/p$ is denoted δ . In the paraxial approximation, the particle coordinates in phase space $(y_2, y'_2, z_2, z'_2, \delta)$ at $s = s_2$ are linear functions of the entrance coordinates in phase space at $s = s_1$. The elements of the transfer matrices are written in short form, e.g. $(y|y) = dy_2/dy_1$.

In this notation, a TF magnet is described by the system of equations:

$$(y|y') = 0, \quad (z|z') = 0, \quad (y|\delta) = 0, \quad (1)$$

where each matrix element depends on the five parameters defining the dipole.

The horizontal and vertical transfer matrices for transport inside the magnet, $\mathbf{H}$ and $\mathbf{V}$, are defined as:

$$\begin{pmatrix} y_2 \\ y'_2 \\ \delta \end{pmatrix} = \mathbf{H} \begin{pmatrix} y_1 \\ y'_1 \\ \delta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_2 \\ z'_2 \end{pmatrix} = \mathbf{V} \begin{pmatrix} z_1 \\ z'_1 \end{pmatrix}$$

and their explicit form is given in [10]:

$$\mathbf{H} = \begin{pmatrix} \frac{\cos(\varphi - \alpha)}{\cos \alpha} & \sin \varphi & 1 - \cos \varphi \\ -\frac{\sin(\varphi - \alpha - \beta)}{\cos \alpha \cos \beta} & \frac{\cos(\varphi - \beta)}{\cos \beta} & \sin \varphi + (1 - \cos \varphi) \tan \beta \\ 0 & 0 & 1 \end{pmatrix}$$

and:

$$\mathbf{V} = \begin{pmatrix} 1 - \varphi \tan \alpha & \varphi \\ -\tan \alpha - \tan \beta + \varphi \tan \alpha \tan \beta & 1 - \varphi \tan \beta \end{pmatrix}$$

The transfer matrices from the object to the image are simply obtained as the product of the matrices of the drift length b , of the magnet, and of the drift length a . Using the notations $t_\alpha = \tan \alpha$ and $t_\beta = \tan \beta$, the system of equations (1) then reads:

$$0 = (y|\delta) = 1 - \cos \varphi + b \sin \varphi + b(1 - \cos \varphi)t_\beta \quad (2a)$$

$$0 = (z|z') = \varphi + (1 - t_\alpha \varphi)a + (1 - t_\beta \varphi)b + (t_\alpha t_\beta \varphi - t_\alpha - t_\beta)ab \quad (2b)$$

$$0 = (y|y') = \sin \varphi + (\cos \varphi + t_\alpha \sin \varphi)a + (\cos \varphi + t_\beta \sin \varphi)b + ((t_\alpha t_\beta - 1) \sin \varphi - (t_\alpha + t_\beta) \cos \varphi)ab \quad (2c)$$

This system of equations fixes three of the five parameters which describe the dipole.

3 Solution of the system of equations

The system of equations (2) is transcendental only in φ and otherwise linear in a , b , t_α , and t_β . Therefore, we found it convenient to solve the system of equations (2) for the entrance and exit angles α and β .

First, we solve (2a) for t_β :

$$t_\beta = - \left(\frac{1}{b} + \cot \left(\frac{\varphi}{2} \right) \right) \quad (3)$$

Next, we substitute the expression obtained for t_β in (2b) and solve for b :

$$b = -2 \frac{a + (1 - at_\alpha)\varphi}{(a + (1 - at_\alpha)\varphi) \cot \left(\frac{\varphi}{2} \right) + 1 - at_\alpha} \quad (4)$$

Then, we insert (3) and (4) into (2c) and solve for t_α :

$$t_\alpha = - \left(\frac{1}{a} + \cot \left(\frac{\varphi}{2} \right) \right) \quad (5)$$

Finally, we substitute the expression obtained for t_α in (4):

$$ab \cot \left(\frac{\varphi}{2} \right) (\varphi \cot \left(\frac{\varphi}{2} \right) + 2) + 2(a + b)(\varphi \cot \left(\frac{\varphi}{2} \right) + 1) + 4\varphi = 0 \quad (6)$$

The system of equations [3,5,6] is explicitly symmetric in entrance and exit, i.e., invariant under the exchange $(a, \alpha) \leftrightarrow (b, \beta)$, obviously because of the symmetry under time reversal.

Introducing the parameter $\kappa = \varphi \cot \left(\frac{\varphi}{2} \right)$ and the arm ratio $g = b/a$, and substituting $u = a/\varphi$, we re-write equation (6) in a more compact form:

$$g\kappa(\kappa + 2)u^2 + 2(g + 1)(\kappa + 1)u + 4 = 0 \quad (7)$$

This second-degree polynomial equation in u has two solutions which depend on the two free parameters g and κ . Figure 3 shows κ as a function of φ . The angle φ obviously varies between 0 and 2π , hence, $\kappa < 2$.

We are looking for solutions of equation (7) describing physical magnets, i.e., magnets with positive arms³ and therefore $g > 0$, and u real and positive. The ratio

³This corresponds to an object and an image outside the magnetic field.

g being positive, the discriminant of equation (7) $\Delta' = (\kappa + 1)^2(g - 1)^2 + 4g$ is always positive. Therefore, equation (7) always has two real solutions u_+ and u_- . In order to study their sign, we form their product p and their sum s from the coefficients of equation (7):

$$p = \frac{4}{g\kappa(\kappa + 2)} \quad s = -\frac{2(g + 1)}{g} \cdot \frac{(\kappa + 1)}{\kappa(\kappa + 2)} \quad (8)$$

We give the sign of p and s , and thus of u_+ and u_- in the following table:

κ	$-\infty$	-2	-1	0	2
φ	360°	232.5°		180°	0
p	$+$	$-$	$+$	$+$	$-$
s	$+$	$-$	$+$	$-$	$+$
branch {0}	$u_+ > 0$	$u_+ > 0$	$u_+ > 0$	$u_+ < 0$	$u_+ < 0$
branch {1}	$u_- > 0$	$u_- < 0$	$u_- < 0$	$u_- < 0$	$u_- < 0$

Physical solutions only exist in the interval $\pi < \varphi < 2\pi$, corresponding to negative values of κ . We obtain exactly one physical solution in the interval $\pi < \varphi < \varphi_0$, where φ_0 is defined by $\varphi_0 \cot(\frac{\varphi_0}{2}) = \kappa_0 = -2$, that is, $\varphi_0 \approx 232.5^\circ$, and two physical solutions in the interval $\varphi_0 < \varphi < 2\pi$.

The two branches of solutions correspond respectively to magnets having either no or exactly one intermediate focal point in the vertical direction, and are therefore denoted {0} and {1}.

Example magnet and general properties

Figure 4 shows an example ray-tracing for parameters φ and g , for which two solutions of equation (7) exist.

One observes that the arms are longer on branch {1} than on branch {0}. It should be noted that all these magnets also have an intermediate focal point in the horizontal plane. As the dispersion at this point is large, momentum selection may be done easily using collimating slits.

We calculate the parameters a , b , α , β , and the dipole angle ω analytically as functions of φ and g using equations (3), (5), and (7), and present them in figure 5. The symmetry between entrance and exit appears clearly in figure 5a. In particular, the dipole angle $\omega = \varphi + \alpha + \beta$ is obviously symmetric under exchange of g and $1/g$.

Figure 5b shows the above parameters as functions of φ for arm ratios greater than unity⁴. One observes that the arms are always longer on branch {1} than on branch {0}. The magnet described in references [3, 4] is indicated by an open square. It is to be noted that the range of ω on branch {1} is quite narrow: $[180^\circ, 207.2^\circ]$.

4 Asymptotic behaviour at finite arm ratio

The parameter κ (and hence φ) may take on values so that equation (7) degenerates into a linear equation. In that case, the solution often corresponds to magnets with

⁴The graphs for arm ratios smaller than unity may be obtained by interchanging a and b , and α and β .

particular optical properties.

Limiting case $\kappa \rightarrow 0$ ($\varphi \rightarrow 180^\circ$)

In this case, the unique solution has no intermediate focal point in the vertical direction (branch $\{0\}$). Equation (7) becomes: $2g\kappa u^2 + 2(g+1)u + 4 = 0$. Since κ tends to zero and the other terms are positive, u must become infinite. Thus, the constant term 4 becomes negligible, and we obtain $a \sim -(g+1)/(g \cot(\frac{\varphi}{2}))$. For a finite arm ratio, b also becomes infinite. The entrance and exit angles tend to zero as $\alpha \sim -\arctan(\cot(\frac{\varphi}{2})/(g+1))$ and $\beta \sim -\arctan(\cot(\frac{\varphi}{2})g/(g+1))$.

The limit $\varphi = 180^\circ$ corresponds to an afocal mirror magnet with a straight entrance/exit face ($\omega = \pi$) and normal entrance and exit of a parallel beam. Yet, this magnet is *not* a solution of the system of equations (1) since it is dispersive, with $(y|\delta) = 2$.

Limiting case $\kappa \rightarrow -2$ ($\varphi \rightarrow \varphi_0 \approx 232.5^\circ$)

In this case, equation (7) becomes: $-2g(\kappa+2)u^2 - 2(g+1)u + 4 = 0$. The trivial solution $u = 2/(g+1)$ is located on branch $\{0\}$, which is continuous in φ at φ_0 . The solution on branch $\{1\}$ diverges as $u \sim -(g+1)/(g(\kappa+2))$. In the latter case, the entrance and exit angles tend to the same value $(\varphi_0 - \pi)/2$, or 26.2° , and the exit arm b also becomes infinite.

As above, let us stress the point that at the limit, $\varphi = \varphi_0$, the afocal magnet is *not* a solution of the system of equations (1).

Limiting case $\kappa \rightarrow -\infty$ ($\varphi \rightarrow 360^\circ$)

Even if this limit may not be of practical interest, it is quite instructive to study the asymptotic behaviour of the solutions. In this case, equation (7) becomes:

$g(\kappa u)^2 + 2(g+1)\kappa u + 4 = 0$. The discriminant of this equation in κu is equal to $\Delta' = (g-1)^2$. The two solutions are $\kappa u = -2$ and $\kappa u = -2/g$, that is, the arms a and b vanish. The smaller of the two solutions is located on branch $\{0\}$, the greater one on branch $\{1\}$.

The entrance angle tends to $\pm 90^\circ$ as $\alpha = -\arctan[(1/u + \kappa)/\varphi]$. Its sign changes at $\kappa u = -1$, and thus, only on branch $\{0\}$ and for $g = 2$. For $g = 2$, the limit $\kappa \rightarrow -\infty$ of equation (7) gives $\alpha \rightarrow \arctan(1/2\pi) \approx 9.0^\circ$. On branch $\{1\}$, α always tends to $+90^\circ$.

By symmetry, the sign of the exit angle β changes at $g = 1/2$, also on branch $\{0\}$. Thus, on branch $\{0\}$, the dipole angle ω tends to 180° for $1/2 < g < 2$, to $\approx 261.0^\circ$ for $g = 1/2$ or $g = 2$, and to 360° elsewhere (see figure 5b).

5 Optical properties of the solutions

Magnification

The horizontal and vertical magnifications are given by:

$$m_y \equiv (y|y) = \frac{\cos(\varphi - \alpha)}{\cos \alpha} - b \frac{\sin(\varphi - \alpha - \beta)}{\cos \alpha \cos \beta} \quad (9)$$

$$m_z \equiv (z|z) = 1 - \varphi t_\alpha + b(-t_\alpha - t_\beta + \varphi t_\alpha t_\beta) \quad (10)$$

Developing the right-hand side of equation (9), then expressing the sines and cosines of φ with $\cot(\frac{\varphi}{2})$, and further using equations (3) and (5) for substituting t_α and t_β , we finally obtain $m_y = g$. Similarly substituting equations (3) and (5) in equation (10), and then simplifying using equation (7), we obtain the following expression for the magnification in the vertical direction:

$$m_z = -g(\kappa + 1) - \frac{2}{u} \quad (11)$$

In a similar way, we obtain the angular magnification $m'_z = (z'|z') = -(\kappa + 1)/g - 2/ug$. Using equation (7) one immediately verifies $m_z \cdot m'_z = 1$.

The vertical magnification is shown in figure 6 as a function of g and φ . Obviously, m_z is positive for branch $\{1\}$, and negative for branch $\{0\}$, depending directly on the existence of an intermediate focal point in the vertical direction.

Of particular interest is the case where the image shows no first-order distortion, that is, where the horizontal and vertical magnifications have the same absolute value: $m_z = \pm m_y$, that is, $m_z = \pm g$. Using the above expression, we obtain $u = -2/g(\kappa + 1 \pm 1)$. Substituting this expression for u in equation (7), we obtain $g = 1$. Hence, the condition for identical magnifications in the horizontal and vertical directions is independent of φ (see figure 6a).

In figure 6b we observe that, for some value of φ , the vertical magnification for branch $\{0\}$ does not depend on the arm ratio, and is about equal to -1 . In fact, if we set the above expression for m_z equal to -1 , then solve for u , and substitute in equation (7), we obtain $(g - 1)(\kappa + 2) = 0$, which implies $\varphi = \varphi_0$.

Note also a simple relation between the vertical magnifications $m_z^{\{0\}}$ and $m_z^{\{1\}}$ on the two branches: $m_z^{\{0\}} \cdot m_z^{\{1\}} = -g$. This is easily proved by direct calculation using the expression of equations (8) for the sum and the product of the solutions of equation (7).

Angular dispersion

The angular dispersion is $(y'|\delta) = \sin \varphi + (1 - \cos \varphi)t_\beta$, that is, $(y'|\delta) = (\cos \varphi - 1)/b$ here, and is negative for $\varphi < 360^\circ$. It is therefore impossible to design strictly achromatic, i.e., non-dispersive and angle achromatic, DF magnets with finite arm lengths.

Time of flight

The variation λ of the time of flight of the particles is described in the same paraxial formalism by a development to first order around the central trajectory.

The coefficients describing the dependence of λ on position and angle in the vertical direction vanish for symmetry reasons.

In the horizontal plane, the coefficient describing the dependence of λ on angle is $(\lambda|y') = (1 - \cos \varphi)$ and is always positive. The coefficient describing the dependence of λ on position is $(\lambda|y) = \sin \varphi + (1 - \cos \varphi)t_\alpha$, that is, $(\lambda|y) = (\cos \varphi - 1)/a$ here, and is negative. The coefficient describing the dependence of λ on momentum is $(\lambda|\delta) = \varphi - \sin \varphi$, and is positive.

We can see that it is impossible to design isochronous TF magnets, with finite arm lengths.

Special case of zero arm ratio ($g \rightarrow 0$)

A case of practical interest is the deflection of a parallel beam with a large energy spread and its focusing on a small spot. This situation corresponds to the limit $g \rightarrow 0$, $u \rightarrow \infty$, and gu finite. We then read equation (7) as an equation in gu : $\kappa(\kappa+2)(gu)^2 + 2(\kappa+1)gu = 0$. The non-trivial solution (branch {1}) is: $gu = -2(\kappa+1)/\kappa(\kappa+2)$.

For the particular value $\varphi = 270^\circ$, that is, $\kappa = -\varphi$, we obtain $b \rightarrow 2(3\pi - 2)/(3\pi - 4) \approx 2.74$, $a \rightarrow \infty$, $\alpha \rightarrow \pi - \varphi/2$, that is, $\alpha \rightarrow 45^\circ$, and $\tan \beta \rightarrow 3\pi/2(3\pi - 2)$, that is, $\beta \rightarrow 32.4^\circ$ (fig. 7). This magnet has been presented in [3, 4].

Special case of a unit arm ratio ($g = 1$)

For $g = 1$, we get $a = b$ and $\alpha = \beta$: the magnet is symmetric. We then get unit absolute magnifications: $m_y = 1$ and $m_z = \pm 1$. The solution of equation (7) is particularly simple here, with $\Delta' = 4$, $u = -2/\kappa$ (branch {0}) and $u = -2/(\kappa + 2)$ (branch {1}).

The particular value $\varphi = 270^\circ$ gives $a = b = 2$, $\alpha = \beta = \arctan(1/2) \approx 26.6^\circ$ on branch {0} used in reference [9].

6 Working with a given magnet (fixed pole angle ω)

In practice, often the question arises, in which configuration a given dipole can be operated as a TF magnet. If we substitute φ by $\alpha + \beta + \omega$, the system of equations (2) becomes transcendental in the three variables α, β, ω . Therefore, we did not attempt an analytical solution, but rather solved the system numerically.

Figure 8 shows the curves of constant ω in the (φ, g) plane. We can see that in the central part of the ω range, say $[230^\circ, 290^\circ]$, a wide deflection angle range is allowed on branch {0}, with reasonable values of the ratio g , i.e., not too far from 1. We also note that for ω around 190° , a large range of arm ratios, and therefore of magnifications, may be covered by only a small variation of φ .

7 Conclusion

In this paper, we have presented the first general analytic study of TF, index-free, single dipole magnets. We have written the paraxial equations describing TF magnets in the approximation of a small air gap.

We have solved the system of three equations in five variables, which leads to a second-degree polynomial equation. The solutions describe a two dimensional surface which is parametrized by the angle of deflection of the particles and by the arm ratio.

Physical solutions exist in the interval $180^\circ < \varphi < 360^\circ$ only. We obtain one physical solution in the interval $180^\circ < \varphi < 232.5^\circ$, and two physical solutions in the interval $232.5^\circ < \varphi < 360^\circ$. The two branches of solutions correspond respectively to magnets having either none or only one intermediate focal point in the vertical direction.

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Figure 1: Dipole parameter definition.

Figure 2: Moving frame of the central ray.

Figure 3: The variation of the parameter κ with φ .

Figure 4: Ray-tracing of example TF magnets for $\varphi = 289^\circ$ and $g = 1.5$ (above: in the horizontal plane, below: in the vertical direction); all the lengths are in units of R , and s denotes the curvilinear coordinate along the central ray. Solid: central ray, dotted: angular focusing, dashed: momentum focusing.

(a) The curves correspond to $\phi = 190, 210, 230, 235, 270, 300, 340^\circ$.

(b) The curves correspond to $g = 1, 1.5, 2, 5, 10$.

Figure 5: Variation of a, b, α, β and ω with g and ϕ . Solid: branch {1}, dashed: branch {0}.

(a)

(b)

Figure 6: Variations of the vertical magnification m_z with g and φ . Solid: branch {1}, dashed: branch {0}.

Figure 7: Ray tracing of a 270° magnet with a parallel beam on entrance (see caption of fig. 4).

Figure 8: Curves of constant dipole angle ω in the (φ, g) plane. Solid: branch {1}, dashed: branch {0}, dotted: lines of maximum ω at given g (branch {0} and {1}).