Altered Stats: Two anyons via path integrals for multiply connected spaces

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Abstract

We apply the formalism of path integrals in multiply connected spaces to the problem of two anyons.
Quantum mechanical systems of identical particles provide an elegant situation in which one can apply the ideas of path integrals in multiply connected configuration spaces [1,2]. The reason is clear if one permits oneself to ignore coincident points in the configuration space, i.e., configurations where two or more of the particles are at the same position [3]. In that case, non-contractible closed loops in configuration space can easily be constructed. For example, a continuous evolution of the system involving an interchange of particles forms a closed path due to the identical nature of the particles, yet the path is non-contractible due to the interchange.

According to the general rules of path integrals in multiply connected configuration space,¹ one can divide the space of paths into homotopy classes, and rewrite the path integral as a sum of sub-integrals, each of which is a path integral over one such class:

\[
\int D\mathbf{x}(t) e^{i\mathcal{S}} = \sum_n \int_{C_n} D\mathbf{x}(t) e^{i\mathcal{S}},
\]

where \(C_n\) denotes the homotopy class labelled by \(n\). Since the main requirement of the path integral is that it satisfy the Schrödinger equation, and since each sub-integral in (1) is itself a solution, any linear combination of the sub-integrals will still satisfy the Schrödinger equation. For consistency of the path integral, we must impose one requirement on the coefficients: that they form a commutative representation of the first homotopy group of the configuration space.

A very instructive example of the above ideas, due to its simplicity, is provided by a free particle constrained to move on a circle [1,4]. There, the paths are classified according to the net number of windings around the circle. The topology involved can be regarded from the point of view of the simply-connected covering space of the configuration space, which is simply a line. Each point on the circle is the image of an infinity of points on the line, and different homotopy classes on the circle correspond to classes of paths on the line with different pre-images of the final point under the mapping between the circle and line. The sub-integral corresponding to any homotopy class on the circle, viewed in the covering space, is an ordinary path integral between the initial point and the corresponding final point on

¹For an excellent discussion, see Ref. [4].
the line. The path integral for the circle, then, becomes an infinite sum of free-particle path integrals in one dimension.

One can, as described above, add relative phases \( \exp(-in\alpha) \), where \( n \) denotes the number of times a path winds around the circle, to the sub-integrals; such liberty, in fact, corresponds to threading a magnetic flux \( \Phi = \alpha/e \) through the circle \([1]\).

Applying these ideas to systems of identical particles, the above reasoning was used to demonstrate that only Bose and Fermi statistics are possible in three space dimensions \([2]\). The reason is essentially that the first homotopy group of the configuration space of \( N \) identical particles in 3 dimensions is the permutation group \( S_N \), which has only two commutative representations: the trivial representation and that representation which assigns \( \pm 1 \) to a path according to whether it corresponds to an even or odd permutation of the particles. The former describes Bose statistics, the latter, Fermi statistics.

The work of Ref. \([2]\) can in fact be turned around slightly: using a path integral formulation, a system of identical spinless fermions could be described by bosons, by dividing the path integral into its homotopy classes and adding a factor \(-1\) to all paths corresponding to odd permutations of the particles. Although this is true in principle, the division of the path integral into homotopy classes appears to be rather difficult, and to our knowledge such an approach has not been put into practice.

More recently, it was realized by completely independent means that fractional statistics (anyons) is allowed in two space dimensions \([5-7]\)\(^2\). A subsequent re-examination of the ideas of Ref. \([2]\) in two space dimensions \([9]\) revealed in an elegant way that indeed the possibility of fractional statistics might have been anticipated via path integrals: the relevant homotopy group is a braid group (rather than a permutation group), and the commutative representations of such groups allow for any relative phase between different homotopy classes.

Anyons are normally described by adding fictitious ("statistical") charges and magnetic fluxes to "normal" particles (bosons or fermions) in such a way that no new forces are introduced but so that windings of each particle around the others are counted in an appropriate

\(^2\)For a comprehensive review including many applications, see Ref. \([8]\).
way. As in three dimensions, a path integral approach provides, in principle, an alternative means of altering the statistics. One can imagine dividing the path integral for identical bosons into homotopy classes, and adding relative phases between different classes to change the way different windings interfere with one another in a way appropriate for the desired statistics.

Once again, putting this program into practice is extremely difficult. The problem is that one needs a set of coordinates which retain a memory of the relative windings of the particles. (For instance, Cartesian coordinates are clearly inappropriate since the coordinates of a given configuration of particles are unique; the winding of particles around one another is lost in the Cartesian description of the motion.) For the case of two particles the resolution is obvious: if, after disposing of trivial center-of-mass motion, the relative motion is described in polar coordinates, the angular coordinate measures the winding of the two particles around one another. The path integral can thus be divided into homotopy sectors where the final point in configuration space is described by final angles differing by a multiple of $\pi$.

Indeed, a program along these lines has been put into practice in the context of a path integral description of the Aharonov-Bohm effect [10,11], which has obvious and well-known similarities with anyons [7], and in the context of the statistical mechanics of anyons [12]. However, these discussions of the Aharonov-Bohm effect are in a sense “less topological” than, for instance, the treatment outlined above of the path integral for a particle on a circle [1,4]. Essentially, a (non-topological) path integral in polar coordinates is written down for a particle moving in the plane [13,14], and a topological constraint on the path integral is added after the fact. The intermediate angular positions are integrated only over the interval $[0, 2\pi]$, in contrast with the general approach discussed in Refs. [1,4].

In this paper, we present an alternative approach to the problem of the relative motion of two anyons, which has a certain aesthetic advantage over that described above, in that it is more faithful to the topology of the situation. The approach consists in regarding the motion from the point of view of the simply-connected covering space for the plane with the origin excised, which can be visualized either as a sort of spiral staircase or as a half plane, and performing the path integral on this space, allowing the angle of intermediate steps in a path to take on any real values in a manner exactly analogous to Refs. [1,4], rather than
restricting them to the range \([0, 2\pi]\). Our approach is similar in spirit to that of Khandekar, Bhagwat and Wiegel [15], who discussed the problem of the motion of a particle in a plane with the origin removed from an equally topological point of view, although the details of our calculation differ from theirs.

We begin with the Lagrangian describing the relative motion of two bosons of unit mass moving in the plane with the origin removed, in polar coordinates:

\[
L = \frac{1}{2}(\dot{r}^2 + r^2 \dot{\theta}^2).
\]  

(2)

Rather than adding an explicit interaction term to (2) to alter the statistics of the particles, we will introduce relative phases between the different homotopy classes of the path integral, as described above.

The propagator for motion from a point \(r' = (r', \theta')\) to \(r'' = (r'', \theta'')\) in a time interval \(T\) is the sum over all paths from the initial to the final point:

\[
K(r''; r'; T) = \int_{r_{1}}^{r_{19}} D\!r(t) e^{iS}.
\]  

(3)

If we write the paths in polar coordinates, \((r(t), \theta(t))\), then \((r(0), \theta(0)) = (r', \theta')\), and \((r(T), \theta(T)) = (r'', \theta'' + n\pi) \equiv r''_n\), reflecting the fact that changing the final angle by an integer multiple of \(\pi\) describes the same final state: the paths between a given initial and final configuration can be divided into homotopy classes labelled by \(n\).

We can therefore rewrite the path integral in the following way:

\[
K(r''; r'; T) = \sum_n \int_{r_{1}}^{r_{19}} D\!r(t) e^{iS} \equiv \sum_n K_n.
\]  

(4)

We are now, according to Ref. [1], free to add phases between the sub-integrals in (4):

\[
K \rightarrow K^{(\alpha)} = \sum_n e^{-i\alpha n} K_n.
\]  

(5)

The phases change the interference between different homotopy sectors, reflecting altered statistics: if \(\alpha = \pi\) (or any odd multiple of \(\pi\)), the amplitudes of paths whose \(n\)'s differ by one subtract rather than adding, as appropriate for Fermi statistics, while if \(\alpha\) is any even multiple of \(\pi\), the additional phases in (5) have no effect, and the particles are bosons. For arbitrary \(\alpha\), the particles are anyons.
We see that anyons can be described by perfectly conventional path integrals within each homotopy sector (i.e., without introducing the statistical interaction), if the total propagator is computed with appropriate phases between the different sectors.

Let us compute, then, a sub-integral, $K_n$. The computation is performed in polar coordinates, which introduces certain subtleties. In particular, it was shown in Ref. [13] that the most naive guess for the path integral must be modified, due to a path integral version of operator ordering ambiguities. In Ref. [16], it was shown that a path integral in curved space requires the inclusion of an additional "effective potential" term to the Hamiltonian; applying that result to the case at hand (where polar coordinates introduce a nontrivial metric, so the formalism of Ref. [16] applies directly), this effective potential is $-1/(8\epsilon^2)$, and the sub-integral becomes

$$K_n = \frac{1}{(2\pi i)^N} \int \prod_{j=1}^{N-1} (r_j dr_j d\theta_j) \exp i \frac{1}{\epsilon} \sum_{j=1}^{N} \left( \frac{(r_j - r_{j-1})^2 + (\bar{r}_j)^2}{2\epsilon} + \frac{\epsilon}{8(\bar{r}_j)^2} \right),$$

where the limit $N \to \infty$ will be understood throughout, $\theta_0 = \theta'$, $\theta_N = \theta'' + n\pi$, and $\bar{r}_j$ is the average position for the interval $(j-1, j)$, which we take to be the geometric mean $\sqrt{r_{j-1} r_j}$.

We can simplify the angular integrals considerably by the following trick. First, we insert a factor $1 = \int d\theta_N \delta(\theta_N - (\theta'' + n\pi))$. Second, we rewrite the argument of the $\delta$-function in terms of the angular differences $\Delta \theta_j \equiv \theta_j - \theta_{j-1}$; it becomes $\Delta \theta_1 + \ldots + \Delta \theta_N - \Delta \theta_n$, where $\delta \theta_n \equiv \theta'' + n\pi - \theta'$ is the total angular change for the propagator for the $n$th sector. Third, we rewrite the Dirac $\delta$-function as an exponential:

$$\delta \left( \sum_{1}^{N} \Delta \theta_j - \delta \theta_n \right) = \int \frac{d\lambda}{2\pi} \exp i\lambda \left( \sum_{1}^{N} \Delta \theta_j - \delta \theta_n \right).$$

The $N$ angular integrals can now be rewritten in terms of integrals over the $N$ angular differences $\Delta \theta_j$; the integrals so obtained are independant Gaussians and we obtain

$$K_n = \frac{1}{(2\pi i)^{N/2}} \int \frac{d\lambda}{2\pi} e^{-i\lambda \delta \theta_n} \int \prod_{1}^{N-1} (r_j dr_j) \prod_{1}^{N} (\bar{r}_j)^{-2} \exp i \frac{\lambda}{2} \sum_{1}^{N} \left( \frac{(r_j - r_{j-1})^2}{\epsilon} - \frac{(\lambda^2 - 1/4)\epsilon}{(\bar{r}_j)^2} \right).$$

The integrand can be usefully rewritten in terms of modified Bessel functions, using the asymptotic form for these; one finds

$$K_n = e^{i(r''^2 + r'^2)/2\epsilon} \left( i\epsilon \right)^N \int \frac{d\lambda}{2\pi} e^{-i\lambda \delta \theta_n} \prod_{1}^{N-1} (r_j dr_j) \prod_{1}^{N} I_{|\lambda|} ((\bar{r}_j)^2/\epsilon) \exp i \sum_{1}^{N-1} r_j^2/\epsilon.$$
The radial integrals can now be performed \cite{14,11}, yielding (taking the limit \( N \to \infty \))

\[
K_n = \frac{e^{i(\alpha^2+\theta^2)/2T}}{iT} \int \frac{d\lambda}{2\pi} e^{-i\lambda \delta \theta_n} \Im \left( \frac{-i\theta' r''}{T} \right),
\]

(10)

our final result for the sub-integral.

Although we have derived this result for a system of two identical bosons, it applies equally well to the problem of a single particle moving in a punctured plane (which is also, of course, described by the Lagrangian (2), if we write \( \delta \theta_n = \theta'' + 2n\pi - \theta' \), since for a single particle we have periodicity of \( 2\pi \) rather than \( \pi \). With this change, our result (10) is then in complete agreement with that of Ref. \cite{15}.

A simple check of this result is to sum the sub-integrals (taking \( \delta \theta_n = \theta'' + 2n\pi - \theta' \)) for all the topological sectors; one must obtain the propagator for a free particle. Instead, we will perform this sum with relative phases added in, thus obtaining the propagator for a charged particle in the presence of a flux tube; the free particle is obtained in the limit where the flux is zero. The propagator (for \( e\phi = x \)) is

\[
K^{(\alpha)} = \sum_n e^{-i\alpha x} K_n = \frac{e^{i(\alpha^2+\theta'/\theta)}/2T}{iT} \int \frac{d\lambda}{2\pi} \sum_n e^{-i(\lambda/\alpha/2\pi)2\pi n} e^{-i(\lambda/\alpha/2\pi)2\pi n} \Im \left( \frac{-i\theta' r''}{T} \right).
\]

(11)

Shifting \( \lambda \to \lambda - \alpha/2\pi \) and using the Poisson summation formula \( \sum_n e^{-i\lambda x} = \sum_m \alpha(\lambda + m) \),

\[
K^{(\alpha)} = \frac{e^{i(\alpha^2+\theta'/\theta)}/2T}{2\pi i T} \sum_m e^{i(\alpha + 2\pi/\theta)2\pi n} \Im \left( \frac{-i\theta' r''}{T} \right).
\]

(12)

This is our final result for the propagator in the presence of a flux tube. It differs from the result of, e.g., Ref. \cite{11} \footnote{Note that Ref. \cite{11} uses a definition of \( \alpha \) which differs by a factor \( 2\pi \) from ours.} by an overall phase \( \exp i(\alpha/2\pi)(\theta'' - \theta') \). This phase is in fact simply a choice of gauge for the problem \cite{10}, and is related to whether or not one chooses to describe the particle by a single-valued wave function (as in \cite{11}) or not (as in this work). Alternatively, our method of describing the interaction of the particle with the flux tube takes into account the additional phase for paths which wind around the solenoid different numbers of times, but it does not take into account the phase due to the change in angle between the initial and final points; if we wanted, we could have taken this into account by...
adding an overall phase $\exp -i\alpha(\theta'' - \theta')/2\pi$ to the propagator, in which case (12) would be in complete agreement with Ref. [11].

We can take the limit $\alpha \to 0$ in (12); the sum is easily evaluated in terms of the generating function for modified Bessel functions and one finds that the free propagator is indeed obtained.

For the application of these ideas to the problem of the relative motion of two anyons, the change is trivial: different homotopy sectors now differ in their final angle by multiples of $\pi$ (rather than $2\pi$), and the change in angle for the $n$th sector is now $\theta_n = \theta'' + n\pi - \theta'$. The propagator becomes

$$K^{(\alpha)} = \frac{e^{i(r^2 + r'^2)/2T}}{\pi i T} \sum_m e^{i(2m + \alpha/\pi)(\theta'' - \theta')} I_{2m + \alpha/\pi}(-ir'r'')/T). \quad (13)$$

As expected, one can easily verify that when $\alpha = 0$ or $\pi$ (corresponding to bosons or fermions) the propagator is

$$K = \frac{1}{2\pi i T} \left(e^{i|r'' - r|^2/2T} \pm e^{i|r'' + r|^2/2T}\right). \quad (14)$$

Unfortunately, applying these ideas to systems whose configuration space is more complicated (e.g., three or more anyons), while in principle straightforward, appears exceedingly difficult because of the difficulty in finding a suitable set of coordinates which retains a memory of the windings of particles around one another. While such coordinates have been found for the case of three anyons, and in fact an expression analogous to (11) can be found, in those coordinates the action is sufficiently horrible that the path integral does not seem feasible.

In summary, we have given a derivation of the propagator for a particle in the plane with origin removed which is in agreement with previous work, but which is more faithful to the topology of the configuration space. We have added permitted phases to different sectors of the path integral in order to describe the Aharonov-Bohm effect and a system of two anyons.

This work was supported by the Natural Science and Engineering Research Council of Canada.
REFERENCES


