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DIMENSIONAL LIE ALGEBRAS

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International Atomic Energy Agency
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**ON THE q -DEFORMATION OF CERTAIN INFINITE
DIMENSIONAL LIE ALGEBRAS**

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ABSTRACT

A representation of the q -deformed centreless Virasoro algebra in terms of the Gauss derivatives D_x and D_y on the quantum plane $\mathcal{C}_q[x, y]$ is given. Moreover, we obtain the deformed version of the algebra of the area-preserving diffeomorphisms of the torus T^2 . In the end, the correspondence between $\text{Psd}(q, p, r)$ and the $\bar{a}^\infty$ algebra is pointed out.

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Infinite dimensional Lie algebras have played a crucial role in the understanding of string theories, two dimensional conformal field theories, and certain statistical models (see e.g. Refs.[1,2,3]). Much interest has then been attributed to their deformed versions. The latter, called also quantum algebras, are generalizations of the ordinary Lie algebras.

In the present work, we obtain a representation of the q -deformed centreless Virasoro algebra [4] in terms of the Gauss derivatives D_x and D_y on the quantum plane $\mathcal{C}_q[x, y]$ (see e.g. Refs.[5]). This representation is nothing but a generalization of the realization

$$L_n \equiv -x^{n+1} \partial_x, \quad \bar{L}_n \equiv -\bar{x}^{n+1} \partial_{\bar{x}}, \quad n \in \mathbb{Z}.$$

Thus, our realization may be useful in the understanding of the underlying geometry of $\text{Diff}_q(S^1)$. More interestingly, this representation shows that the centreless Virasoro algebra can be deformed in a more general way than the one described in Ref.[4].

Furthermore, we study the algebra $\text{Psd}(q, p, r)$ of formal pseudo-differential operators [6] on the extended quantum plane $\mathcal{C}_q[[x, y]]$. As a result we obtain the q -deformation of the algebra of the area-preserving diffeomorphisms of the torus T^2 [11]. This q -algebra reads as:

$$[\hat{T}_n, \hat{T}_m]_{(q^{2n \wedge m}, q^{-2n \wedge m})} = [n \wedge m]_q \hat{T}_{m+n}$$

where $n = (n_1, n_2)$, $m = (m_1, m_2)$, $m \wedge n = \varepsilon^{ab} m_a n_b$ and $\varepsilon^{12} = -\varepsilon^{21} = 1$, $\varepsilon^{22} = \varepsilon^{11} = 0$. $n_a, n_a \in \mathbb{Z}$ and $[x]_q = (q^x - q^{-x}) / (q - q^{-1})$.

At the end of this paper, we point out a correspondence between $\text{Psd}(q, p, r) \equiv \text{Psd}$ and the known matrix algebra $\bar{a}^\infty$ [7].

Let x and y be two formal variables, q a complex non vanishing parameter and $\mathcal{C}\{x, y\}$ the free associative algebra constructed on x, y and the empty word $\emptyset$. Consider then the two sided ideal I_q of $\mathcal{C}\{x, y\}$ generated by the quadratic element $xy - qyx$. The quantum plane $\mathcal{C}_q[x, y]$ is defined to be the factor algebra $\mathcal{C}_q\{x, y\} / I_q$. Each element of $\mathcal{C}_q[x, y]$ is a finite linear combination of the monomes $y^n x^m$; $n, m \in \mathbb{N}$, satisfying

$$x^m y^n = q^{nm} y^n x^m; \quad n, m \in \mathbb{N} \quad (1)$$

Recall that many differential calculi related to the quantum plane $\mathcal{C}_q[x, y]$ have been found [5]. For our purpose, we shall consider the following case:

$$\begin{aligned} x \, dx &= p \, dx \, x^{-1}, & x \, dy &= q \, dy \, x^{-1} \\ y \, dx &= q^{-1} \, dx \, y^{-1}, & y \, dy &= r \, dy \, y^{-1} \\ dx \, dy &= -q \, dy \, dx^{-1}, \end{aligned} \quad (2)$$

where p and r are arbitrary complex numbers.

Let $f(x,y)$ be an arbitrary element of $\mathcal{C}_q[x, y]$, the first derivatives $D_x f$ and $D_y f$ satisfy:

$$df(x, y) = dx \, D_x f(x,y) + dy \, D_y f(x,y)$$

and consistent with Eqs.(2) have the form

$$D_x f(x,y) = \frac{1}{x} \frac{f(px,y) - f(x,y)}{(p-1)} \quad (3)$$

$$D_y f(x, y) = \frac{1}{y} \frac{f(x, ry) - f(x, y)}{(r-1)} \quad (4)$$

D_x and D_y are simply the Gauss derivatives in direction x and y . Symbols $\frac{1}{x}$ and $\frac{1}{y}$ appear only formally and mean decreasing of the power of x and y respectively in the series defining the function $f(x, y)$, so that we have not, for instance, assumed the existence of x^{-1} and y^{-1} .

Using Eqs.(3-4), one can see that D_x and D_y generate a non commutative algebra A , with defining relation

$$D_x D_y = q \, D_y D_x, \quad (5)$$

homomorphic with Manin's plane. An arbitrary element D of A can be written as

$$D = \sum_{n,m \geq 0} f_{nm}(x, y) D_x^n D_y^m \quad (6)$$

where (n, m) runs over a finite subset of $\mathbf{N}^2$. The m derivative in direction y of x is obtained by iterations, for example

$$D_x^m f(x, y) = D_x (D_x^{m-1} f(x,y)) \quad (7)$$

However, we shall consider a larger algebra than A namely the algebra $\mathcal{Psd}$ of formal D_x and differential operators [6]. An element of $\mathcal{Psd}$ is an expression:

$$\sum_{n=-\infty}^k \sum_{m=-\infty}^l f_{nm}(x, y) D_x^{-n} D_y^m \quad (8)$$

for some integers k and l , and where $f_{nm}(x, y)$ are now formal power series in x and y . This led us to deal with the set $\mathcal{C}_q[[x, y]]$ of all laurent series in x and y , such that Eq.(1) is valid for also negative values of n and m . The rule of multiplying two elements of $\mathcal{Psd}$ is determined by the relation:

$$D_x f = f D_x + (D_x f) \quad (9)$$

and for example

$$D_x^{-1} f = f D_x^{-1} - (D_x f) D_x^{-2} + (D_x^2 f) D_x^{-3} + \dots \quad (10)$$

The equation (5) can be generalized to

$$D_x^n D_y^m = q^{nm} D_y^m D_x^n \quad (11)$$

In what follows we show that certain subalgebras of $\mathcal{Psd}$ can be endowed with a Lie algebra structure and provide a realization of a class of infinite dimensional lie algebras as well as deformed ones.

Consider then the subalgebras A_l and A_R generated respectively by the sets:

$\{J_n = D_x^n, n \in \mathbf{Z}\}$ and $\{S_m = D_y^m, m \in \mathbf{Z}\}$. Their defining commutation relation reads as

$$[J_n, J_m] = 0 \quad [S_n, S_m] = 0 \quad (12)$$

for $m, n \in \mathbf{Z}$. A_l and A_R are two isomorphic Lie algebras in one to one correspondence with two copies of non commuting trivial Heisenberg algebras H_l and H_R . Notice that A_l and A_R are freely generated by J_1 and S_1 respectively. The property that A_l and A_R do not commute allows to consider a new kind of deformation of symmetries of system described by left and right commuting structures, that is to perform a deformation breaking also the chirality of the theory. An example on the centreless Virasoro algebra is in order. The latter is the infinite dimensional Lie algebra generated by the $\{L_n, \bar{L}_n, n \in \mathbf{Z}\}$ with defining relations:

$$\begin{aligned} [L_n, L_m] &= (n-m) L_{n+m} \\ [L_n, \bar{L}_m] &= (n-m) \bar{L}_{m+n} \end{aligned} \quad (13)$$

$$L_n \bar{L}_m = \bar{L}_m L_n \quad (14)$$

Deformation of this algebra has been investigated in many occasions [4,8,9]. The one deformation parameter commutation relations is given by

$$[L_n, L_m]_{(q^{n-m}, q^{m-n})} = [n-m]_q L_{n+m}, \quad q \in \mathcal{C} \quad (14)$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. The generators $\bar{L}_n, n \in \mathbb{Z}$ satisfy similar relations as Eq.(14) but actually the equality (14) is conserved that is we have $L_n \bar{L}_m = \bar{L}_m L_n$ in the deformed case. However, we shall show using a representation of the algebra Eq.(15) that Eq.(14) can be again deformed. To this end, consider the following operators by the identification with the discrete basis of vector fields on the unit circle S^1 :

$$\begin{aligned} d_n &= -u \cdot x^{n+1} \cdot D_x \\ \bar{d}_n &= -v \cdot y^{n+1} \cdot D_y \end{aligned} \quad (15)$$

where $u = p^{1/2}$, $v = r^{1/2}$ and $n \in \mathbb{Z}$. It is not hard to check that the d_n 's (and similarly $\bar{d}_n$ with change $u \rightarrow v$) satisfy the deformed commutation relation Eq.(15) with u as the deformation parameter i.e.

$$\{d_n, d_m\}_{(q^{n-m}, q^{m-n})} = [n-m]_u d_{n+m} \quad (17)$$

This shows that the d_n 's provide a representation of the algebra Eq.(15). More interestingly, this representation gives also a deformation of the equality (14), that is

$$d_n \bar{d}_m = q^{nm} \bar{d}_m d_n \quad (18)$$

The important remark here is that the parameters p, q and r are independent. This fact leads to consider various kinds of deformation of the Virasoro algebra Eqs.(13-14).

These are obtained by appending in different ways the value 1 to p, q and r . The known deformation Eq.(15) corresponds to $p \neq 1$, $r \neq 1$ and $q=1$ in the realization Eq.(16)

Now, we turn to determine the deformation of the algebra of area-preserving diffeomorphisms in the 2-d torus. Recall that this algebra [11] plays a central role in the theory of membranes and is generated by the operators $L_{n1, n2} : n_1, n_2 \in \mathbb{Z}$ satisfying the following commutation relations:

$$[L_m, L_n] = (m \wedge n) L_{m+n} \quad (19)$$

where $n = (n_1, n_2)$, $m = (m_1, m_2)$, $m \wedge n = \varepsilon^{ab} m_a n_b$ and $\varepsilon^{12} = -\varepsilon^{21} = 1$, $\varepsilon^{22} = \varepsilon^{11} = 0$.

Let us consider the family of the generators:

$$T_{n_1, n_2} = q^{n_1 \cdot n_2 / 2} S_{n_1} J_{n_2}, \quad n_1, n_2 \in \mathbb{Z} \quad (20)$$

in which the Lie algebra reads as:

$$[T_n, T_m] = (q^{m \wedge n / 2} - q^{-m \wedge n / 2}) T_{m+n} \quad (21)$$

Eq.(21) plays a kind of "affine" algebra on the torus T^2 [10] with a new feature that the structure constants are now generator dependent. An interesting case is obtained for special values of q that is when q is a N -th root of the unity: $q^N = 1$. We assume that N is an odd integer and write q as $q = \exp(\frac{4i\pi}{N})$. On then rewrite Eq.(20) as follows

$$[T_n, T_m] = 2i \sin\left(\frac{2\pi}{N} m \wedge n\right) T_{m+n} \quad (22)$$

$m = (m_1, m_2)$, $n = (n_1, n_2)$, or equivalently that the rescaled generators $K_{n_1, n_2} = \frac{iN}{4\pi} T_{n_1, n_2}$

satisfy

$$[K_m, K_n] = \frac{N}{2\pi} \sin\left(\frac{2\pi}{N} m \wedge n\right) K_{m+n} \quad (23)$$

This shows that the generators $\hat{K}_m \equiv \lim_{N \rightarrow +\infty} K_m$ provide a realization of the algebra of area-preserving diffeomorphisms on the torus T^2 [10]. The generators $\hat{K}_m$ are then in correspondence with:

$$A_m = -i \varepsilon^{ab} m_a \exp(m_1 \sigma_1 + m_2 \sigma_2) \partial_b, \quad m_1, m_2 \in \mathbb{Z} \quad (24)$$

where $\sigma_i = (\sigma_1, \sigma_2)$ are local coordinates on T^2 , and $\partial_b = \partial/\partial \sigma_b$.

This correspondence permits to construct a Virasoro algebra out of the $\hat{K}_m$, $m = (m_1, m_2)$. Indeed, it is not hard to check that the generators $\hat{V}_m \equiv \lim_{N \rightarrow +\infty} V_m$, $m \in \mathbb{Z}$ where:

$$V_m = \sum_{j \neq 0} \frac{1}{j} K_{j, m-j} = \frac{iN}{4\pi} \sum_{j \neq 0} \frac{1}{j} q^{j(m-j)/2} D_y^j D_x^{m-j} \quad (25)$$

satisfy the following commutation relation:

$$[\hat{V}_m, \hat{V}_{m'}] = (m-m') \hat{V}_{m+m'}, \quad m, m' \in \mathbb{Z} \quad (26)$$

Concerning the central extension of this algebra, one can show that the two-cocycle Ω of two generators $\hat{V}_m$ and $\hat{V}_n$ is obtained by linearity from the following equation [10]

$$\Omega(D_x^{m_1} D_y^{m_2}, D_x^{n_1} D_y^{n_2}) = f(N) \delta_{m_1-n_2, n_1-m_2} (m_2-m_1) \exp\left(\frac{4i\pi m_1 m_2}{N}\right), \quad (27)$$

where $f(N)$ is an arbitrary function of N , the computation of $\Omega(\hat{V}_m, \hat{V}_n)$ shows that the central extension we are looking for is trivial.

Now, we give the deformed version of the algebra of the area-preserving diffeomorphisms of the torus T^2 , generated by $\hat{K}_{m_1, m_2}$ and characterized by the following commutation relations:

$$[\hat{K}_m, \hat{K}_n] = (m \wedge n) \hat{K}_{n+m} \quad (28)$$

Starting from Eq.(20), it is easy to show that the generators T_{n_1, n_2} , satisfy the relation

$$[T_{n_1, n_2}, T_{m_1, m_2}]_{(q^{2n \wedge m}, q^{-2n \wedge m})} = (q^{n \wedge m} - q^{m \wedge n}) \cdot T_{n_1+m_1, n_2+m_2} \quad (29)$$

This shows that the generators $\hat{T}_{n_1, n_2} = (q - q^{-1})^{-1} T_{n_1, n_2}$ obey the equality

$$[\hat{T}_n, \hat{T}_m]_{(q^{2n \wedge m}, q^{-2n \wedge m})} = [n \wedge m]_q \hat{T}_{m+n} \quad (30)$$

This is exactly the deformed version of the algebra of the area-preserving diffeomorphisms of the torus. Indeed, the limit $q \rightarrow 1$ in Eq.(30) reproduces the algebra Eq.(28). By comparing, Eq.(30) and Eq.(15), one remarks that they differ by the square 2 in Eq.(30). This fact seems to be related to the dimension of the underlying manifold i.e. the unit circle S^1 for Eq.(15) and the 2-d torus T^2 for Eq.(30).

At the end of this paper we examine a correspondence between $\text{Psd}(q, p, r)$ and the $\bar{a}_\infty$ algebra. Let X and Y be the $N \times N$ matrices given by

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & & & & \ddots & \\ \vdots & & & & & \ddots \\ 0 & \dots & \dots & \dots & \dots & 1 \\ 1 & 0 & \dots & \dots & \dots & 0 \end{pmatrix}, \quad (31)$$

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & q & 0 & 0 & \dots & \dots \\ 0 & 0 & q^2 & 0 & \dots & \dots \\ \vdots & & & & \ddots & \\ \vdots & & & & & \ddots \\ 0 & \dots & \dots & \dots & \dots & q^{N-1} \end{pmatrix}$$

where $q^N = 1$.

It is easy to see that $XY = qYX$. Thus we have the following correspondence:

$$D_x \rightarrow X \quad ; \quad D_y \rightarrow Y \quad (32)$$

This correspondence can be forced to have a morphism structure. However, this is true on just a subspace of Psd defined as the factor algebra $\text{Psd} / (D_x^N - 1 ; D_y^N - 1)$. This fact is due to the property of X and Y namely: $X^N = Y^N = I$.

However, the limit $N \rightarrow +\infty$ seems to be interesting. Indeed, in this case we have the identification of $J_1 = D_x$ (or $S_1 = D_y$) with the infinite matrix

$$A_1 = \sum_{i \in \mathbb{Z}} E_{i, i+1} \quad (33)$$

where $(E_{i,j})_{i,j \in \mathbb{Z}}$ denotes the usual basis of the $gl(\infty, \mathbb{C})$ algebra. Thus the algebra Psd is in correspondence with the algebra $\bar{a}_\infty$ of infinite matrices having a finite non vanishing number of diagonals [7].

This correspondence and the deformation of the algebra of area-preserving diffeomorphisms on higher dimensional manifold will be studied on a future occasion.

References

- [1] M. Green, J. Schwarz and E. Witten, Superstrings (Cambridge University, Cambridge, England, 1987) Vd 1.
S. Randjbar-Daemi and J. Strathdee, Lectures delivered at the Summer Workshop on High Energy, Trieste, 1989.
- [2] A. Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys. B241 (1984) 333.
P. Ginsparg, HUTP 1988/A054.
- [3] J.L. Cardy, "Conformal Invariance", in Phase Transition and Critical Phenomena, Vol. 11, edited by C. Domb and J.L. Lobowitz (Academic, London 1985).
- [4] T. Curtright and C. Zachos, Phys. Lett. B243 (1990) 237.
- [5] Christian Kassel, Cours sur les groupes quantiques 1^{ère} partie. Institut de R.M.A. Strasbourg (1991).
T. Brzezinski, H. Dabrowski and J. Rembiński, J. Math. Phys. 33(1) (1992) 19.
J. Wess and B. Zumino, preprint CERN.TH-5697/90. LAPP-TH-284/90.
- [6] See e.g. I.M. Gelfand and L.A. Dickii, Funct. Anal. Appl. 10 (1976) 259.
- [7] V.G. Kac and A.K. Raina; Bombay lectures on Highest Weight representation of infinite dimensional Lie algebras (World Scientific, Singapore, 1987).
- [8] N. Aizawa and H. Sato, Phys. Lett. B256 (1991) 185.
- [9] A.P. Polychronakos, Phys. Lett. B256 (1991) 35.
M. Chaichian, D. Ellinas and Z. Popowicz, Phys. Lett. B248 (1990) 95.
- [10] M. Zakkari, J. Math. Phys. 35 (07) (1994) 3761.
- [11] E.G. Floratos, J. Iliopoulos, Phys. Lett. B201 (1988) 237.

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