0A

CERN LIBRARIES, GENEVA

SCAN-9509143

RIMS-1026

SW 85 89

**Huge Space-Dependent Symmetry in the
Lightcone-Gauge Two-Dimensional Quantum Gravity**

By

Mitsuo ABE

and

Noboru NAKANISHI

August 1995

京都大学 数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

HUGE SPACE-DEPENDENT SYMMETRY IN THE LIGHTCONE-GAUGE TWO-DIMENSIONAL QUANTUM GRAVITY

MITSUO ABE and NOBORU NAKANISHI

*Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-01, Japan*

August 1995

ABSTRACT

The lightcone-gauge two-dimensional quantum gravity, i.e., the local version of Polyakov's "induced" quantum gravity, is analyzed in the canonical operator formalism. An extremely huge x^+ -dependent symmetry algebra is found to exist in this model. Both Polyakov's $SL(2, \mathbf{R})$ current algebra and residual general-coordinate invariance are very very tiny subalgebras of it.

1. Introduction

Two-dimensional quantum gravity furnishes to us interesting models for nonperturbative analysis of quantum gravity. In a series of papers,¹⁻⁸ we have quite successfully investigated two-dimensional quantum gravity in the de Donder gauge in the canonical operator formalism. Furthermore, in a previous paper,⁹ which is referred to as I hereafter, the operator-formalism analysis has extensively been made for the corresponding lightcone-gauge model.

This lightcone-gauge two-dimensional quantum gravity is nothing but the local version of Polyakov's "induced" quantum gravity,¹⁰ which was investigated from different motivation on the basis of the path-integral formalism. He found the existence of $SL(2, \mathbf{R})$ current algebra. Subsequently, Knizhnik, Polyakov, and Zamolodchikov¹¹ and Helayël-Neto, Mokhtari, and Smith¹² investigated residual general-coordinate invariance which involves two arbitrary transformation functions of x^+ . They regarded this invariance as gauge symmetry acting on the $SL(2, \mathbf{R})$ currents, that is, they do not consider $SL(2, \mathbf{R})$ and residual general-coordinate invariance on an equal footing.

In the present paper, we demonstrate how powerful the canonical operator formalism is for the analysis of symmetry. Indeed, the results which we have obtained are extremely surprising!

We begin our analysis by finding seven x^+ -dependent symmetry generators, whose integrands are seven x^- -independent operators, denoted by $j^{(-1)}$, $j^{(0)}$, $j^{(1)}$, c , a , f , and e , multiplied by an arbitrary c -number function of x^+ . The first three ones form the $SL(2, \mathbf{R})$ algebra, while $j^{(-1)}$ and a linear combination of e , $j^{(0)}$, and f produce the residual general-coordinate transformations. The algebra formed by the above seven generators is closed.

Our next step is a really remarkable one. We find that symmetry generators of c , a , f , and e can be extended to those of c^m , a^n , f^p , and e^q , where m , n , p , and q

are arbitrary integers. By taking their commutators successively, we produce many new generators, such as $c^m a^n f^p e^q (\partial_+ c)^{m'} (\partial_+ a)^{n'}$. It can explicitly be confirmed that they are really symmetry generators of our model. We then prove that any arbitrary function of x^+ and c, a, f, e , together with their derivatives is the integrand of a symmetry generator.

The three quantities $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$ are nonlinearly expressible in terms of $c, a, f, e, \partial_+ c$, and $\partial_+ a$. Hence an arbitrary function of x^+ and $\{j^{(-1)}, j^{(0)}, j^{(1)}\}$ is also the integrand of a symmetry generator, provided that the definitions of $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$ are the appropriate ones consistent with the above consideration; we see that the naive definitions of them are inadequate for constructing nonlinear functions of them.

The present paper is organized as follows. In Sec.2, we present field equations and commutators in the model. In Sec.3, we find seven fundamental x^+ -dependent symmetry generators. Extension to power-type ones for c, a, f , and e is made in Sec.4. In Sec.5, we combine them to obtain product-type generators. The most general symmetry generator is given in Sec. 6. Nonlinear extension of Polyakov's $SL(2, \mathbf{R})$ is considered in Sec.7. Anomaly and spontaneous symmetry breakdown are considered in Sec.8. The final section is devoted to discussion. In the Appendix, we discuss the generator of the residual general-coordinate transformation in connection with Refs. 11 and 12.

2. Field equations and commutation relations

Polyakov's two-dimensional "induced" quantum gravity is based on the nonlocal Lagrangian density quadratic in the scalar curvature R . Introducing a scalar field $\tilde{b}$, it can be rewritten as *

$$\mathcal{L} = \sqrt{-g}R\tilde{b} + \frac{\alpha}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\tilde{b}\cdot\partial_\nu\tilde{b}, \quad (2.1)$$

where $g \equiv \det g_{\mu\nu}$ and α is a nonzero parameter. In the lightcone gauge, (2.1) reduces to

$$\mathcal{L} = \partial_-\tilde{b}\cdot\partial_-h - \frac{1}{2}\alpha(\partial_-\tilde{b})^2h + \frac{1}{2}\gamma\alpha\partial_+\tilde{b}\cdot\partial_-\tilde{b} \quad (2.2)$$

with $\gamma = 1$. We have introduced a new parameter γ in order to stress that the relative ratios of the three terms are arbitrary as long as the first and second terms are nonvanishing. Our starting point is (2.2); we discuss its mathematical consequences, forgetting any possible "prehistory" of it.

The field equations which follow from (2.2) are

$$\partial_-(\partial_-h - \alpha\partial_-\tilde{b}\cdot h + \gamma\alpha\partial_+\tilde{b}) = 0, \quad (2.3)$$

$$\partial_-^2\tilde{b} + \frac{1}{2}\alpha(\partial_-\tilde{b})^2 = 0. \quad (2.4)$$

Setting

$$\tilde{b} = \frac{2}{\alpha} \log \rho, \quad \text{i.e.,} \quad \partial_\pm\tilde{b} = \frac{2}{\alpha} \cdot \frac{\partial_\pm\rho}{\rho}, \quad (2.5)$$

we rewrite (2.4) as

$$\partial_-^{-2}\rho = 0. \quad (2.6)$$

Hence we may write

$$\rho(x) = c(x^+)x^- + a(x^+) \quad (2.7)$$

with

$$\partial_- c = \partial_- a = 0. \quad (2.8)$$

We set

$$f \equiv \partial_- h - \alpha \partial_- \tilde{b} \cdot h + \gamma \alpha \partial_+ \tilde{b}, \quad (2.9)$$

then (2.3) becomes

$$\partial_- f = 0. \quad (2.10)$$

From (2.9) with (2.5), we deduce

$$\partial_-^2 (\rho \partial_- h - 2 \partial_- \rho \cdot h + 2 \gamma \partial_+ \rho) = \partial_-^2 (f \rho). \quad (2.11)$$

Using (2.6) and (2.10), we find

$$\partial_-^3 h = 0. \quad (2.12)$$

Hence, following Polyakov,¹⁰ we set

$$h(x) = j^{(-1)}(x^+)(x^-)^2 - 2j^{(0)}(x^+)x^- + j^{(1)}(x^+) \quad (2.13)$$

with

$$\partial_- j^{(-1)} = \partial_- j^{(0)} = \partial_- j^{(1)} = 0. \quad (2.14)$$

Then (2.3) is rewritten as

$$a^2 j^{(-1)} + 2acj^{(0)} + c^2 j^{(1)} + \gamma(a\partial_+ c - c\partial_+ a) = 0. \quad (2.15)$$

In terms of $j^{(-1)}$, $j^{(0)}$, $j^{(1)}$, c , and a , the quantity f is expressed as

$$\begin{aligned} f &= \theta \frac{2}{c} (a j^{(-1)} + c j^{(0)} + \gamma \partial_+ c) \\ &\quad + (1 - \theta) \frac{2}{a} (-a j^{(0)} - c j^{(1)} + \gamma \partial_+ a), \end{aligned} \quad (2.16)$$

where θ is an arbitrary operator.

Canonical quantization can be straightforwardly carried out by regarding x^- as time variable. Because of (2.7) with (2.8) and (2.13) with (2.14), it is easy to calculate the commutator between any two of $j^{(-1)}$, $j^{(0)}$, $j^{(1)}$, c and a . As shown in I, nonvanishing commutators are as follows:

$$[j^{(-1)}(x^+), j^{(0)}(y^+)] = i\frac{\alpha}{2}j^{(-1)}\delta(x^+ - y^+), \quad (2.17)$$

$$[j^{(-1)}(x^+), j^{(1)}(y^+)] = i\alpha j^{(0)}\delta(x^+ - y^+) + i\frac{\gamma\alpha}{2}\delta'(x^+ - y^+), \quad (2.18)$$

$$[j^{(0)}(x^+), j^{(0)}(y^+)] = -i\frac{\gamma\alpha}{4}\delta'(x^+ - y^+), \quad (2.19)$$

$$[j^{(0)}(x^+), j^{(1)}(y^+)] = i\frac{\alpha}{2}j^{(1)}\delta(x^+ - y^+), \quad (2.20)$$

$$[j^{(-1)}(x^+), a(y^+)] = -i\frac{\alpha}{2}c\delta(x^+ - y^+), \quad (2.21)$$

$$[j^{(0)}(x^+), c(y^+)] = -i\frac{\alpha}{4}c\delta(x^+ - y^+), \quad (2.22)$$

$$[j^{(0)}(x^+), a(y^+)] = i\frac{\alpha}{4}a\delta(x^+ - y^+), \quad (2.23)$$

$$[j^{(1)}(x^+), c(y^+)] = i\frac{\alpha}{2}a\delta(x^+ - y^+). \quad (2.24)$$

We can then calculate the commutators involving f as follows:

$$[\rho(x), f(y^+)] = i\frac{\alpha}{2}\rho(x)\delta(x^+ - y^+), \quad (2.25)$$

$$[h(x), f(y^+)] = 0, \quad (2.26)$$

$$[f(x^+), f(y^+)] = i\gamma\alpha\delta'(x^+ - y^+). \quad (2.27)$$

As will be found in Sec.3, there is another conserved quantity defined by

$$e \equiv \frac{\alpha}{2}[\partial_+ \tilde{b} \cdot \partial_- h - (\partial_- \partial_+ \tilde{b} + \alpha \partial_+ \tilde{b} \cdot \partial_- \tilde{b})h + \frac{\gamma\alpha}{2}(\partial_+ \tilde{b})^2]. \quad (2.28)$$

Indeed, from (2.3) and (2.4), we see

$$\partial_- e = 0. \quad (2.29)$$

In terms of $j^{(-1)}$, $j^{(0)}$, $j^{(1)}$, c , and a , we have

$$e = \theta \frac{1}{c^2} [\partial_+(ca) \cdot j^{(-1)} + 2c\partial_+ c \cdot j^{(0)} + \gamma(\partial_+ c)^2]$$

$$+(1-\theta)\frac{1}{a^2}[-2a\partial_+a\cdot j^{(0)}-\partial_+(ca)\cdot j^{(1)}+\gamma(\partial_+a)^2], \quad (2.30)$$

where θ is an arbitrary operator. The commutators involving e are as follows:

$$[\rho(x), e(y^+)] = i\frac{\alpha}{2}\partial_+\rho(x)\cdot\delta(x^+-y^+), \quad (2.31)$$

$$[h(x), e(y^+)] = i\frac{\alpha}{2}\partial_+^x[h(x)\delta(x^+-y^+)], \quad (2.32)$$

$$[f(x^+), e(y^+)] = i\frac{\alpha}{2}\partial_+^x[f(x^+)\delta(x^+-y^+)], \quad (2.33)$$

$$[e(x^+), e(y^+)] = i\frac{\alpha}{2}[e(x^+)+e(y^+)]\delta'(x^+-y^+), \quad (2.34)$$

3. Seven space-dependent symmetry generators

Before studying the symmetry properties of our model, we briefly review the general framework of the symmetry properties of the Lagrangian quantum field theory.

Given a time-independent operator Q , a generic field φ_A is infinitesimally transformed into $\varphi_A + \delta\varphi_A$, where

$$\delta\varphi_A = -i\varepsilon[\varphi_A, Q], \quad (3.1)$$

ε being an infinitesimal c -number constant. Let $\mathcal{L}(\varphi_A, \partial_\mu\varphi_A)$ be the Lagrangian density and $\delta\mathcal{L}$ be the infinitesimal change of $\mathcal{L}$ under (3.1); then Q is a symmetry generator if and only if $\delta\mathcal{L}$ can be written as

$$\delta\mathcal{L} = \varepsilon\partial_\mu\mathcal{F}^\mu, \quad (3.2)$$

where $\mathcal{F}^\mu$'s are local quantities. The Noether current J^μ is defined by

$$J^\mu \equiv \sum_A \delta\varphi_A \cdot \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)} - \mathcal{F}^\mu. \quad (3.3)$$

which is conserved:

$$\partial_\mu J^\mu = 0. \quad (3.4)$$

Hence the spatial integral of J^0 is time-independent and reproduces Q . In the above reasoning, it is of paramount importance that *in order to show (3.2) one must not use the field equations*

$$\frac{\partial \mathcal{L}}{\partial \varphi_A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_A)} = 0, \quad (3.5)$$

while one may freely use them in (3.1) and (3.3). Hence, in calculating $\delta \mathcal{L}$, one should choose an appropriate expression among various possible ones for $\delta \varphi_A$. We have no general rule for finding this “appropriate” expression.

Now, we go back to our model. The original action given by (2.1) is, of course, general-coordinate invariant. After the lightcone-gauge fixing is made, this invariance still partially remains. It is called the residual general-coordinate invariance, which involves two arbitrary transformation functions $\eta(x^+)$ and $\lambda(x^+)$ depending on the “space” coordinate x^+ only. The infinitesimal transformations are as follows:^{11, 12}

i) η -transformation

$$x'^- = x^- + \varepsilon \eta(x^+), \quad x'^+ = x^+; \quad (3.6)$$

$$\delta \tilde{b} = \varepsilon \eta \partial_- \tilde{b}, \quad (3.7)$$

$$\delta h = \varepsilon (\eta \partial_- h + \gamma \partial_+ \eta); \quad (3.8)$$

$$\delta \mathcal{L} = \varepsilon \partial_- (\eta \mathcal{L}). \quad (3.9)$$

ii) λ -transformation

$$x'^- = x^- - \varepsilon \partial_+ \lambda(x^+) \cdot x^-, \quad x'^+ = x^+ + \varepsilon \lambda(x^+); \quad (3.10)$$

$$\delta \tilde{b} = \varepsilon (\lambda \partial_+ \tilde{b} - \partial_+ \lambda \cdot x^- \partial_- \tilde{b}), \quad (3.11)$$

$$\delta h = \varepsilon [\lambda \partial_+ h - \partial_+ \lambda \cdot (x^- \partial_- h - 2h) - \gamma \partial_+^2 \lambda \cdot x^-]; \quad (3.12)$$

$$\delta \mathcal{L} = \varepsilon [\partial_+ (\lambda \mathcal{L}) - \partial_- (\partial_+ \lambda \cdot x^- \mathcal{L} + \gamma \partial_+^2 \lambda \cdot \tilde{b})]. \quad (3.13)$$

On the other hand, Polyakov¹⁰ found the $SL(2, \mathbf{R})$ current algebra. As pointed out in I, this symmetry can be described in the language of symmetry generators. Let

$\Lambda(x^+)$ be a c -number function.

1) $j^{(-1)}$ -transformation

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) j^{(-1)}(x^+); \quad (3.14)$$

$$\delta \tilde{b} = \varepsilon \Lambda \partial_- \tilde{b}, \quad (3.15)$$

$$\delta h = \varepsilon (\Lambda \partial_- h + \gamma \partial_+ \Lambda); \quad (3.16)$$

$$\delta \mathcal{L} = \varepsilon \partial_- (\Lambda \mathcal{L}); \quad (3.17)$$

$$J^- = \frac{2}{\alpha} \Lambda j^{(-1)} + \partial_+ (\gamma \Lambda \partial_- \tilde{b}), \quad (3.18)$$

$$J^+ = -\partial_- (\gamma \Lambda \partial_- \tilde{b}). \quad (3.19)$$

2) $j^{(0)}$ -transformation

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) j^{(0)}(x^+); \quad (3.20)$$

$$\delta \tilde{b} = \varepsilon \Lambda \left(x^- \partial_- \tilde{b} - \frac{1}{\alpha} \right), \quad (3.21)$$

$$\delta h = \varepsilon [\Lambda (x^- \partial_- h - h) + \gamma \partial_+ \Lambda \cdot x^-]; \quad (3.22)$$

$$\delta \mathcal{L} = \varepsilon \partial_- \left[\Lambda x^- \mathcal{L} - \frac{1}{2} \gamma \Lambda \partial_+ \tilde{b} \right] + \varepsilon \partial_+ \left(\frac{1}{2} \gamma \Lambda \partial_- \tilde{b} \right); \quad (3.23)$$

$$J^- = \frac{2}{\alpha} \Lambda j^{(0)} + \partial_+ (\gamma \Lambda x^- \partial_- \tilde{b}), \quad (3.24)$$

$$J^+ = -\partial_- (\gamma \Lambda x^- \partial_- \tilde{b}). \quad (3.25)$$

3) $j^{(1)}$ -transformation

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) j^{(1)}(x^+); \quad (3.26)$$

$$\delta \tilde{b} = \varepsilon \Lambda \left[(x^-)^2 \partial_- \tilde{b} - \frac{2}{\alpha} x^- \right], \quad (3.27)$$

$$\delta h = \varepsilon [\Lambda ((x^-)^2 \partial_- h - 2x^- h) + \gamma \partial_+ \Lambda \cdot (x^-)^2]; \quad (3.28)$$

$$\delta \mathcal{L} = \varepsilon \partial_- \left[\Lambda (x^-)^2 \mathcal{L} - \frac{2}{\alpha} \Lambda h - \gamma \Lambda x^- \partial_+ \tilde{b} \right] + \varepsilon \partial_+ (\gamma \Lambda x^- \partial_- \tilde{b}); \quad (3.29)$$

$$J^- = \frac{2}{\alpha} \Lambda j^{(1)} + \partial_+ (\gamma \Lambda (x^-)^2 \partial_- \tilde{b}), \quad (3.30)$$

$$J^+ = -\partial_- (\gamma \Lambda (x^-)^2 \partial_- \tilde{b}). \quad (3.31)$$

It is evident that the $j^{(-1)}$ -transformation is identical with the η -transformation.

Since $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$ are the expansion coefficients of h in power of x^- , it is natural to expect that the expansion coefficients, c and a , of ρ yield symmetry transformation. This is indeed the case.

4) c -transformation

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) c(x^+); \quad (3.32)$$

$$\delta \tilde{b} = 0, \quad (3.33)$$

$$\delta h = \varepsilon \Lambda \rho; \quad (3.34)$$

$$\delta \mathcal{L} = 0; \quad (3.35)$$

$$J^- = \frac{2}{\alpha} \Lambda c, \quad J^+ = 0. \quad (3.36)$$

5) a -transformation

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) a(x^+); \quad (3.37)$$

$$\delta \tilde{b} = 0, \quad (3.38)$$

$$\delta h = -\varepsilon \Lambda x^- \rho; \quad (3.39)$$

$$\delta \mathcal{L} = \varepsilon \partial_- \left(-\frac{2}{\alpha} \Lambda \rho \right); \quad (3.40)$$

$$J^- = \frac{2}{\alpha} \Lambda a, \quad J^+ = 0. \quad (3.41)$$

Since $\mathcal{L}$ is evidently invariant under $\tilde{b} \rightarrow \tilde{b} + \text{const}$, we can calculate the generator of this transformation, which turns out to be given by f . Hence we have the following symmetry:

6) f -transformation

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) f(x^+); \quad (3.42)$$

$$\delta \tilde{b} = \varepsilon \frac{2}{\alpha} \Lambda, \quad (3.43)$$

$$\delta h = 0; \quad (3.44)$$

$$\delta \mathcal{L} = \varepsilon \partial_- (\gamma \partial_+ \Lambda \cdot \tilde{b}) = \varepsilon \partial_- (-\gamma \Lambda \partial_+ \tilde{b}) + \varepsilon \partial_+ (\gamma \Lambda \partial_- \tilde{b}); \quad (3.45)$$

$$J^- = \frac{2}{\alpha} \Lambda f, \quad J^+ = 0. \quad (3.46)$$

Finally, we reconsider the λ -transformation. Its generator can be simplified by adding the $j^{(0)}$ -transformation with $\Lambda = \partial_+ \lambda$ and f -transformation with $\Lambda = \frac{1}{2} \partial_+ \lambda$. We call the resulting transformation “ e -transformation”.

7) e -transformation

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) e(x^+); \quad (3.47)$$

$$\delta \tilde{b} = \varepsilon \Lambda \partial_+ \tilde{b}, \quad (3.48)$$

$$\delta h = \varepsilon \partial_+ (\Lambda h); \quad (3.49)$$

$$\delta \mathcal{L} = \varepsilon \partial_+ (\Lambda \mathcal{L}); \quad (3.50)$$

$$J^- = \frac{2}{\alpha} \Lambda e + \partial_+ (\Lambda \partial_- \tilde{b} \cdot h), \quad (3.51)$$

$$J^+ = -\partial_- (\Lambda \partial_- \tilde{b} \cdot h). \quad (3.52)$$

We find that e is given by (2.28).

The above seven generators form a closed algebra. The commutators can be easily calculated from (2.17)-(2.27) and (2.31)-(2.34). Schematically, we obtain the following table:

	$j^{(-1)}$	$j^{(0)}$	$j^{(1)}$	c	a	f	e
$j^{(-1)}$	0						
$j^{(0)}$	$j^{(-1)}$	*					
$j^{(1)}$	$j^{(0)} + *$	$j^{(1)}$	0				
c	0	c	a	0			
a	c	a	0	0	0		
f	0	0	0	c	a	*	
e	$j^{(-1)}$	$j^{(0)}$	$j^{(1)}$	c	a	f	e

Here $*$ denotes a central charge.

4. Power-type extensions

Do the seven generators presented in Sec.3 exhaust the symmetries of our model? No, this is certainly not. We can take the Casimir operator, $j^{(-1)}j^{(1)} - (j^{(0)})^2$, of $SL(2, \mathbf{R})$ as the integrand of a new symmetry generator. Furthermore, it can be shown that

$$j^{(-1)}j^{(1)} - (j^{(0)})^2 = \gamma\epsilon - \frac{1}{4}f^2. \quad (4.1)$$

Hence f^2 must be able to be the integrand of a symmetry generator. This fact suggest the existence of the power-type symmetric generators. This turns out to be indeed the case.

For

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) [c(x^+)]^m, \quad (4.2)$$

we have

$$\delta \tilde{b} = 0, \quad (4.3)$$

$$\delta h = \epsilon \Lambda \rho \cdot m c^{m-1}, \quad (4.4)$$

where c is *defined* by

$$c \equiv \partial_- \rho \quad (4.5)$$

and it is *not* regarded as a x^- -independent quantity, because $\partial_- c = 0$ is a consequence of a field equation. We can then show that

$$\delta \mathcal{L} = \epsilon \partial_- \left(\frac{2}{\alpha} \Lambda (m-1) c^m \right). \quad (4.6)$$

Likewise, we consider

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) [a(x^+)]^n, \quad (4.7)$$

for which we have

$$\delta \tilde{b} = 0, \quad (4.8)$$

$$\delta h = -\varepsilon \Lambda x^- \rho n a^{n-1}, \quad (4.9)$$

where a is *defined* by

$$a \equiv \rho - x^- \partial_- \rho. \quad (4.10)$$

Then we obtain

$$\delta \mathcal{L} = \varepsilon \partial_- \left[-\frac{2}{\alpha} \Lambda (x^- \partial_- \rho \cdot n a^{n-1} + a^n) \right]. \quad (4.11)$$

We note that the following identities hold without using $\partial_- c = \partial_- a = 0$:

$$a + x^- c = \rho, \quad (4.12)$$

$$\partial_- a + x^- \partial_- c = 0. \quad (4.13)$$

It is natural to expect that similar symmetry generators can be constructed from $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$. Unfortunately, however, this is *not* the case, as long as the following naturally-looking definitions are adopted:

$$j^{(-1)} = \frac{1}{2} \partial_-^2 h, \quad (4.14)$$

$$j^{(0)} = \frac{1}{2} x^- \partial_-^2 h - \frac{1}{2} \partial_- h, \quad (4.15)$$

$$j^{(1)} = \frac{1}{2} (x^-)^2 \partial_-^2 h - x^- \partial_- h + h. \quad (4.16)$$

The “appropriate” definitions of them will be found in Sec.7.

Contrary to $j^{(k)}$, f and e are appropriately defined by (2.9) and by (2.28), respectively.

For

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) [f(x^+)]^p, \quad (4.17)$$

we have

$$\delta \tilde{b} = \varepsilon \frac{2}{\alpha} \Lambda \cdot p f^{p-1}, \quad (4.18)$$

$$\delta h = 0. \quad (4.19)$$

Then we find

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left[\frac{2}{\alpha} \Lambda (p-1) f^p - \gamma \Lambda \partial_+ \tilde{b} \cdot p f^{p-1} \right] \\ & + \varepsilon \partial_+ (\gamma \Lambda \partial_- \tilde{b} \cdot p f^{p-1}). \end{aligned} \quad (4.20)$$

For

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) [e(x^+)]^q, \quad (4.21)$$

we have

$$\delta \tilde{b} = \varepsilon \Lambda \partial_+ \tilde{b} \cdot q e^{q-1}, \quad (4.22)$$

$$\delta h = \varepsilon \partial_+ (\Lambda h \cdot q e^{q-1}). \quad (4.23)$$

Then we find

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left(\frac{2}{\alpha} \Lambda (q-1) e^q \right) \\ & + \varepsilon \partial_+ [\Lambda \mathcal{L} \cdot q e^{q-1} + \Lambda \partial_- \tilde{b} \cdot h \partial_- (q e^{q-1})]. \end{aligned} \quad (4.24)$$

5. Combining the symmetry generators

In Sec.4, we have found the existence of an infinite number of x^+ -dependent symmetry generators. This fact is already of great surprise, but extreme surprise comes further.

As is well known, a commutator between any two symmetry generators is again a symmetry generator. If we apply this rule to two power-type symmetry generators,

we obtain a new symmetry generator. Repeating this procedure, we endlessly obtain new symmetry generators, whose integrand contains a product of c^m , a^n , f^p , and ϵ^q . Furthermore, since commutation relations (2.27) and (2.32)-(2.34) contain a δ' term, we encounter $\partial_+ c$, $\partial_+ a$, $\partial_+ f$, and $\partial_+ \epsilon$. Repeating the procedure, we obtain products of derivatives and arbitrarily higher derivatives. It is, however, a rather subtle problem to explicitly construct a desired quantity through the procedure of taking commutators repeatedly. Hence we directly show the invariance of the action under the transformation induced by the expected symmetry generator.

We first present two derivative-type examples. For

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) [\partial_+ c(x^+)]^{m'}. \quad (5.1)$$

we have

$$\delta \tilde{b} = 0, \quad (5.2)$$

$$\delta h = -\varepsilon \rho \partial_+ [\Lambda \cdot m' (\partial_+ c)^{m'-1}]. \quad (5.3)$$

Then we find

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left[\frac{2}{\alpha} \Lambda(m'-1) (\partial_+ c)^{m'} \right] \\ & + \varepsilon \partial_+ \left[-\frac{2}{\alpha} \Lambda c \partial_- (m' (\partial_+ c)^{m'-1}) \right]. \end{aligned} \quad (5.4)$$

For

$$Q = \frac{2}{\alpha} \int dx^+ \Lambda(x^+) [\partial_+ a(x^+)]^{n'}. \quad (5.5)$$

we have

$$\delta \tilde{b} = 0, \quad (5.6)$$

$$\delta h = \varepsilon \rho x^- \partial_+ [\Lambda \cdot n' (\partial_+ a)^{n'-1}]. \quad (5.7)$$

Then we find

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left[-\frac{2}{\alpha} \Lambda ((\partial_+ a)^{n'} + x^- \partial_+ \partial_- \rho \cdot n' (\partial_+ a)^{n'-1}) \right] \\ & + \varepsilon \partial_+ \left[\frac{2}{\alpha} \Lambda \partial_- \rho \cdot \partial_- (x^- n' (\partial_+ a)^{n'-1}) \right]. \end{aligned} \quad (5.8)$$

Now, we proceed to discussing product-type generators. The generator which we consider is

$$Q = \frac{2}{\alpha} \int dx^+ \Phi(x^+), \quad (5.9)$$

where

$$\Phi \equiv \Lambda c^m a^n f^p e^q (\partial_+ c)^{m'} (\partial_+ a)^{n'}. \quad (5.10)$$

In order to simplify the expressions, we introduce such formal derivatives as

$$\frac{\partial \Phi}{\partial c} = \Lambda \cdot m c^{m-1} a^n f^p e^q (\partial_+ c)^{m'} (\partial_+ a)^{n'}. \quad (5.11)$$

For (5.9), we have

$$\delta \tilde{b} = \varepsilon \left(\frac{2}{\alpha} \frac{\partial \Phi}{\partial f} + \partial_+ \tilde{b} \cdot \frac{\partial \Phi}{\partial e} \right), \quad (5.12)$$

$$\delta h = \varepsilon \left[\rho \left(\frac{\partial \Phi}{\partial c} - \partial_+ \frac{\partial \Phi}{\partial (\partial_+ c)} \right) - \rho x^- \left(\frac{\partial \Phi}{\partial a} - \partial_+ \frac{\partial \Phi}{\partial (\partial_+ a)} \right) + \partial_+ \left(h \frac{\partial \Phi}{\partial e} \right) \right]. \quad (5.13)$$

Then we find

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left[\frac{2}{\alpha} \left(-\Phi + c \frac{\partial \Phi}{\partial c} + \partial_+ c \cdot \frac{\partial \Phi}{\partial (\partial_+ c)} + f \frac{\partial \Phi}{\partial f} + e \frac{\partial \Phi}{\partial e} \right) \right. \\ & \left. - \frac{2}{\alpha} x^- \left(c \frac{\partial \Phi}{\partial a} + \partial_+ c \cdot \frac{\partial \Phi}{\partial (\partial_+ a)} \right) - \gamma \partial_+ \tilde{b} \cdot \frac{\partial \Phi}{\partial f} \right] \\ & + \varepsilon \partial_+ \left[-\frac{2}{\alpha} c \partial_- \left(\frac{\partial \Phi}{\partial (\partial_+ c)} - x^- \frac{\partial \Phi}{\partial (\partial_+ a)} \right) + \gamma \partial_- \tilde{b} \cdot \frac{\partial \Phi}{\partial f} \right. \\ & \left. + \mathcal{L} \frac{\partial \Phi}{\partial e} + \partial_- \tilde{b} \cdot h \partial_- \left(\frac{\partial \Phi}{\partial e} \right) \right]. \end{aligned} \quad (5.14)$$

It is more convenient to rewrite (5.14) as

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left\{ \frac{2}{\alpha} \left[-\Phi + c \left(\frac{\partial \Phi}{\partial c} - \partial_+ \frac{\partial \Phi}{\partial (\partial_+ c)} \right) - c x^- \left(\frac{\partial \Phi}{\partial a} - \partial_+ \frac{\partial \Phi}{\partial (\partial_+ a)} \right) \right. \right. \\ & \left. \left. + f \frac{\partial \Phi}{\partial f} + e \frac{\partial \Phi}{\partial e} \right] - \gamma \partial_+ \tilde{b} \cdot \frac{\partial \Phi}{\partial f} \right\} \\ & + \varepsilon \partial_+ \left[\gamma \partial_- \tilde{b} \cdot \frac{\partial \Phi}{\partial f} + \mathcal{L} \frac{\partial \Phi}{\partial e} + \partial_- \tilde{b} \cdot h \partial_- \left(\frac{\partial \Phi}{\partial e} \right) \right. \\ & \left. + \frac{2}{\alpha} \left(\partial_- c \cdot \frac{\partial \Phi}{\partial (\partial_+ c)} + \partial_- a \cdot \frac{\partial \Phi}{\partial (\partial_+ a)} \right) \right] \end{aligned} \quad (5.15)$$

by moving double-derivative terms and by using (4.13).

6. Huge symmetry

It is noteworthy that the concrete expression, (5.10), for Φ is totally implicit in (5.12)-(5.15). This fact implies that (5.12)-(5.15) are actually valid for any function

$$\Phi \equiv \Phi(x^+; c, a, f, e, \partial_+ c, \partial_+ a). \quad (6.1)$$

We can now extend our consideration to the most general symmetry generator

$$Q = \frac{2}{\alpha} \int dx^+ F(x^+; c, a, f, e), \quad (6.2)$$

where

$$c = \{c, \partial_+ c, \partial_+^2 c, \dots, \partial_+^L c\}, \quad (6.3)$$

etc., L being an arbitrary nonnegative integer. We introduce the following notation:

$$\left(\frac{\delta}{\delta c}\right)_+ F \equiv \sum_{t=0}^L (-1)^t \partial_+^t \left(\frac{\partial F}{\partial(\partial_+^t c)}\right), \quad (6.4)$$

etc.

For (6.2), we have

$$\delta \tilde{b} = \varepsilon \left[\frac{2}{\alpha} \left(\frac{\delta}{\delta f}\right)_+ F + \partial_+ \tilde{b} \cdot \left(\frac{\delta}{\delta e}\right)_+ F \right], \quad (6.5)$$

$$\delta h = \varepsilon \left\{ \left[\rho \left(\frac{\delta}{\delta c}\right)_+ F - \rho x^- \left(\frac{\delta}{\delta a}\right)_+ F \right] + \partial_+ \left[h \left(\frac{\delta}{\delta e}\right)_+ F \right] \right\}. \quad (6.6)$$

Then we find

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left\{ \frac{2}{\alpha} \left[-F + c \left(\frac{\delta}{\delta c}\right)_+ F - c x^- \left(\frac{\delta}{\delta a}\right)_+ F + f \left(\frac{\delta}{\delta f}\right)_+ F \right. \right. \\ & \left. \left. + e \left(\frac{\delta}{\delta e}\right)_+ F \right] - \gamma \partial_+ \tilde{b} \cdot \left(\frac{\delta}{\delta f}\right)_+ F \right\} \\ & + \varepsilon \partial_+ \left\{ \gamma \partial_- \tilde{b} \cdot \left(\frac{\delta}{\delta f}\right)_+ F + \mathcal{L} \left(\frac{\delta}{\delta e}\right)_+ F + \partial_- \tilde{b} \cdot h \partial_- \left[\left(\frac{\delta}{\delta e}\right)_+ F \right] \right. \\ & \left. + \frac{2}{\alpha} \sum_{\varphi=c, a, f, e} \sum_{t=1}^L \sum_{m=1}^t (-1)^{m+1} \partial_+^{t-m} \varphi \cdot \partial_+^{m-1} \frac{\partial F}{\partial(\partial_+^t \varphi)} \right\}. \quad (6.7) \end{aligned}$$

The Noether current is given by

$$J^- = \frac{2}{\alpha} F + \partial_+ \left[\partial_- \tilde{b} \cdot h \left(\frac{\delta}{\delta e}\right)_+ F \right], \quad (6.8)$$

$$J^+ = -\partial_- \left[\partial_- \tilde{b} \cdot h \left(\frac{\delta}{\delta e}\right)_+ F \right]. \quad (6.9)$$

Our huge algebra is closed; indeed the commutator is given by

$$\begin{aligned}
& \left[\frac{2}{\alpha} \int dx^+ F_1(x^+), \frac{2}{\alpha} \int dy^+ F_2(y^+) \right] \\
&= i \frac{2}{\alpha} \int dx^+ \left[\sum_{\varphi=c, a} \left(\varphi \left(\frac{\delta}{\delta \varphi} \right)_+ F_1 \cdot \left(\frac{\delta}{\delta f} \right)_+ F_2 + \partial_+ \varphi \cdot \left(\frac{\delta}{\delta \varphi} \right)_+ F_1 \cdot \left(\frac{\delta}{\delta e} \right)_+ F_2 \right) \right. \\
&\quad + \gamma \left(\frac{\delta}{\delta f} \right)_+ F_1 \cdot \left(\frac{\delta}{\delta f} \right)_+ F_2 + f \left(\frac{\delta}{\delta e} \right)_+ F_1 \cdot \partial_+ \left(\frac{\delta}{\delta f} \right)_+ F_2 \\
&\quad \left. + e \left(\frac{\delta}{\delta e} \right)_+ F_1 \cdot \partial_+ \left(\frac{\delta}{\delta e} \right)_+ F_2 - (F_1 \longleftrightarrow F_2) \right]. \tag{6.10}
\end{aligned}$$

7. Generalized Polyakov symmetry

Now, we return to the problem of constructing the symmetry generators nonlinear in $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$.

We can regard (2.15), (2.16), and (2.30) as simultaneous linear algebraic equations for $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$. Assuming that operator inversion is possible, we find

$$j^{(-1)} = K[c\partial_+ c \cdot f - c^2 e - \gamma(\partial_+ c)^2], \tag{7.1}$$

$$j^{(0)} = K[-\frac{1}{2}\partial_+(ca) \cdot f + ace + \gamma\partial_+ a \cdot \partial_+ c], \tag{7.2}$$

$$j^{(1)} = K[a\partial_+ a \cdot f - a^2 e - \gamma(\partial_+ a)^2] \tag{7.3}$$

with

$$K \equiv (a\partial_+ c - c\partial_+ a)^{-1}. \tag{7.4}$$

Thus $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$ are expressible in terms of c , a , f , e , $\partial_+ c$, and $\partial_+ a$. Hence an arbitrary function of $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$ can be regarded as a special case of Φ considered in (6.1). Accordingly, the formulae corresponding to

$$Q = \frac{2}{\alpha} \int dx^+ \tilde{\Phi}(x^+; j^{(-1)}, j^{(0)}, j^{(1)}) \tag{7.5}$$

can be calculated from (5.12)-(5.14) by using the chain rule

$$\frac{\partial \tilde{\Phi}}{\partial c} = \sum_{k=-1}^1 \frac{\partial \tilde{\Phi}}{\partial j^{(k)}} \cdot \frac{\partial j^{(k)}}{\partial c}, \tag{7.6}$$

etc. We find

$$\delta \tilde{b} = \varepsilon(\partial_- \tilde{b} \cdot Y - \frac{2}{\alpha} Z), \quad (7.7)$$

$$\delta h = \varepsilon(\partial_- h \cdot Y - 2hZ + \gamma \partial_+ Y), \quad (7.8)$$

and

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left[\frac{2}{\alpha} \left(-\hat{\Phi} + \sum_{k=-1}^1 j^{(k)} \frac{\partial \hat{\Phi}}{\partial j^{(k)}} \right) + \mathcal{L}Y - \frac{2}{\alpha} h \frac{\partial \hat{\Phi}}{\partial j^{(1)}} - \gamma \partial_+ \tilde{b} \cdot Z \right] \\ & + \varepsilon \partial_+ [\gamma \partial_- \tilde{b} \cdot (Y + Z)], \end{aligned} \quad (7.9)$$

where

$$Y \equiv \sum_{k=-1}^1 (x^-)^{k+1} \frac{\partial \hat{\Phi}}{\partial j^{(k)}}, \quad (7.10)$$

$$Z \equiv \frac{1}{2} \frac{\partial \hat{\Phi}}{\partial j^{(0)}} + x^- \frac{\partial \hat{\Phi}}{\partial j^{(1)}}. \quad (7.11)$$

Thus we have succeeded in generalizing Polyakov's $SL(2, \mathbf{R})$ symmetry to a nonlinear one.

From the above analysis, we have learnt that (7.1)-(7.3) are "appropriate" definitions of $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$. Substituting the appropriate definitions of c , a , f , and e , i.e., (4.5), (4.10), (2.9), and (2.28), into (7.1)-(7.3), we obtain

$$\begin{aligned} j^{(-1)} &= \frac{\alpha}{2} \partial_- \tilde{b} \cdot \partial_- h - \left(\frac{\alpha}{2} \partial_- \tilde{b} \right)^2 h - \frac{\gamma \alpha}{2} \partial_+ \partial_- \tilde{b} \\ &= \frac{\alpha}{2} \mathcal{L} - \gamma \frac{\partial_+ \partial_- \rho}{\rho}, \end{aligned} \quad (7.12)$$

$$j^{(0)} = x^- j^{(-1)} - \frac{1}{2} \partial_- h, \quad (7.13)$$

$$j^{(1)} = (x^-)^2 j^{(-1)} - x^- \partial_- h + h. \quad (7.14)$$

Once we have found those expressions, it makes the derivation of (7.7)-(7.9) easier to use (7.12)-(7.14) directly.

Because of the presence of δ' terms in the commutation relations (2.18) and (2.19), the algebra formed by (7.5) is not closed, that is, Q must include derivatives of $j^{(k)}$:

$$Q = \frac{2}{\alpha} \int dx^+ \tilde{F}(x^+; \mathbf{j}^{(-1)}, \mathbf{j}^{(0)}, \mathbf{j}^{(1)}). \quad (7.15)$$

We employ the notation analogous to (6.3) and (6.4). We find

$$\delta \tilde{b} = \varepsilon(\partial_- \tilde{b} \cdot \mathbf{Y} - \frac{2}{\alpha} \mathbf{Z}), \quad (7.16)$$

$$\delta h = \varepsilon(\partial_- h \cdot \mathbf{Y} - 2h \mathbf{Z} + \gamma \partial_+ \mathbf{Y}), \quad (7.17)$$

and

$$\begin{aligned} \delta \mathcal{L} = & \varepsilon \partial_- \left[\frac{2}{\alpha} \left(-\hat{F} + \sum_{k=-1}^1 j^{(k)} \left(\frac{\delta}{\delta j^{(k)}} \right)_+ \hat{F} \right) + \mathcal{L} \mathbf{Y} \right. \\ & \left. - \frac{2}{\alpha} h \left(\frac{\delta}{\delta j^{(1)}} \right)_+ \hat{F} - \gamma \partial_+ \tilde{b} \cdot \mathbf{Z} \right] \\ & + \varepsilon \partial_+ \left[\gamma \partial_- \tilde{b} \cdot (\mathbf{Y} + \mathbf{Z}) \right. \\ & \left. + \frac{2}{\alpha} \sum_{k=-1}^1 \sum_{\ell=1}^L \sum_{m=1}^{\ell} (-1)^{m+1} \partial_+^{\ell-m} \partial_- j^{(k)} \cdot \partial_+^{m-1} \frac{\partial \hat{F}}{\partial (\partial_+^{\ell} j^{(k)})} \right] \quad (7.18) \end{aligned}$$

where $\mathbf{Y}$ and $\mathbf{Z}$ are obtained from (7.10) and (7.11), respectively, by replacing $\frac{\partial \Phi}{\partial j^{(k)}}$ by $\left(\frac{\delta}{\delta j^{(k)}} \right)_+ \hat{F}$.

8. Anomaly and spontaneous symmetry breakdown

Symmetries found at the operator level may not survive as genuine ones at the level of the representation in terms of state vectors. One reason is possible appearance of anomaly and the other is the spontaneous breakdown of symmetry.

In the general framework, let Q be a symmetric generator expressed as

$$Q = \int d\mathbf{x} F(x), \quad (8.1)$$

where $F(x)$ is a local function of the primary fields $\varphi_A(x)$, and $x^\mu = (x^0, \mathbf{x})$. At the operator level, we can prove

$$\partial_0 Q = 0 \quad (8.2)$$

by using the field equations for φ_A . We say that Q is suffered from anomaly if there exists a vacuum expectation value such that

$$\partial_0^x \int d\mathbf{x} \langle F(x) \varphi_1(y_1) \cdots \varphi_n(y_n) \rangle \neq 0. \quad (8.3)$$

We say that Q is spontaneously broken if there exists a vacuum expectation value such that

$$\int d\mathbf{x} \langle [F(x), \varphi_1(y_1) \cdots \varphi_n(y_n)] \rangle \neq 0. \quad (8.4)$$

Here, it is very important to note the following fact: Given Q , $F(x)$ is not uniquely determined, and both (8.3) and (8.4) are dependent on the choice of $F(x)$.

Now, we go back to our model. In I, we have explicitly constructed all n -point functions and shown that the corresponding τ -functions satisfy Polyakov's recurrence formula.¹⁰ We have also shown, however, that a new type of anomaly arises, that is, the field equation (2.3) is violated:

$$\langle \partial_- (\partial_- h - \alpha \partial_- \tilde{b} \cdot h + \gamma \alpha \partial_+ \tilde{b})(x) h(y) \rangle \neq 0, \quad (8.5)$$

though $\partial_-^2 \rho = 0$ and $\partial_-^3 h = 0$ are all right.

One might conclude, therefore, that $\int dx^+ \Lambda f$ is suffered from anomaly. This is not the case, however. We first note that c and a , defined by (4.5) and (4.10), satisfy $\partial_- c = \partial_- a = 0$ because of $\partial_-^2 \rho = 0$. Likewise, $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$, if defined by (4.14)-(4.16), satisfy $\partial_- j^{(k)} = 0$ ($k = -1, 0, 1$) because of $\partial_-^3 h = 0$. In the discussion of anomaly, we need *not* worry about the fact that (4.14)-(4.16) are not the appropriate definitions in calculating $\delta \mathcal{L}$. We define f by (2.16) (with $\theta = 0^{+1}$) and ϵ by (2.30) (with $\theta = 0$), $j^{(-1)}$, $j^{(0)}$, and $j^{(1)}$ being defined by (4.14)-(4.16). Then we have $\partial_- f = \partial_- \epsilon = 0$ without using the field equation (2.3). Thus all generators of our huge symmetry can be made free of anomaly.

*1) See below.

Next, we discuss the problem of spontaneous symmetry breakdown. We assume that $\langle \rho \rangle$ and $\langle h \rangle$ are independent of x^μ , but $\langle \rho \rangle \neq 0$ because of (2.5). We then see

$$\langle c \rangle = \langle j^{(-1)} \rangle = \langle j^{(0)} \rangle = 0, \quad (8.6)$$

$$\partial_\pm \langle a \rangle = \partial_\pm \langle j^{(1)} \rangle = 0. \quad (8.7)$$

From the above anomaly consideration, f and e should be given by

$$f = \frac{2}{a}(-aj^{(0)} - cj^{(1)} + \gamma\partial_+ a), \quad (8.8)$$

$$e = \frac{1}{a^2}[-2a\partial_+ a \cdot j^{(0)} - \partial_+(ac) \cdot j^{(1)} + \gamma(\partial_+ a)^2], \quad (8.9)$$

where we have avoided the presence of c in the denominator. From (8.8) and (8.9), we find

$$\langle f \rangle = \langle e \rangle = 0. \quad (8.10)$$

Spontaneous breakdown takes place if either (or both) of $\langle \delta \tilde{b} \rangle$ and $\langle \delta h \rangle$ is non-vanishing. Almost all symmetries discussed in Sec.3 are spontaneously broken. Only unbroken ones are translations given by

$$\frac{2}{\alpha} \int dx^+ j^{(-1)}, \quad \frac{2}{\alpha} \int dx^+ e. \quad (8.11)$$

and dilatation given by

$$\frac{2}{\alpha} \int dx^+ (j^{(0)} + x^+ e + \frac{1}{2}f). \quad (8.12)$$

We can see, however, that most of the symmetries of product-type generators are not spontaneously broken owing to (8.6) and (8.7).

9. Discussion

In the present paper, we have extensively analyzed the symmetry properties of the lightcone-gauge two-dimensional quantum gravity. We have found a huge symmetry, which has never been encountered in quantum field theory so far. We do not know what it means; we have just studied the mathematical consequences of the model.

It looks quite surprising that the relative ratios of the three terms in the Lagrangian density (2.2) are arbitrary, because then one supposes that each term would be invariant under our huge symmetry. This is not the case, however; the arbitrariness is superficial. By scale transformations of $\tilde{b}$, h and x^+ , we can always normalize all coefficients to be unity. The important restriction is the nonvanishing of the first and second terms.

It is natural to ask whether or not our model can be extended without spoiling our huge symmetry. The key to the model is that $\tilde{b}$ and h satisfy $\partial_-^2 \rho = 0$ and $\partial_-^3 h = 0$ in spite of the nonlinearity of the theory. In order to have $\partial_-^2 \rho = 0$, the form of the first two terms of $\mathcal{L}$ is crucial. Hence we should consider the following generalization:

$$\tilde{\mathcal{L}} = \partial_- \tilde{b} \cdot \partial_- h - \frac{1}{2} \alpha (\partial_- \tilde{b})^2 h + F(\tilde{b}, \partial_+ \tilde{b}, \partial_- \tilde{b}), \quad (9.1)$$

as long as we do not introduce any new field. In order to have $\partial_-^3 h = 0$, it is crucial that the Euler equation for $\tilde{b}$ is of the form $\partial_- (\ast) = 0$ as in (2.3). Therefore, we require

$$\frac{\partial F}{\partial \tilde{b}} - \partial_+ \frac{\partial F}{\partial (\partial_+ \tilde{b})} = \partial_- (\ast). \quad (9.2)$$

If $\tilde{\mathcal{L}}$ is assumed to be at most quadratic in $\partial_- \tilde{b}$, the F satisfying (9.2) is a linear combination of

$$\tilde{b}^n \partial_- \tilde{b}, \quad \partial_+ \tilde{b} \cdot \partial_- \tilde{b}, \quad (\partial_- \tilde{b})^2. \quad (9.3)$$

The first one is trivial because it is a divergence term. The third one can be absorbed into the second term of $\tilde{\mathcal{L}}$ by redefining h . Thus only the second one survives. In this way, we see that the third term of (2.2) is uniquely characterized.

On the other hand, the dimensionality of the model is not restricted to two dimensions. We may consider

$$\mathcal{L} = \partial_0 \tilde{b} \cdot \partial_0 h - \frac{1}{2} \alpha (\partial_0 \tilde{b})^2 h + \frac{\alpha}{2} \sum_{k=1}^{d-1} \gamma_k \partial_k \tilde{b} \cdot \partial_0 \tilde{b} \quad (9.4)$$

in d dimensions.

Our model is, of course, not probabilistically interpretable. From the physical point of view, it is more reasonable to make BRS invariant by introducing gauge-fixing and Faddeev-Popov ghost terms into the Lagrangian density. It is an interesting problem whether or not our huge symmetry survives in such formulation of the model.

Appendix. λ -transformation generator

We make some comments on the other papers^{11,12} concerning the λ -transformation. As shown in Sec.3, the symmetry generator of the λ -transformation is given by

$$\begin{aligned} & \frac{2}{\alpha} \int dx^+ (\lambda e - \partial_+ \lambda \cdot j^{(0)} - \frac{1}{2} \partial_+ \lambda \cdot f) \\ &= \frac{2}{\alpha} \int dx^+ \lambda (e + \partial_+ j^{(0)} + \frac{1}{2} \partial_+ f). \end{aligned} \quad (A.1)$$

From (4.1) with $\gamma = 1$, we can rewrite the operator part of (A.1) as

$$e + \partial_+ j^{(0)} + \frac{1}{2} \partial_+ f = j^{(-1)} j^{(1)} - (j^{(0)})^2 + \partial_+ j^{(0)} + \ell \quad (A.2)$$

with

$$\ell \equiv \frac{1}{2} \partial_+ f + \frac{1}{4} f^2. \quad (A.3)$$

From (A.3) and (2.26), we have

$$[j^{(k)}(x^+), \ell(y^+)] = 0. \quad (A.4)$$

Hence, as a generator *acting only on the currents* $j^{(k)}$, (A.1) may be replaced by

$$\frac{2}{\alpha} \int dx^+ \lambda(j^{(-1)}j^{(1)} - (j^{(0)})^2 + \partial_+ j^{(0)}). \quad (\text{A.5})$$

The operator part of (A.5) is nothing but the quantity encountered in connection with the energy-momentum tensor “ T_{++} ” in Knizhnik et al.¹¹ or “ $\tilde{T}_{++}$ ” in Helayël-Neto et al.¹² apart from a normalization factor. In the former, there is no room for considering ℓ , while in the latter ℓ is set equal to zero by hand (our f corresponds to their “ $(2/\zeta)g_+$ ”). We emphasize that ℓ is a nontrivial operator.

Helayël-Neto et al.¹² discarded a x^- -dependent term $-x^- \partial_+ j^{(-1)} [= -\frac{1}{2} \partial_+ \partial_- h - \partial_+ j^{(0)}]$ in $\tilde{T}_{++}$ by assuming the x^+ -independence of $j^{(-1)}$. But this assumption contradicts the fact that the two-point function $\langle j^{(-1)}(x^+) j^{(1)}(y^+) \rangle$, which they use, is explicitly dependent on x^+ . Knizhnik et al.¹¹ mentioned nothing about this term. Kuramoto¹³ directly wrote down the generator’s expression, “ T_{gravity} ”, with no mention about the relation to T_{++} . We thus conclude that T_{++} is a quantity different from the generator of the λ -transformation.

References

- 1) M. Abe and N. Nakanishi, Int. J. Mod. Phys. **A6** (1991), 3955.
- 2) M. Abe and N. Nakanishi, Prog. Theor. Phys. **86** (1991), 517.
- 3) M. Abe and N. Nakanishi, Prog. Theor. Phys. **86** (1991), 1087.
- 4) M. Abe and N. Nakanishi, Prog. Theor. Phys. **87** (1992), 495.
- 5) M. Abe and N. Nakanishi, Prog. Theor. Phys. **87** (1992), 757.
- 6) M. Abe and N. Nakanishi, Int. J. Mod. Phys. **A7** (1992), 6405.
- 7) M. Abe and N. Nakanishi, Prog. Theor. Phys. **90** (1993), 1091.
- 8) M. Abe and N. Nakanishi, Prog. Theor. Phys. **92** (1994), 449.
- 9) M. Abe and N. Nakanishi, preprint RIMS-1018.
- 10) A. M. Polyakov, Mod. Phys. Lett. **A2** (1987), 893.
- 11) V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988), 819.
- 12) J. A. Helayël-Neto, S. Mokhtari and A. W. Smith, Phys. Lett. **B236** (1990), 12.
- 13) T. Kuramoto, Phys. Lett. **B233** (1989), 363.