On the q-Deformed N=2 SUSY Algebra

Won-Sang Chung,

Theory Group,

Department of Physics,

College of Natural Sciences,

Gyeongsang National University,

Jinju, 660-701, Korea

To be published in Phys.Lett.A

Abstract

In this paper an alternative example of the q-deformed N=2 SUSY algebra is suggested. This algebra is different from the earlier q-deformed versions of SUSY algebra proposed by Parthasarathy et.al and Spiridonov. The q-deformed boson operators and undeformed fermion operators are used in deforming SUSY algebra.

1 Introduction

Quantum groups or q-deformed Lie algebra implies some specific deformations of classical Lie algebras. From a mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

The q-deformation of Heisenberg algebra was made by Arik and Coon [3], Macfarlane [4] and Biedenharn [5]. Recently there has been some interest in more general deformations involving an arbitrary real functions of weight generators and including q-deformed algebras as a special case [6-10].

In the mean time some theoretical physicists [11,12] constructed the q-deformed version of the fermion algebra for the purpose of deforming the N=2 SUSY algebra.

In this paper we suggest an alternative example of q-deformed SUSY algebra which can not be realized in terms of the superpotentials. Instead we introduce the deformed bosonic mode operators and undeformed fermionic mode operators to realize the q-deformed supercharges and construct the new q-deformed N=2 SUSY algebra.

The ordinary (undeformed) N=2 SUSY algebra [13] takes the following
form

\[ \{Q, Q^+\} = H, \]
\[ Q^2 = (Q^+)^2 = 0, \]
\[ [H, Q] = [H, Q^+] = 0, \]

where all operators are assumed to be well defined on the relevant Hilbert space. Here \( Q \) and \( Q^+ \) are Grassmann odd generators called supercharges and \( H \) Grassmann even generators called Hamiltonian. The physics related to the algebra (1) is explained in detail in ref [13].

It is Parthasarathy and Viswanathan [11] who constructed a q-analogue of N=2 SUSY algebra first. In deforming the algebra (1), they made use of the q-deformed boson algebra and q-deformed fermion algebra. Then the supercharges are given by

\[ Q^+ = a f^+, \quad Q = a^+ f, \]

where \( a, a^+ \) and \( f, f^+ \) satisfy

\[ a a^+ - q a^+ a = q^{-N_B}, \quad [N_B, a^+] = a^+, \quad [N_B, a] = -a, \]

and

\[ f f^+ + q f^+ f = q^{-N_F}, \quad [N_F, f^+] = f^+, \quad [N_F, f] = -f. \]

The relations between mode operators and number operators are given by

\[ a^+ a = [N_B]_B = \frac{q^{N_B} - q^{-N_B}}{q - q^{-1}}, \quad f^+ f = [N_F]_F = \frac{q^{-N_F} - (-)^{N_F} q^{N_F}}{q + q^{-1}}. \]

Then the Hamiltonian is written as

\[ H = q^{-N_B} [N_F]_F + q^{-N_F} [N_B]_B \]

Then the deformed supercharges \( Q, Q^+ \) and Hamiltonian \( H \) satisfy the deformed N=2 SUSY algebra

\[ \{Q, Q^+\} = H, \quad [H, Q] = 0, \quad [H, Q^+] = 0 \]

The algebra looks like the undformed algebra. However, the supercharges do not satisfy the nilpotency condition any more, which makes the algebra proposed by authors of ref [11] differ from the ordinary (undeformed) SUSY algebra.

The second example of q-deformed SUSY algebra is proposed by Spiridonov [14]. He defined the q-deformed factorization operators

\[ A^+ = \frac{1}{\sqrt{2}}(p - i W(x)) T_q, \quad A = \frac{q^{-1}}{\sqrt{2}} T_q^{-1}(p + i W(x)), \]

where \( W(x) \) is called a superpotential and \( x \) and \( p \) satisfy the Heisenberg relation \([x, p] = i\). Here q-scaling operator \( T_q \) is defined as

\[ T_q = q^{x \partial / \partial x}, \quad T_q f(x) = f(q x) \]

Then the q-deformed Hamiltonian and supercharges are written in the following form

\[ H = \begin{pmatrix} q A^+ A & 0 \\ 0 & q^{-1} A A^+ \end{pmatrix}, \quad Q = A \sigma_-, \quad Q = A^+ \sigma_+ \]
where the Pauli matrix $\sigma_-$ and $\sigma_+$ is given by

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and satisfy the undeformed fermionic algebra

$$\sigma_-^2 = \sigma_+^2 = 0, \quad \{\sigma_-, \sigma_+\} = 1$$

Then the q-deformed version of the N=2 SUSY algebra takes the following form

$$\{Q, Q^+\}_q = H, \quad Q^2 = (Q^+)^2 = 0, \quad [H, Q]_q = [Q^+, H]_q = 0,$$

where the q-brackets are defined as

$$[X, Y]_q = qXY - q^{-1}YX, \quad \{X, Y\}_q = qXY + q^{-1}YX.$$

$$[N_B, a^+]_q = a^+, \quad [a, N_B]_q = a.$$  \hspace{1cm} (15)

This algebra seems to be somewhat different from the standard q-oscillator algebra. But, if we define

$$a = \frac{b}{\sqrt{q}}, \quad a^+ = \frac{b^+}{\sqrt{q}},$$

$$N_B = \frac{1 - \frac{1}{q} - q^{-2N}}{q} = \frac{1 - q^{-2N}}{1 - q^{-2}},$$

the algebra (15) then reduces to the standard form of the q-oscillator algebra

$$bb^+ - q^{-2}b^+b = 1,$$

$$[N, b^+] = b^+, \quad [N, b] = -b$$

where $N$ is the number operator associated to $b$ and $b^+$.

In eq.(15) $a$ is assumed to be a hermitian conjugate of $a^+$. Then we can write $a^+a$ and $aa^+$ in terms of the number operator $N_B$, which are given by

$$a^+a = N_B,$$  \hspace{1cm} (18)

$$aa^+ = q^{-2}N_B + q^{-1}.$$  \hspace{1cm} (19)

The proof of eq.(18) and eq.(19) is easy. Now we can assume that the number operator $N_B$ is a smooth function in $a^+a$;

$$N_B = h(a^+a).$$  \hspace{1cm} (20)
If we demand that
\[ [N_B, a^+] = a^+, \] (21)
we find
\[ h(aa^+) = q^{-2}N_B + q^{-1}. \] (22)
If we write the inverse function of \( h \) as \( H \), then the eq.(22) is rewritten as
\[ aa^+ = H(q^{-2}N_B + q^{-1}). \] (23)

From the eq.(20) we also have
\[ a^+a = H(N_B). \] (24)

Inserting eq.(23) and eq.(24) into the first equation of eq.(15) leads to
\[ qH(q^{-2}N_B + q^{-1}) - q^{-1}H(N_B) = 1, \] (25)
which gives the unique solution for \( H(N_B) \);
\[ H(N_B) = N_B. \] (26)

Then we reach the following relation;
\[ a^+a = N_B. \] (27)

Similarly we have q-deformed super charges given by
\[ Q = a^+f^+. \] (28)

\[ Q^+ = a^+f. \] (29)

We can easily check that these two fermionic operators are nilpotent;
\[ Q^2 = (Q^+)^2 = 0, \] (30)
which result from the nilpotency of the fermionic mode operators.\(^1\)

Thus we have the following relation
\[ \{Q, Q^+\}_q = H, \] (31)
where Hamiltonian \( H \) is given by
\[ H = N_F + q^{-1}N_B \] (32)

Another relations of the q-deformed N=2 SUSY algebra are given by
\[ [H, Q]_q = 0, \] (33)
\[ [Q^+, H]_q = 0. \] (34)

These two relations can be easily proved from the definition of the q-deformed boson algebra (15) and the ordinary fermion algebra. Therefore, from the realization (28, 29), we have the following q-deformed N=2 SUSY algebra
\[ \{Q, Q^+\}_q = H, \]

\(^1\) At this stage we stress that the author of ref [14] defined the supercharges in terms of Pauli matrix; \( Q = A\sigma^-, \ Q^+ = A^+\sigma^+ \). Then the nipotency of supercharges results from the fact that \( \sigma^2 = \sigma^+ = 0 \).
\[ [H,Q] = [Q^+, H] = 0, \]
\[ \{Q, Q\} = \{Q^+, Q^+\} = 0. \]  \hspace{1cm} (35)

If we introduce the two even generators \( N \) and \( S \) by
\[ N = \frac{1}{2}(qN_F + N_B), \]  \hspace{1cm} (36)
\[ S = \frac{1}{2}(qN_F - N_B), \]  \hspace{1cm} (37)

Then the following relations provide a realization of a q-deformed superalgebra.
\[ [N, Q^+] = 0, \]
\[ [Q, N] = 0, \]
\[ [Q, S] = Q, \]
\[ [S, Q^+] = -Q^+, \]
\[ \{Q, Q^+\} = H = 2q^{-1}N, \]
\[ [N, S] = 0. \]  \hspace{1cm} (38)

These relations illustrate the general structure of a superalgebra. It is seen that this algebra is generated by the set \( \{N, S, Q, Q^+\} \). \( N \) and \( S \) generate commuting \( U(1) \) group, while the q-deformed anticommutator of two odd generators contains \( N \) only. The last comment holds because the total number operator \( N \) is proportional to the total Hamiltonian.

\section{Fock Space Representation of q-Deformed SUSY Algebra}

In this section we discuss the Fock space representation of the q-deformed N=2 SUSY algebra (35). Now consider the Fock space basis \( |n\> \) defined by
\[ \mathcal{N}|n\> = n|n\>, \hspace{1cm} n = 0, 1, 2, \cdots. \]  \hspace{1cm} (39)

From the relation between \( N_B \) and \( \mathcal{N} \), we can obtain
\[ N_B|n\> = q^{-1}|n\> - |n\>, \]  \hspace{1cm} (40)
where
\[ |n\> = \frac{1 - q^{-2n}}{1 - q^{-2}}. \]  \hspace{1cm} (41)

From the fact that \( a^+a = N_B \), we get
\[ a|n\> = q^{-1/2}\sqrt{|n\>}|n\> - 1 > \]  \hspace{1cm} (42)
where we used the fact that \( [\mathcal{N}, a] = -a \). Similarly, from the definition of q-oscillator algebra (15), we find
\[ a^+|n\> = q^{-1/2}\sqrt{|n+1\>}|n+1\> \]  \hspace{1cm} (43)

If we define the q-bosonic vacuum defined as
\[ a|0\> = 0 \]  \hspace{1cm} (44)
we can construct an orthonormal $n$ q-boson state:

$$|n> = \frac{q^{n/2}(a^+)^n |0>}{\sqrt{|n|!}}$$

(45)

The fermionic sector is simple because it is not deformed. The eigenvalue of the fermionic number operator $N_F$ is 0 or 1. So there exists only two fermionic states. We write the state corresponding to the eigenvalue 0 as the down state $|\downarrow>$ and the state to the eigenvalue 1 as the up state $|\uparrow>$. From the fermionic algebra we have the following relations for the raising and lowering operators

$$f^+ |\uparrow> = 0, \quad f |\uparrow> = |\downarrow>, \quad N_F |\uparrow> = |\uparrow>$$

$$f^+ |\downarrow> = |\uparrow>, \quad f |\downarrow> = 0, \quad N_F |\downarrow> = 0.$$  

(46)

So the down state $|\downarrow>$ is the vacuum of the fermionic sector.

Therefore there exist two types of state for the supercharges and Hamiltonian. Acting the supercharges on them yields

$$Q|n > |\downarrow> = q^{-1/2}\sqrt{|n|} [n - 1] |\uparrow>, \quad Q|n > |\uparrow> = 0,$$

(47)

$$Q^+|n > |\uparrow> = q^{-1/2}\sqrt{|n+1|} [n + 1] |\downarrow>, \quad Q^+|n > |\downarrow> = 0.$$  

(48)

Similarly, applying the deformed Hamiltonian to two types of states yields

$$H|n > |\uparrow> = [n + 1]|n > |\uparrow>,$$

(49)

$$H|n > |\downarrow> = q^{-2}|n| |n > |\downarrow>$$

(50)

which implies that the Hamiltonian is some combination of q-boson number operator $N_B$ and fermionic number operator $N_F$;

$$H = q^{-1}N_B + N_F.$$  

(51)

4 Conclusion

In this paper we suggested an alternative example to the q-deformed $N = 2$ SUSY algebra. We realized the q-deformed supercharges in terms of deformed bosonic mode operators and undeformed fermionic mode operators.

Here we are now going to compare our SUSY algebra (35) with the Spiridonov’s case. Two algebra takes a very similar form. But a closer investigation of two algebras indicates us that one is different from another; that is to say, our algebra (35) is an alternative example to the q-deformed N=2 SUSY algebra. Our algebra (35) has the inherent additional property:

$$[aa^+, a^+a] = 0,$$

(52)

while the SUSY algebra generated by a superpotential (as it happens in the Spiridonov’s case) does not obey always in this restriction. In the case of ref [14], the corresponding operator to the destruction operator $a$ is the operator $A$ as is shown in eq.(9). Then the commutator of $AA^+$ and $A^+A$
is given by

\[ [A^+A, AA^+] \]

\[ = \frac{1}{4}[p^2, q^{-2}W^2(q^{-1}x) - W^2(x)] - \frac{1}{4}[p^2, q^{-1}W'(q^{-1}x) + W'(x)] \]  (53)

where \( \prime \) means the derivative with respect to \( x \). This shows that the right hand side of eq.(53) does not always vanish.

It will be interesting to obtain the q-deformed parasupersymmetric algebra by use of some deformed paraboson and fermion algebra. I hope that this problem and its related topics will become clear in the near future.

Acknowledgement

I give many thanks to the referees for their pointing out some mistakes of my earlier manuscript.

This paper was supported in part by NON DIRECTED RESEARCH FUND, Korea Research Foundation (1994) and in part by the KOSEF through C.T.P. at S.N.U. And the Present Studies were supported in part by Basic Science Research Program, Ministry of Education, 1994 (BSRI-94-2413).

References
