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O(N)-SCALAR MODEL IN CURVED SPACETIME WITH BOUNDARIES: A RENORMALIZATION GROUP APPROACH

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ABSTRACT

We discuss the volume and surface running couplings for $O(N)$ scalar theory in curved spacetime with boundaries. The IR limit of the theory - in which it becomes asymptotically conformally invariant - is studied, and the existence of IR fixed points for all couplings (also in $D = 4 - \epsilon$ dimensions) is shown. For $N = 4$ the behaviour of some gravitational couplings in the IR limit is changing qualitatively, from a growth for $N \leq 4$ to a decrease for $N > 4$. The non-local RG improved effective action, account being taken of the boundary terms, is found. For $O(N)$ scalar theory and for scalar electrodynamics the RG improved effective action in the spherical cap is constructed. The relevance of surface effects for the effective equations of motion for the spherical cap is considered (what maybe important in quantum cosmology). Some preliminary remarks on the connection with Casimir theory are also given.

1. INTRODUCTION

One of the main motivations to study the quantum field theory in curved spacetime with boundaries is connected with the Casimir effect (for an introduction and list of refs, see for example the book [1]). According to the by now already classic result, two neutral conducting surfaces with a small gap between them experience an attractive force. The explicit calculation of the Casimir effect may be done by a variety of methods, among them the zeta-regularization method is the most elegant one (for a recent discussion in the case of a nondispersive material, see [2]).

From another viewpoint, quantum field theory in spaces with boundaries is very useful in quantum cosmology, in particular, in the study of the wave function of the Universe [3, 4], where the choice of the boundary conditions [5, 6] for fields of different spins is also of relevance. Moreover, there are many formal similarities between Casimir theory and quantum cosmology.

The explicit calculations of the boundary divergences of the effective action (or, conformal anomaly in space with boundaries) have been intensively studied recently [6 - 9] (and references herein). They are useful for discussing the wave function of the Universe and also for the construction of the renormalization group. The point is that in order to make a theory to be multiplicatively renormalizable in curved spacetime with boundaries one has to include the surface Lagrangian with the arbitrary coupling constants in the total Lagrangian. When the renormalization group (RG) is constructed, each coupling becomes a running effective coupling. The running surface couplings have been introduced in refs. [7, 10]. Some applications of the running surface couplings have been discussed recently in refs. [10, 11].

The purpose of this work is to discuss RG of $O(N)$ scalar theory in a space with boundaries and, in particular, the surface running couplings in this theory. The paper is organised as follows. In the next section, the «volume» effective couplings in curved space without boundaries are found. Their relevance for the calculation of quantum corrections to the Newtonian potential is discussed. The IR limit where the theory is becoming asymptotically conformally invariant is described. The existence of the barrier $N = 4$ where some gravitational couplings qualitatively change their behaviour in the IR limit is shown. Finally, the IR fixed point is found in another way, via ϵ -expansion in $(4 - \epsilon)$ dimensions.

In section 3 we find the surface running couplings of $O(N)$ scalar theory in a space with boundaries (assuming Dirichlet boundary conditions). The IR behaviour of surface couplings is considered. As an application of these couplings the RG improved non-local effective action in a weak but rapidly varying gravitational field is constructed. Section 4 is devoted to the calculation of the RG improved effective action for $O(N)$ scalar theory and scalar electrodynamics in a spherical cap. Analyzing the effective equations of motion, the influence of the boundary effects to the self-consistent quantum solution (De Sitter) is shown.

In the last section, having in mind the possible extension of the RG approach to studies on the Casimir effect, we make some remarks on electromagnetic boundary conditions and permittivity properties for dispersive media.

2. RENORMALIZATION GROUP EQUATIONS AND EFFECTIVE COUPLINGS IN $O(N)$ -MODEL IN CURVED SPACETIME

Let us start by discussing the $O(N)$ model in curved spacetime. The corresponding Lagrangian has the form (we use Euclidean notations)

$$\begin{aligned} L &= L_m + L_{\text{ext}} , \\ L_m &= \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^a + \frac{1}{2} \xi R \varphi^2 + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 , \\ L_{\text{ext}} &= a_1 R^2 + a_2 C_{\mu\nu\alpha\beta}^2 + a_3 G + \Lambda - \frac{1}{16\pi G} R , \end{aligned} \tag{1}$$

where L_m is the Lagrangian of the matter sector, $\varphi^2 = \varphi^a \varphi^a$, $a = 1, \dots, N$, and L_{ext} is the Lagrangian of the external gravitational field. $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, G is the Gauss-Bonnet invariant.

In flat space such a model is a very popular toy laboratory for studying the scalar bound states (mainly, in $1/N$ -expansion) [12 - 14] (for a recent discussion, see [15, 16] and references therein). Note that the addition of L_{ext} is necessary to make the theory multiplicatively renormalizable in curved spacetime (see [17] for a review).

First of all, we will recall the behaviour of the effective couplings in this theory:

$$\lambda(t) = \frac{\lambda}{1 - \frac{(N+8)\lambda t}{3(4\pi)^2}}, \quad \xi(t) = \frac{1}{6} + \left(\xi - \frac{1}{6} \right) \left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2} \right)^{\frac{N+2}{N+8}},$$

$$m^2(t) = m^2 \left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2} \right)^{\frac{N+2}{N+8}}, \quad (2)$$

where classical scaling dimensions for dimensional parameters will not be included in RG equations; t is the RG parameter.

For the gravitational effective couplings we have ($N \neq 4$)

$$a_1(t) = a_1 - \frac{3N}{2\lambda} \frac{\left(\xi - \frac{1}{6} \right)^2}{(4-N)} \left[\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2} \right)^{\frac{4-N}{N+8}} - 1 \right],$$

$$a_2(t) = a_2 + \frac{Nt}{120(4\pi)^2},$$

$$a_3(t) = a_3 - \frac{Nt}{360(4\pi)^2},$$

$$\Lambda(t) = \Lambda - \frac{3}{2} \frac{Nm^4}{\lambda(4-N)} \left[\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2} \right)^{\frac{4-N}{N+8}} - 1 \right],$$

$$G^{-1}(t) = G_0^{-1} \left\{ 1 + \frac{48\pi G_0 m^2 \left(\xi - \frac{1}{6} \right) N}{\lambda(4-N)} \left[\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2} \right)^{\frac{4-N}{N+8}} - 1 \right] \right\}. \quad (3)$$

Note that the above expression for the gravitational coupling constant gives the possibility to estimate the matter quantum corrections to the Newtonian potential. Starting from the classical Newtonian potential

$$V(r) = -\frac{Gm_1 m_2}{r}, \quad (4)$$

we should substitute the classical gravitational constant in (4) by the effective coupling constant from (3) with $t = \ln \frac{r}{r_0}$ (for more details and list of refs, see [18]). Expanding $G(t)$ and keeping only the linear t terms we will get the leading (logarithmic) corrections to the Newtonian potential.

One can also say a few words about the behaviour of the theory at strong and weak curvature (high and low energies). When $t \rightarrow \infty$ (ultraviolet limit) we have the standard Landau pole; the theory is not asymptotically free. However, at $t \rightarrow -\infty$ (infrared limit), $\lambda(t) \rightarrow 0$ (asymptotic freedom). In this limit $\xi(t) \rightarrow \frac{1}{6}$, hence $\xi_* = \frac{1}{6}$ is an IR fixed point. In the matter sector, the theory is asymptotically free and asymptotically conformally invariant [19] in IR.

The gravitational coupling constants $a_2(t)$, $a_3(t)$ grow linearly with t at $t \rightarrow \pm\infty$ as it normally happens (see [17]). At the same time, the gravitational coupling constants $a_1(t)$, $\Lambda(t)$ and $G^{-1}(t)$ grow for $N < 4$, and for $N > 4$ they decrease with the growth of t .

For $N = 4$ the gravitational coupling constants have a logarithmic t -dependence. For example

$$\begin{aligned} a_1(t) &= a_1 - \frac{4\left(\xi - \frac{1}{6}\right)^2}{8\lambda} \ln\left(1 - \frac{4\lambda t}{(4\pi)^2}\right), \\ \Lambda(t) &= \Lambda - \frac{4m^4}{8\lambda} \ln\left(1 - \frac{4\lambda t}{(4\pi)^2}\right), \\ G^{-1}(t) &= G_0^{-1} \left\{ 1 + \frac{8\pi G_0 m^2 \left(\xi - \frac{1}{6}\right)}{\lambda} \ln\left(1 - \frac{4\lambda t}{(4\pi)^2}\right) \right\}. \end{aligned} \quad (5)$$

Hence, the behaviour of the gravitational coupling constants is drastically changing at the barrier $N = 4$. For $O(4)$ the gravitational coupling constants still grow at $t \rightarrow -\infty$, but only logarithmically.

Let us now study the properties of the theory under discussion in the infrared in more detail, using ϵ -expansion technique which has been discussed in refs. [20 - 22, 16]. We will

work in $D = (4 - \epsilon)$ -dimensional spacetime. As usual [20 - 22], to study the critical behaviour of such a system it is enough to consider the renormalizable subset of the theory where masses are extremely small or zero. Also the explicit μ -dependence should be introduced for some coupling constants in order to keep them dimensionless in $D = 4 - \epsilon$ dimensions. With these remarks the Lagrangian to be considered is

$$\begin{aligned} L &= L_m + L_{\text{ext}} , \\ L_m &= \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^a + \frac{1}{2} \xi R \varphi^2 + \frac{\lambda \mu^\epsilon}{4!} \varphi^4 , \\ L_{\text{ext}} &= \mu^{-\epsilon} (a_1 R^2 + a_2 C_{\mu\nu\alpha\beta}^2 + a_3 G) . \end{aligned} \quad (6)$$

The RG equations for the coupling constants in $4 - \epsilon$ dimensions are

$$\begin{aligned} \frac{d\lambda}{dt} &= -\epsilon\lambda + \frac{N+8}{(4\pi)^2 3} \lambda^2 , \\ \frac{d\xi}{dt} &= \frac{1}{(4\pi)^2} \frac{N+2}{3} \lambda \left(\xi - \frac{1}{6} \right) , \\ \frac{da_1}{dt} &= \epsilon a_1 + \frac{N}{2(4\pi)^2} \left(\xi - \frac{1}{6} \right)^2 , \\ \frac{da_2}{dt} &= \epsilon a_2 + \frac{N}{120(4\pi)^2} , \\ \frac{da_3}{dt} &= \epsilon a_3 - \frac{N}{360(4\pi)^2} . \end{aligned} \quad (7)$$

Let us consider the matter sector. There are two fixed points:

$$\lambda_1^* = 0 , \quad \lambda_2^* = \frac{3\epsilon(4\pi)^2}{N+8} , \quad (8)$$

where the trivial fixed point $\lambda_1^* = 0$ is unstable. At the IR fixed point λ_2^* we have $\xi_2^* = \frac{1}{6}$.

Hence, we showed that there is an IR stable fixed point in the matter sector. It means that a model near a phase transition shows the second order transition behaviour. Moreover , the

system at the above fixed point is approximately conformally invariant (asymptotic conformal invariance [19]).

Now, one can turn to the analysis of the fixed points in the gravitational sector. We get at the stable IR fixed point (λ_2^*, ξ_2^*)

$$a_1^* = 0, \quad a_2^* = -\frac{N}{120(4\pi)^2 \epsilon}, \quad a_3^* = \frac{N}{360(4\pi)^2 \epsilon}. \quad (9)$$

Thus, there exists the non-trivial IR fixed point for all coupling constants of the theory under discussion. At this fixed point the asymptotic conformal invariance [17] is realized. For large N (but finite ϵ) we find that λ_2^* is decreasing while a_2^*, a_3^* are increasing. It is interesting to remark that quantum gravity effects (in frames of R^2 -gravity) may drastically change the above picture. In particular, as was shown recently in ref. [23], many new IR fixed points appear even at the limit of large N . We will finish the discussion of renormalization group properties of the $O(N)$ -theory in the IR limit at this point. It is interesting to note that RG for composite boundstates in the theory under discussion is getting to be trivial (because the corresponding renormalizations are finite [24], unlike the well-known case of four-fermion theories [25]).

3. RUNNING SURFACE COUPLINGS IN THE $O(N)$ MODEL

We will turn now to the study of $O(N)$ scalar theory (massless, for simplicity) in curved spacetime M with boundaries ∂M . The renormalization of such a theory maybe done [7] similarly as in curved spacetime without boundaries. Suppose that for the theory under discussion the Dirichlet boundary conditions (see [6], for example)

$$\varphi(x) = 0, \quad x \in \partial M \quad (10)$$

maybe chosen.

In curved spacetime with boundaries, in order to have a renormalizable theory [7], one has to add the boundary action S_b to the total action of the theory (1) (with $m^2 = \Lambda = \frac{1}{G} = 0$).

It is convenient to introduce two invariants expressed in terms of $R_{\mu\nu\alpha\beta}$ and the extrinsic curvature of the boundary $K_{\mu\nu}$ [8, 9]:

$$\begin{aligned}
q &= \frac{8}{3}K^3 + \frac{16}{3}K_\mu^\nu K_\nu^\alpha K_\alpha^\mu - \\
&- 8KK_{\mu\nu}K^{\mu\nu} + 4KR - 8R_{\mu\nu}(Kn^\mu n^\nu + K^{\mu\nu}) + \\
&+ 8R_{\mu\nu\alpha\beta}K^{\mu\alpha}n^\nu n^\beta, \\
g &= K_\mu^\nu K_\nu^\alpha K_\alpha^\mu - KK_{\mu\nu}K^{\mu\nu} + \frac{2}{9}K^3.
\end{aligned} \tag{11}$$

Then, one can write the boundary classical action as

$$\begin{aligned}
S_b &= \int_{\partial M} d^3x \sqrt{\gamma} \text{tr} \{ \alpha_D q + \beta_D g + \gamma_D RK + \\
&+ \delta_D n^\mu \nabla_\mu R + \zeta_D C_{\mu\nu\alpha\beta} K^{\mu\alpha} n^\nu n^\beta \},
\end{aligned} \tag{12}$$

where $\gamma_{\alpha\beta}$ is the induced metric on the boundary, n^μ is an outward normal for ∂M and α_D ,, ζ_D are surface coupling constants. From the RG viewpoint, these surface coupling constants are becoming the running surface couplings [7, 10]. Note that the subscript D means that we use the Dirichlet boundary conditions.

One can study now the behaviour of running coupling constants in the theory under discussion. The «volume» effective couplings have been presented already in the previous section (see Expr. (2) and (3)). Using the results of the explicit calculation of the one-loop boundary divergences [8, 9] one finds

$$\begin{aligned}
\alpha_D(t) &= \alpha_D - \frac{Nt}{360(4\pi)^2}, \\
\beta_D(t) &= \beta_D + \frac{2Nt}{35(4\pi)^2}, \\
\gamma_D(t) &= \gamma_D + \frac{N\left(\xi - \frac{1}{6}\right)}{6\lambda} \left(\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2} \right)^{\frac{6}{N+8}} - 1 \right),
\end{aligned}$$

$$\begin{aligned}\delta_D(t) &= \delta_D + \frac{N\left(\xi - \frac{1}{6}\right)}{4\lambda} \left(\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2}\right)^{\frac{6}{N+8}} - 1 \right), \\ \zeta_D(t) &= \zeta_D + \frac{Nt}{15(4\pi)^2}.\end{aligned}\tag{13}$$

In the IR limit ($t \rightarrow -\infty$) where theory is asymptotically free the behaviour of surface running couplings is similar to the behaviour of gravitational coupling constants. They grow when $t \rightarrow -\infty$.

The infrared behaviour of the theory may also be described using the ϵ -expansion in $(4-\epsilon)$ -dimensions. In particular, in the stable IR fixed point (λ_2^*, ξ_2^*) in addition to (9) we will find:

$$\begin{aligned}\alpha_D^* &= \frac{N}{360(4\pi)^2\epsilon}, \\ \beta_D^* &= -\frac{2N}{35(4\pi)^2\epsilon}, \\ \gamma_D^* &= 0, \\ \delta_D^* &= 0, \\ \zeta_D^* &= -\frac{N}{15(4\pi)^2\epsilon}.\end{aligned}\tag{14}$$

It would be of interest to take into account the quantum gravitational effects (along the lines of [23]) to these IR fixed points. Thus, we defined the IR fixed point for all volume and surface couplings in $(4-\epsilon)$ -dimensions.

Now, we will turn to the study of the applications of the running coupling constants in curved spacetime, (for an introduction to RG in curved spacetime, see [17]). We will discuss the RG improved effective Lagrangian which implies the summation of leading logarithms of perturbation theory. The technique of RG improvement (or Wilsonian procedure [20]) is quite well-known and it is not necessary to repeat all details here. It has been applied successfully to the study of the effective potential in flat space [26] (especially, for Standard Model effective

potential) and in curved space [27]. In flat space the correctness of the RG improvement maybe shown also via explicit loop calculations in the symmetry-broken phase [28].

Let us briefly recall how the RG improved effective action maybe obtained. First, one has to write the RG equation for the effective action $\Gamma(\lambda_i, \varphi, g_{\mu\nu}, \mu)$ where μ is mass parameter, λ_i is the set of all couplings, and φ is the background field. Solving RG equation by means of characteristics method one gets

$$\Gamma(\lambda_i, \varphi, g_{\mu\nu}, \mu) = \Gamma(\lambda_i(t), \varphi(t), g_{\mu\nu}, \mu e^t), \quad (15)$$

where $\lambda_i(t)$ are the running coupling constants, and $\varphi(t)$ is the background running field. The functional form of the right-hand side of Eq. (15) is defined by using some boundary condition at $t = 0$. The standard choice is to take the classical action as such a boundary condition.

The next problem in the explicit calculation of (15) is the choice of the RG parameter t . In order to make this choice, one has to specify the conditions at which the RG improved effective action is going to be found (for explicit examples, see papers [26, 27, 11] and refs. therein).

First of all, we will consider the purely gravitational background where the gravitational field is weak but rapidly varying, i.e.

$$\nabla\nabla R \gg R^2. \quad (16)$$

One can also require similar conditions for surface invariants.

The calculations of the one-loop effective action as an expansion in curvature invariants at such conditions has been considered in refs. [29, 30] (and refs. therein) in spaces without boundaries. Our result is beyond one-loop as it makes summation of the leading logarithms of the whole perturbation theory (RG improvement). At the same time, it will be also an expansion in the curvature invariants although we cannot go up to terms higher than the invariants which are quadratic in curvature. From the explicit one-loop calculations and by dimensional arguments

$$t \simeq \frac{1}{2} \ln \frac{-\square + \xi R}{\mu^2}. \quad (17)$$

Moreover, due to condition (16) the ξR -term in (17) is not important. Using such a form of t in the RG improved effective action (15) with the explicit running couplings (2), (3) and (13) we will obtain

$$\begin{aligned} \Gamma_{\text{RG}} = & \int d^4x \sqrt{g} \left\{ R \left(a_1 - \frac{3N \left(\xi - \frac{1}{6} \right)^2}{2\lambda(4-N)} \times \right. \right. \\ & \times \left[\left(1 - \frac{(N+8)\lambda \ln \left(-\frac{\square}{\mu^2} \right)}{6(4\pi)^2} \right)^{\frac{4-N}{N+8}} - 1 \right] R + \\ & \left. \left. + C_{\mu\nu\alpha\beta} \left(a_2 + \frac{N \ln \left(-\frac{\square}{\mu^2} \right)}{240(4\pi)^2} \right) C^{\mu\nu\alpha\beta} \right\} + \right. \\ & + \int_{\partial M} d^3x \sqrt{\gamma} \text{tr} \left\{ \left(\alpha_D - \frac{N \ln \left(-\frac{\square}{\mu^2} \right)}{720(4\pi)^2} \right) q + \right. \\ & + \left(\beta_D + \frac{N \ln \left(-\frac{\square}{\mu^2} \right)}{35(4\pi)^2} \right) g + K \left(\gamma_D + \right. \\ & \left. \left. + \frac{N \left(\xi - \frac{1}{6} \right)}{6\lambda} \left[\left(1 - \frac{(N+8)\lambda \ln \left(-\frac{\square}{\mu^2} \right)}{6(4\pi)^2} \right)^{\frac{6}{N+8}} - 1 \right] R \right) \right\} \end{aligned}$$

$$+C_{\mu\nu\alpha\beta}\left\{\zeta_D + \frac{N \ln\left(-\frac{\Pi}{\mu^2}\right)}{30(4\pi)^2}K^{\mu\alpha}n^\nu n^\beta\right\}. \quad (18)$$

Note that some total derivative terms in (18) have been dropped. For $N = 4$, in first term of (18) the $a_1(t)$ (5) should be used.

Hence, we have got the non-local RG improved gravitational effective action with account being taken of boundary terms. It is interesting to note that the effective action (18) maybe relevant to black hole physics where boundary terms may give some additional contribution to different quantities like black hole entropy or vacuum radiation flux through the future null infinity [30].

4. RG IMPROVED EFFECTIVE ACTION IN A SPHERICAL CAP

One of the main motivations to study quantum field theory in curved spacetime with boundaries is connected with quantum cosmology, or more precisely with the study of the quantum state of the Universe [3, 4]. In such a case one can limit oneself to constant curvature spaces: $R_{\mu\nu} = \Lambda g_{\mu\nu}$. It is well-known that the quasilocal approximation for effective action works quite well in spaces of constant curvature.

We will consider the situation when for non-zero scalar background the curvature is bigger than the effective mass, i.e. $\Lambda \gg \varphi^2$. Then the RG parameter t has the form:

$$t = \frac{1}{2} \ln \frac{R}{\mu^2} \equiv \frac{1}{2} \ln \frac{4\Lambda}{\mu^2}. \quad (19)$$

The background which is considered below is a spherical cap C , i.e. the region of the four-sphere with maximum colatitude θ . Using the results of the calculation of conformal anomaly in this case [9] (for Dirichlet boundary conditions), one may find the RG improved effective action:

$$\begin{aligned}
\Gamma &= \int_M d^4x \sqrt{g} S_{V, RG} + \int_{\partial M} d^3x \sqrt{\lambda} S_{b, RG} = \\
&= 24\pi^2 \left\{ \left(16a_1(t) + \frac{8}{3}a_3(t) + \frac{\lambda(t)\varphi^4}{4!\Lambda^2} + 2\xi(t)\frac{\varphi^2}{\Lambda} \right) \times \right. \\
&\quad \times \left(\frac{1}{2} - \frac{3}{4}\cos\theta + \frac{1}{4}\cos^3\theta \right) + 2\cos^3\theta\alpha_D(t) + \\
&\quad \left. + \frac{9}{2}\cos\theta\sin^2\theta\gamma_D(t) \right\}. \tag{20}
\end{aligned}$$

For the pure four-sphere S^4 the corresponding expression has the form:

$$\Gamma = 24\pi^2 \left\{ 16a_1(t) + \frac{8}{3}a_3(t) + \frac{\lambda(t)}{4!} \frac{\varphi^4}{\Lambda^2} + 2\xi(t) \frac{\varphi^2}{\Lambda} \right\} \tag{21}$$

(for the study of the one-loop effective action in De Sitter space, see [31]). It is well-known that the four-sphere S^4 (De Sitter space) describes the inflationary stage of the early Universe. The scale factor is given as

$$a(T) = \frac{\sqrt{3}\beta}{2} \left[\exp\left(\frac{T}{\sqrt{3}\beta}\right) + K \exp\left(-\frac{T}{\sqrt{3}\beta}\right) \right], \tag{22}$$

where T is physical time, $K = 1, 0, -1$ and $\beta = \Lambda^{-\frac{1}{2}}$.

Let us analyse now the effective equations of motion (for $N = 1$, see also [11])

$$\frac{\partial \Gamma}{\partial \varphi^2} = \frac{\partial \Gamma}{\partial \Lambda} = 0. \tag{23}$$

Putting for simplicity $\varphi^2 = 0$ in (21) one can see that there maybe some quantum self-consistent solution $\Lambda \neq 0$ for $\xi \neq \frac{1}{6}$. In particular, for $\lambda t \ll 1$ (keeping only linear terms in t)

we get

$$\ln \frac{4\Lambda}{\mu^2} \simeq \frac{3(4\pi)^2}{\lambda(2N+4)} \left[\frac{1}{1080 \left(\xi - \frac{1}{6} \right)^2} - 1 \right]. \quad (24)$$

This is the quantum De Sitter solution (the parameter β in (22) is defined via theory parameters).

One can see the importance of boundary terms in effective equations in this example. For Γ (20) ($\varphi = 0$ again) we get the equation of motion which leads to the following solution (λt is again small):

$$\begin{aligned} -\ln \frac{4\Lambda}{\mu^2} = & \left\{ \left(8 \left(\xi - \frac{1}{6} \right)^2 - \frac{1}{135} \right) \left(\frac{1}{2} - \frac{3}{4} \cos \theta + \frac{1}{4} \cos^3 \theta \right) \right. \\ & \left. - \frac{\cos^3 \theta}{180} - \frac{3}{2} \cos \theta \sin^2 \theta \left(\xi - \frac{1}{6} \right) \right\} \left\{ \frac{8 \left(\xi - \frac{1}{6} \right)^2 \lambda(2N+4)}{3(4\pi)^2} \times \right. \\ & \left. \times \left(\frac{1}{2} - \frac{3}{4} \cos \theta + \frac{1}{4} \cos^3 \theta \right) - \frac{3}{2} \cos \theta \sin^2 \theta \left(\xi - \frac{1}{6} \right) \frac{\lambda(2+N)}{3(4\pi)^2} \right\}^{-1}. \end{aligned} \quad (25)$$

One may compare (24) and (25) in order to see the explicit influence of the boundary effects upon the self-consistent quantum solution.

Thus, we have, shown how the RG improvement procedure maybe used to take into account the boundary effects in the back-reaction problem. Similarly, one can consider other backgrounds. Moreover, there are not any problems to find RG improved effective action on similar backgrounds for any other theory.

As another example let us consider massless scalar electrodynamics. The necessary effective coupling are (see, for example [17, 27] and first ref. of. [26] for flat space):

$$\begin{aligned} \lambda(t) = & \frac{1}{10} e^2(t) \left[\sqrt{719} \tan \left(\frac{1}{2} \sqrt{719} \ln e^2(t) + C \right) + 19 \right], \\ C = & \arctan \left[\frac{1}{\sqrt{719}} \left(\frac{10\lambda}{e^2} - 19 \right) \right] - \frac{1}{2} \sqrt{719} \ln e^2, \end{aligned}$$

$$\begin{aligned}
e^2(t) &= e^2 \left(1 - \frac{2e^2 t}{3(4\pi)^2} \right)^{-1}, \\
\varphi^2(t) &= \varphi^2 \left(1 - \frac{2e^2 t}{3(4\pi)^2} \right)^{-9}, \\
\xi(t) &= \frac{1}{6} + \left(\xi - \frac{1}{6} \right) \left[\frac{e^2(t)}{e^2} \right]^{-26/5} \frac{\cos^{2/5} \left(\frac{1}{2} \sqrt{719} \ln e^2 + C \right)}{\cos^{2/5} \left(\frac{1}{2} \sqrt{719} \ln e^2(t) + C \right)}, \\
a_1(t) &= a_1 + \left(\xi - \frac{1}{6} \right)^2 \int_0^t \frac{dt}{(4\pi)^2} \left[\frac{e^2(t)}{e^2} \right]^{-52/5} \times \frac{\cos^{4/5} \left(\frac{1}{2} \sqrt{719} \ln e^2 + C \right)}{\cos^{4/5} \left(\frac{1}{2} \sqrt{719} \ln e^2(t) + C \right)}, \\
a_3(t) &= a_3 - \frac{8t}{45(4\pi)^2}, \tag{26}
\end{aligned}$$

where $e^2 \equiv e^2(0)$ is the electrical charge and $\varphi^2 = \varphi^a \varphi^a$, $a = 1, 2$.

The surface couplings $\alpha_D(t)$ and $\gamma_D(t)$ maybe found using the corresponding beta-functions calculated, for example, in ref. [9]:

$$\begin{aligned}
\alpha_D(t) &= \alpha_D - \frac{7t}{315(4\pi)^2}; \\
\gamma_D(t) &= \gamma_D - \frac{31t}{270(4\pi)^2} - \frac{1}{3(4\pi)^2} \int_0^t \left(\xi(t) - \frac{1}{6} \right) dt. \tag{27}
\end{aligned}$$

Then the RG improved effective action in the spherical cap is

$$\begin{aligned}
\Gamma &= 24\pi^2 \left\{ \left(16a_1(t) + \frac{8}{3}a_3(t) + \frac{\lambda(t)\varphi^4(t)}{4!\Lambda^2} + \right. \right. \\
&\quad \left. \left. + \frac{2\xi(t)\varphi^2(t)}{\Lambda} \right) \left(\frac{1}{2} - \frac{3}{4}\cos\theta + \frac{1}{4}\cos^3\theta \right) + \right. \\
&\quad \left. + 2\cos^3\theta\alpha_D(t) + \frac{9}{2}\cos\theta\sin^2\theta\gamma_D(t) \right\}. \tag{28}
\end{aligned}$$

Using this effective action, where the explicit values for coupling constants are given by (26) and (27), one can again study the effective equations at high curvature $\left(t = \frac{1}{2} \ln \frac{4\Lambda}{\mu^2}\right)$ or at small curvature limit $\left(t = \frac{1}{2} \ln \frac{\phi^2}{\mu^2}\right)$. Note that in the small curvature limit the expression (28) reduces to the universal effective potential which maybe written only with the help of the corresponding beta-functions (see last ref. in [27]).

As a final remark note that in a similar way the effective action maybe obtained for an arbitrary renomalizable theory (including GUTS); see the discussion in ref. [11].

5. DISCUSSION

In this work we have considered RG properties of O(N)-scalar theory in curved spacetime (with boundaries). The effective volume and surface coupling constants have been derived. Using the RG equation the improved non-local effective action in curved spacetime with boundaries has been found. The local RG improved effective action in a spherical cap has also been found. The effective equations show the way boundary effects may be present in self-consistent solutions (back-reaction problem).

Let us mention other possible applications of quantum field theory in curved spacetime with boundary to quantum cosmology. The wave function of the Universe is given by

$$\psi(\Lambda, \phi) = e^{-\Gamma}, \quad (29)$$

where the classical background (solution of the effective equations of motion) should be considered and Γ for any specific hypersurface maybe found as in the previous section for a spherical cap. Then, it is not difficult to find numerically a probability distribution for the set of different boundary conditions (using the wave function above).

From another viewpoint, the calculation of transition amplitudes between field configurations at initial and final hypersurfaces has direct analogy with Casimir theory. Moreover, the effective action in curved spacetime with boundaries is directly analogous to the Casimir energy in solid state physics. Having in mind the possible extension of the above

formalism to Casimir theory it is reasonable to give some remarks on some general features of conventional Casimir theory.

The most familiar case is that of electrodynamics in which perfect conducting boundary conditions are imposed on the material surfaces. These conditions imply that the normal component of the magnetic field and the tangential component of the electric field vanish.

However, some care has to be taken as regards the applicability of conditions of this sort in practice. There is no real material satisfying the perfect boundary conditions at all frequencies. When the frequencies become much higher than the absorption (or plasma) frequencies, then the dispersion effect implies that the permittivity, and also the permeability, tend to unity. The photons do not «see» the medium any more. Hence, one must bear in mind that the imposition of perfectly conducting boundary conditions is an extreme idealization. Quite analogously, one must be aware that the adoption of the Dirichlet or Neumann boundaries in scalar field theories is also an extreme idealization.

Note in this respect, that it may well be so that the infinities which plague Casimir energy calculations in dielectric media when the geometry is non-trivial (curved surfaces, typically) owe their presence to the neglect of the dispersion in the dielectric material (cf. for instance, Refs. [32, 33]).

In the application of field theory in curved space with boundaries to quantum cosmology the conditions are to some extent quite analogous. For example, in the discussion of ref. [9], when the ghost satisfies Dirichlet boundary conditions, the emerging boundary conditions are formally exactly those holding at the surface of a perfectly conducting body.

Let us make a rough estimate of the magnitude of the «absorption» frequency ω_0 above which the standard high-conductivity assumption of quantum field theory is definitely inappropriate (using ordinary electromagnetism as an example). We model the medium as a high-permittivity dielectric material endowed with a single absorption frequency $\omega = \omega_0$, playing the role of a soft high-energy cutoff. Adopting a Lorentz dispersion model with one single damping constant γ , we may write the permittivity $\epsilon(\omega)$ as

$$\epsilon(\omega) = 1 + \frac{\chi_s}{1 - \frac{\omega^2}{\omega_0^2} - \frac{i\omega\gamma}{\omega_0^2}}, \quad (30)$$

χ_s being the static susceptibility.

The presence of the damping constant γ in (30) is of physical importance since the fundamental quantity appearing in the two-point function for the fluctuating electromagnetic field is the imaginary part of the retarded Green function [34]. In practice, the calculation of physical quantities like vacuum energy or surface stress tensor is accomplished by a complex frequency rotation whereby the central material property becomes $\epsilon(i\zeta)$ evaluated at the positive imaginary frequency axis, $\zeta = -i\omega$.

For an ordinary material we may typically choose $\omega_0 \approx 5 \times 10^{16} \text{ s}^{-1}$, $\gamma \approx 4 \times 10^{13} \text{ s}^{-1}$ (copper, for instance). Accordingly, the ratio $\gamma / \omega_0 \approx 0.8 \times 10^{-3}$ becomes a small quantity. This is the reason why the γ – term in Eq. (30) is usually dropped when integrating over the imaginary frequencies. Actually, we see that for $\omega \gg \omega_0$ we get the plasma dispersion relation

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad (31)$$

where ω_p is the plasma frequency, $\frac{\omega_p}{\omega_0} \approx \sqrt{\chi_s} \frac{\omega_0}{\omega}$. For ordinary susceptibilities, $\omega_p < \omega_0$, whereas for a high susceptibility material the ratio ω_p / ω_0 is according to the above formula indeterminate.

The conclusion from the above simple remarks is that the dispersive effect is, in principle, an indispensable element in a proper Casimir analysis. It also should be taken into account in the attempts to generalise the RG approach discussed in this work for the Casimir effect. Note also that only in some very special cases like the calculation of the attractive force between two plane dielectric surfaces can one neglect the dispersive effect.

Notice finally that the results of this work maybe easily extended so as to include quantum gravity effects (at least, for renormalizable models of quantum gravity like R^2 -gravity); see [17] for a review. One only has to do the explicit calculation of the boundary counterterms for such theory. We hope to address some of the questions discussed in this section in the near future.

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