QUASI-PERIODIC SOLUTIONS
OF HAMILTONIAN EVOLUTION EQUATIONS

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The problem discussed here is the persistency of quasi-periodic solutions of linear or integrable equations after Hamiltonian perturbation. This subject is closely related to the well known “KAM-theory” on invariant tori in smooth dynamical systems. Most of the research has been achieved within the last ten years by various authors (including J. Fröhlich, T. Spencer, E. Wayne, S. Kuksin, and more recently W. Craig, E. Wayne and myself). The main difference with KAM is the fact that one considers (finite dimensional) tori in an infinite dimensional phase space. Considering PDE’s in space dimension > 1 , the most significant new difficulty arising is the presence of large clusters of normal frequencies. This feature which seems incompatible with the use of a “standard” KAM technology as applied by E. Wayne [W] and S. Kuksin [K1] has lead to the development of a new method to approach the persistency problems, also in finite dimensional phase space. The first PDE-results obtained along these lines appear in the work of W. Craig and E. Wayne [C-W] on time-periodic solutions of 1D-nonlinear wave equations. Later on the author extended their method to cover the full setting of quasi-periodic solutions, leading in particular to a new proof of the KAM theorem and Melnikov theorem in finite dimensional phase space under weaker nonresonance hypothesis (not excluding multiplicities in the normal frequencies). This understanding lead to significant progress on the PDE problems. Presently the persistency problem is in 1D (1 space dimension) in satisfactory way understood (under the Dirichlet and periodic boundary conditions). In higher space dimension several cases can be treated (especially the nonlinear Schrödinger equation (NSE) and the case of time periodic solutions). Many problems remain and are today an active research topic.

The purpose of this exposé is to give a brief account of the model problems, the techniques and some of the results.

Consider an equation of the form

$$iu_t + Au + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0 \quad (1)$$

where A is a selfadjoint operator and $H = H(u, \bar{u})$ is polynomial (or real analytic). We assume that A is diagonal in an eigenfunction basis which is “well localized” wrt the exponentials. The following are the main examples

(i)

$$A = -\Delta + M_\sigma \quad \text{where} \quad M_\sigma e^{i\langle n, x \rangle} = \sigma_n e^{i\langle n, x \rangle}, \quad \sigma_n \in \mathbb{R}$$

obtained by adding to the Laplacian a Fourier multiplier M_σ . In this case, the eigenfunction basis are the exponentials and the eigenvalues

$$\lambda_n = |n|^2 + \sigma_n. \quad (2)$$

We assume

$$\lim_{|n| \rightarrow \infty} \sigma_n = 0 \quad (3)$$

or

$$\lim_{|n| \rightarrow \infty} |\sigma_{n_1} - \sigma_{n_2}| = 0. \quad (4)$$

(ii) Consider the 1D Sturm Liouville operator

$$A = -\frac{d^2}{dx^2} + V(x) \quad (5)$$

where V is a real analytic 1-periodic potential.

The periodic spectrum of A is a sequence

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots < \lambda_{2n-1} \leq \lambda_{2n} < \dots \quad (6)$$

which satisfies

$$\lambda_{2n-1}, \lambda_{2n} = \pi^2 n^2 + \int V dx + o(n^{-2}) \quad (7)$$

$$\lambda_{2n} - \lambda_{2n-1} \rightarrow 0 \quad \text{rapidly.} \quad (8)$$

The corresponding eigenfunctions $\varphi_{2n-1}, \varphi_{2n}$ are periodic or antiperiodic according to the parity of n . Expanding φ_j in a Fourier series

$$\varphi_j(x) = \sum_m \hat{\varphi}_j(m) e^{i\pi m x}. \quad (9)$$

There is the localization property wrt exponentials

$$|\hat{\varphi}_j(m)| < C e^{-c| |m| - j/2 |} \quad (10)$$

for some constants $c > 0$, $C < \infty$ depending on V .

Conversely, one has an expansion

$$e^{i\pi m x} = \sum_j \hat{e}_m(j) \varphi_j(x) \quad (11)$$

with again

$$|\hat{e}_m(j)| < C e^{-c| |m| - j/2 |}. \quad (12)$$

Assume V even i.e. $V(x) = V(-x)$. Consider the spectrum of $A = -\frac{d^2}{dx^2} + V$ subject to Dirichlet boundary conditions on $[0,1]$. This gives a sequence $\{\mu_n\}$ interlacing the periodic spectrum

$$\lambda_0 < \lambda_1 \leq \mu_1 \leq \lambda_2 < \dots < \lambda_{2n-1} \leq \mu_n \leq \lambda_{2n} < \dots \quad (13)$$

and the corresponding eigenfunctions $\{\psi_n\}$ form a basis for the 2-periodic odd functions. Thus

$$\psi_n(x) = \sum_{m=1}^{\infty} \hat{\psi}_n(m) \sin \pi m x \quad (14)$$

$$\sin \pi n x = \sum_{m=1}^{\infty} \hat{s}_n(m) \psi_m(x) \quad (15)$$

where again

$$|\hat{\psi}_n(m)|, |\hat{s}_n(m)| < C e^{-c|n-m|}. \quad (16)$$

(iii) The analogue of the theory in 1D described above fails in higher dimension. One may however, consider the special case of a potential of the form

$$V(x) = V_1(x_1) + \dots + V_d(x_d). \quad (17)$$

For periodic boundary conditions, one gets for eigenvalues and eigenfunctions

$$\{\lambda_{j_1}^1 + \dots + \lambda_{j_d}^d, \varphi_{j_1}^1(x_1) \dots \varphi_{j_d}^d(x_d)\}. \quad (18)$$

In the Dirichlet case, one gets

$$\{\mu_{n_1}^1 + \dots + \mu_{n_d}^d, \psi_{n_1}^1(x_1) \dots \psi_{n_d}^d(x_d)\} \quad (19)$$

where $\{\lambda_j; \varphi_j\}$ and $\{\mu_n; \psi_n\}$ are as above.

(iv)

$$A = (-\Delta + \rho)^{1/2} \quad (20)$$

with eigenvalues

$$\lambda_n = (|n|^2 + \rho)^{1/2} \quad (21)$$

and the exponentials as eigenfunctions.

Observe that the sequence $\{(k + \rho)^{1/2} \mid k \in \mathbb{Z}_+\}$ consists of rationally independent numbers for typical ρ .

Coming back to (1), denote $\{\varphi_n, \mu_n\}$ the eigenfunctions and corresponding eigenvalues for A , i.e.

$$A \varphi_n = \mu_n \varphi_n.$$

Fix some specific indices $n_1, n_2, \dots, n_b$ and denote $\lambda_j = \mu_{n_j}$ ($j = 1, \dots, b$).

Then

$$u_0(x, t) = \sum_{j=1}^b a_j e^{i\lambda_j t} \varphi_{n_j}(x) \quad (22)$$

yields a solution of the linear equation

$$iu_t^0 + Au^0 = 0. \quad (23)$$

This solution is quasi-periodic and corresponds to a flow on a b -torus $\mathbf{T}^b = \mathcal{T}_0\{|a_j|\}$. The problem we study is which of the solutions (22) “persist” for the perturbed equation

$$iu_t + Au + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0. \quad (24)$$

The meaning of persistency is as follows. One has a quasi-periodic solution

$$u_\varepsilon(x, t) = \sum_{n, k} \hat{u}_\varepsilon(n, k) e^{i(\lambda', k)t} \varphi_n(x) \quad (25)$$

of (24), where k is a $\mathbf{Z}^b$ -index

$$\hat{u}_\varepsilon(n_j, e_j) = a_j \quad (e_j = j^{\text{th}} \text{ unit vector}) \quad (26)$$

$$\sum_{(n, k) \notin \mathcal{S}} \omega(n, k) |\hat{u}_\varepsilon(n, k)| < \sqrt{\varepsilon} \quad (27)$$

$$|\lambda' - \lambda| < C \varepsilon. \quad (28)$$

In (27),

$$\mathcal{S} = \{(n_j, e_j) \mid j = 1, \dots, b\} \quad (29)$$

is the “resonant set”.

We let ω be a weight function. For instance

$$\omega(n, k) = e^{c(|n|+|k|)} \quad (30)$$

corresponds to analytic solutions. We will mostly consider weights of the form

$$\omega(n, k) = e^{(|n|+|k|)^c}. \quad (31)$$

For some $c > 0$ (because they lead to some technical simplifications later on). There is dependence of the new frequency λ' on $\{a_j\}$ and the perturbation $\varepsilon \frac{\partial H}{\partial \bar{u}}$. In fact, assuming $\lambda'_1, \dots, \lambda'_b$ rationally independent, a time shift permits to assume $a_j (1 \leq$

$j \leq b$) real and hence λ' depends only on $\{|a_j|\}$. The solution (25) corresponds by (27) to a perturbed torus $\mathcal{T}_\varepsilon\{|a_j|\}$ of $\mathcal{T}_0\{|a_j|\}$ in the phase space.

Consider $\lambda = (\lambda_1, \dots, \lambda_b)$ as a parameter taken in a parameter set Λ . Persistency of quasi-periodic solutions (22) after the Hamiltonian perturbation as described above will occur for λ in a large subset of the frequency set. More precisely, the persistency will hold provided

$$\lambda \notin \Lambda'_\varepsilon(|a_j| \ (1 \leq j \leq b), \text{ perturbation}) \quad (32)$$

where

$$\text{mes } \Lambda'_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (33)$$

The preceding deals with perturbations of linear equations. In order to obtain families of quasi-periodic solutions with b frequencies, one needs to consider a parameter dependent linear equation with b -parameters $p = (p_1, \dots, p_b)$, such that if $\{\lambda_n(p)\}$ are the eigenvalues of $A(p)$, then

$$\det \left(\frac{\partial \lambda_{n_j}}{\partial p_k} \right)_{1 \leq j, k \leq b} \neq 0. \quad (34)$$

In the previous examples, such parameters are obtained via the Fourier multiplier M_σ or by letting V range in some b -manifold of potentials. Recall here that if

$$A(V) = - \frac{d^2}{dx^2} + V(x) \quad (35)$$

then from first order variation

$$\frac{\partial \lambda}{\partial V} = |\varphi|^2 \quad (36)$$

where φ is the eigenfunction with eigenvalue λ . The use of parameter dependent equations may be avoided in various ways.

First, writing

$$\lambda' = \lambda + \varepsilon(\lambda, a) \quad (37)$$

and assuming initial nonresonance conditions (1° Melnikov condition)

$$\sum_{j=1}^b k_j \lambda_j + \mu_n \neq 0 \quad (n \notin \{n, \dots, n_b\}) \quad (38)$$

(expressing nonresonance of the normal frequencies μ_n with ($\mathbf{Z}$ -combinations of) the tangential frequencies $\lambda_1, \dots, \lambda_b$), conditions appearing later on in the process may be ensured by restricting $|a|$ to an appropriate Cantor set. This requires the

dependence (37) of λ' on $|a|$ to be sufficiently nondegenerate (*amplitude-frequency modulation*).

Secondly, one may sometimes extract parameters out of the nonlinearity by putting the Hamiltonian

$$H(\phi) = \langle A\phi, \phi \rangle + \varepsilon H_1(\phi) \quad (39)$$

in an appropriate *Birkhoff normal form*.

Assume that one may write (1) in the form

$$\begin{cases} i\dot{q}_{n_j} + \lambda_j q_{n_j} + \varepsilon q_{n_j} |q_{n_j}|^2 + \varepsilon \delta \frac{\partial H_2}{\partial \bar{q}_{n_j}} = 0 & (j = 1, \dots, b) \\ i\dot{q}_n + \mu_n q_n + \varepsilon \delta \frac{\partial H_2}{\partial \bar{q}_n} = 0 & (n \neq n_1, \dots, n_b) \end{cases} \quad (40)$$

where

$$u_n(x, t) = \sum_n q_n(t) \varphi_n(x).$$

Define for $j = 1, \dots, b$

$$I_j = |q_{n_j}|^2 \quad (41)$$

(action variables). From (40)

$$\dot{I}_j = 2 \operatorname{Re} \dot{q}_{n_j} \bar{q}_{n_j}, \quad i\dot{I}_j = \varepsilon \delta \left(\frac{\partial H_2}{\partial q_{n_j}} q_{n_j} - \frac{\partial H_2}{\partial \bar{q}_{n_j}} \bar{q}_{n_j} \right). \quad (42)$$

Fix some values $\{I_j^0\}_{j=1}^b$ and write

$$I_j = I_j^0 + \sqrt{\delta} I'_j. \quad (43)$$

From (40), (42), (43) we get the system

$$\begin{cases} i\dot{q}_{n_j} + (\lambda_j + \varepsilon I_j^0) q_{n_j} + \varepsilon \sqrt{\delta} I'_j q_{n_j} + \varepsilon \delta \frac{\partial H_2}{\partial \bar{q}_{n_j}} = 0 & (j = 1, \dots, b) \\ i\dot{I}_j - \varepsilon \sqrt{\delta} \left(\frac{\partial H_2}{\partial q_{n_j}} q_{n_j} - \frac{\partial H_2}{\partial \bar{q}_{n_j}} \bar{q}_{n_j} \right) = 0 & (j = 1, \dots, b) \\ i\dot{q}_n + \mu_n q_n + \varepsilon \delta \frac{\partial H_2}{\partial \bar{q}_n} = 0 & (n \neq n_1, \dots, n_b) \end{cases} \quad (44)$$

which appears as $(\varepsilon \sqrt{\delta})$ -perturbation of the linear system

$$\begin{cases} i\dot{q}_{n_j} + (\lambda_j + \varepsilon I_j^0) q_{n_j} = 0 & (j = 1, \dots, b) \\ i\dot{I}_j = 0 & (j = 1, \dots, b) \\ i\dot{q}_n + \mu_n q_n = 0 \end{cases} \quad (45)$$

with a b -parameter set of frequencies $\{\lambda_j + \varepsilon I_j^0\}_{j=1}^b$ obtained by variation of $\{I_j^0\}$. Hence the problem is reduced to a perturbation of a linear system as above.

A similar approach is applied in the study of perturbations of integrable systems. For instance, perturbations of the KdV-equation

$$u_t + u_{xxx} + uu_x = 0 \quad (46)$$

of the form

$$u_t + u_{xxx} + uu_x + \varepsilon f(u)_x = 0 \quad (47)$$

($f(u)$ polynomial or real analytic). The process of extracting the normal form here is rather involved since it is based on the Riemann surface correspondence (cf. [K2]).

Coming back to (1), we deal with the problem of persistency of finite dimensional tori in an infinite dimensional phase space. This is a generalization of the more classical KAM (Kolmogorov–Arnold–Moser) setting of persistency of n tori in $2n$ -dimensional phase space. As a first method to approach (1), one may however, apply the basic KAM scheme, which consists in removing the perturbation εH in (39) by composing with canonical transformations. Given a Hamiltonian flow

$$iq_t = \varepsilon \frac{\partial F}{\partial \bar{q}} \quad F = F(q, \bar{q}) \quad \text{Hamiltonian} \quad (48)$$

denote $q \rightarrow q'$ the time-1 shift, which is a symplectic transformation of phase space. Then the system

$$iq_t = \frac{\partial H}{\partial \bar{q}} \quad (49)$$

is transformed into

$$iq'_t = \frac{\partial H'}{\partial \bar{q}'} \quad (50)$$

where $H(q) = H'(q')$. Denote in complex coordinates the Poisson bracket

$$\{F_1, F_2\} = \text{Im} \sum_n \frac{\partial F_1}{\partial q_n} \frac{\partial F_2}{\partial \bar{q}_n}. \quad (51)$$

Then

$$H' = H + \varepsilon \{H, F\} + \frac{\varepsilon^2}{2!} \{\{H, F\}, F\} + \dots = H + \varepsilon \{H, F\} + 0(\varepsilon^2). \quad (52)$$

Hence, for H given by (39)

$$H(q, \bar{q}) = \sum \mu_n |q_n|^2 + \varepsilon H_1(q, \bar{q}) = H_0 + \varepsilon H_1 \quad (53)$$

we find

$$H'(q', \bar{q}') = \sum \mu_n |q_n|^2 + \varepsilon H_1(q, \bar{q}) + \varepsilon \{H_0, F\} + 0(\varepsilon^2). \quad (54)$$

The main idea is to choose F in order to reduce $H_1 + \{H_0, F\}$ and iterate the process. The drawback of this method is that it requires the spectrum of A to satisfy nonresonance conditions of the form (2^e Melnikov conditions)

$$\sum_{j=1}^b k_j \lambda_j + \mu_n - \mu_{n'} \neq 0 \quad (n \neq n' \notin \{n_1, \dots, n_b\}) \quad (55)$$

besides (38). Condition (55) is violated in the case of multiplicities in the normal frequencies and impairs the use of KAM in this situation (which in the context of (1) appears in 1D under periodic boundary conditions and in higher dimension).

To illustrate the appearance of (55), assume F of the form

$$F(q, \bar{q}) = \langle Bq, q \rangle, \quad B = B^*. \quad (56)$$

Then, by (51)

$$\{H_0, F\} = \frac{1}{2i} \sum_{n, n'} (\mu_n - \mu_{n'}) b_{nn'} \bar{q}_n q_{n'} \sim \langle [A, B]q, q \rangle \quad (57)$$

where $[A, B] = AB - BA$.

If one rewrites (1) in the Fourier version (*wrt* $e^{i(\lambda', k) \cdot t} \varphi_n(x)$), one gets

$$(-\langle \lambda', k \rangle + \mu_n) \hat{u}(n, k) + \varepsilon \frac{\partial \widehat{H}}{\partial u}(n, k) = 0. \quad (58)$$

Hence (38) is the only type of condition which appears if one tries to solve (58) by a direct perturbative approach. If one considers such a ε -series, considerable problems appear due to the small divisors

$$\mu_n - \langle \lambda', k \rangle. \quad (59)$$

They yield approximative solutions by truncation, introducing only divisors such that

$$|\mu_n - \langle \lambda', k \rangle| \gg \varepsilon. \quad (60)$$

To perform their actual summation is a very delicate issue (leading to renormalization problems) and was even in finite dimensional phase space (i.e. n ranges in a finite set) only recently fully understood^(*). The method followed here is to solve (58) by a Newton iteration scheme, converging much faster (double exponentially fast) and therefore less affected by the presence of small divisors. On the other

(*) cf. [E]

hand, it will require to control the inverse of the linearization of (58) which is a non-diagonal operator.

We first perform a Lyapunov-Schmidt type decomposition. Consider the b equations

$$(-\lambda'_j + \lambda_j) a_j + \varepsilon \frac{\widehat{\partial H}}{\partial \bar{u}}(n_j, e_j) = 0 \quad (j = 1, \dots, b) \quad (61)$$

obtained by taking $(n, k) \in \mathcal{S}$ of (26), (29).

They form the (finite) system of Q -equations. The remaining (infinite) system

$$(-\langle \lambda', k \rangle + \mu_n) \hat{u}(n, k) + \varepsilon \frac{\widehat{\partial H}}{\partial \bar{u}}(n, k) = 0 \quad (n, k) \notin \mathcal{S} \quad (62)$$

are called the P -equations. The general procedure is to determine $\hat{u}|_{(n,k) \notin \mathcal{S}}$ from (62) (depending on λ') and then substitute in (61) to obtain the new frequencies $\lambda' = (\lambda'_1, \dots, \lambda'_b)$. Now these frequencies $\lambda'_1, \dots, \lambda'_b$ need to be real. Thus (61) expresses, in fact, $2b$ conditions with b parameters. The formal solvability is a consequence of the Hamiltonian nature of (1) (result due to Poincaré). Assume $H(u, \bar{u})$ is a sum of monomials $u^j \bar{u}^k$ with real coefficients. In proving the persistency result, we may assume $a_j \in \mathbf{R}$ ($j = 1, \dots, b$), considering time shifts as observed earlier. Hence, the system $\hat{u}|_{(n,k) \notin \mathcal{S}}$ produced from solving (62) will be real and so will be $\frac{\widehat{\partial H}}{\partial \bar{u}}(n_j, e_j)$ in (61). Thus (61) yields a real solution in λ' and the formal solvability is clear in this case.

The system (62) cannot be treated by a standard implicit function theorem because of the appearance of small divisors. We denote

$$v = \bar{u}. \quad (63)$$

Assuming the eigenfunction basis for A in (1) given by exponentials, we have thus

$$\hat{v}(n, k) = \overline{\hat{u}(-n, -k)} \quad (64)$$

(if one would consider a real eigenfunction basis $\{\varphi_n\}$, then clearly $\hat{v}(n, k) = \overline{\hat{u}(n, -k)}$).

Duplicate the equations (62) considering the system

$$\begin{cases} (-\langle \lambda', k \rangle + \mu_n) \hat{u}(n, k) + \varepsilon \frac{\widehat{\partial H}}{\partial \bar{v}}(n, k) = 0 & (n, k) \notin \mathcal{S} \\ (\langle \lambda', k \rangle + \mu_{-n}) \hat{v}(n, k) + \varepsilon \frac{\widehat{\partial H}}{\partial u}(n, k) = 0 & (n, k) \notin -\mathcal{S}. \end{cases} \quad (65)$$

We solve this system in $(\hat{u}|_{(n,k) \notin \mathcal{S}}, \hat{v}|_{(n,k) \notin -\mathcal{S}})$ by Newton's algorithm. The relation (64) will be preserved for the approximative solutions.

Recall the formal scheme. Consider the equation

$$F(y) = 0. \quad (66)$$

Starting from y_0 (here 0), the consecutive approximations are defined by

$$\Delta_{i+1}y \equiv y_{i+1} - y_i = -[F'(y_i)]^{-1} F(y_i) \quad (67)$$

and

$$\|F(y_{i+1})\| = 0 \quad (\|y_{i+1} - y_i\|^2) = 0 \quad (\|[F'(y_i)]^{-1} F(y_i)\|^2). \quad (68)$$

In our case (65), the linearized operator T is given by

$$T = D + \varepsilon S \quad (69)$$

where D is diagonal

$$D = \begin{pmatrix} D^+ & 0 \\ 0 & D^- \end{pmatrix} \quad \begin{aligned} D_{n,k}^+ &= -\langle \lambda', k \rangle + \mu_n \\ D_{n,k}^- &= \langle \lambda', k \rangle + \mu_{-n} \end{aligned} \quad (70)$$

and

$$S = \begin{pmatrix} S_{\frac{\partial^2 H}{\partial \mathbf{u} \partial \mathbf{v}}} & S_{\frac{\partial^2 H}{\partial \mathbf{v}^2}} \\ S_{\frac{\partial^2 H}{\partial \mathbf{u}^2}} & S_{\frac{\partial^2 H}{\partial \mathbf{u} \partial \mathbf{v}}} \end{pmatrix}. \quad (71)$$

S_ϕ expresses ϕ -multiplication in Fourier, thus here

$$S_\phi((n, k) (n', k')) = \widehat{\phi}(n - n', k - k'). \quad (72)$$

The operator S is selfadjoint and depends on the given approximative solution (u, v) . Along the construction, we satisfy an estimate on their Fourier coefficients of the form

$$|\hat{u}(n, k)| < C e^{-(|n|+|k|)^c} \quad (73)$$

for some fixed $c > 0$. This fact clearly yields a corresponding off-diagonal decay estimate for each of the 4 matrixes S_ϕ appearing in (71)

$$S_\phi(x, x') < C e^{-|x-x'|^c}. \quad (74)$$

The main difficulty consists in controlling the inverse T^{-1} , due to the fact that the diagonal elements may be arbitrarily small. Fixing some $\varepsilon \ll \delta < 1$, call a site (n, k) singular if

$$|D_{n,k}^+| = |-\langle \lambda', k \rangle + \mu_n| < \delta$$

or

$$|D_{n,k}^-| = |\langle \lambda', k \rangle + \mu_{-n}| < \delta. \quad (75)$$

The geometric structure of the singular sites plays an important role in the study of T^{-1} (in particular, one exploits essentially their separation properties). Assume that one succeeds to ensure for these operators T bounds

$$(T_N)^{-1} < B(N) \quad (76)$$

on the inverse of the restriction

$$T_N = T|_{|n|+|k|<N} \quad (77)$$

where, for instance, $B(N)$ grows at most polynomially

$$B(N) < N^C \quad (78)$$

or at least much slower than exponential.

Assume $H(u, v)$ is a polynomial expression. At the j^{th} step of the Newton iteration, an approximative solution y is obtained and we assume that

$$\text{supp } \hat{u}_j \text{ is contained in an } (n, k) - \text{ball of radius } M^j \quad (79)$$

for some constant M . Letting $y_j = (\hat{u}_j, \hat{v}_j)$ and $F(y)$ the left number of (65), assume

$$\|F(y_j)\| < \varepsilon_j. \quad (80)$$

Since F is expressed as a polynomial in $\hat{u}, \hat{v}$, we get

$$\text{supp } \widehat{F(y_j)} \subset CM^j. \quad (81)$$

Put $N = M^{j+1}$ and define y_{j+1} by (cf. (67))

$$y_{j+1} - y_j = -T_N^{-1}(F(y_j)). \quad (82)$$

Hence, by (78), (80)

$$\|y_{j+1} - y_j\| < N^C \varepsilon_j \quad (83)$$

and by (82)

$$\|F(y_{j+1})\| \leq \|(T - T_N)(y_{j+1} - y_j)\| + 0(\|y_{j+1} - y_j\|^2). \quad (84)$$

From (80), (83) and off-diagonal decay of T and T_N^{-1} (cf. (74)), the first term in (84) may be bounded by

$$\|(T - T_N)(y_{j+1} - y_j)\| < e^{-\frac{1}{2}N^c} \varepsilon_j. \quad (85)$$

Hence, from (84)

$$\varepsilon_{j+1} < e^{-\frac{1}{2}(M^{j+1})^c} \varepsilon_j + M^{2C(j+1)} \varepsilon_j^2. \quad (86)$$

Letting $c > 0$ be sufficiently small, one may conclude that

$$\varepsilon_j < e^{-4(M^j)^c} \quad (87)$$

$$\|y_{j+1} - y_j\| < e^{-3(M^j)^c} < e^{-2(M^{j+1})^c} \quad (88)$$

which is compatible with (73), (79).

The main difficulty is to obtain the estimate (76) and the off-diagonal estimate, say

$$|T_N^{-1}(x, x')| < e^{-\frac{1}{2}|x-x'|^c} \text{ for } |x-x'| > \frac{N}{100}. \quad (89)$$

Coming back to (38), assume (except for $(n, k) \in \mathcal{S}$)

$$|-\langle k, \lambda \rangle + \mu_n| > (1 + |k|)^{-C} \quad (90)$$

for some constant C . Assuming λ' as in (28), it follows that also

$$|-\langle k, \lambda' \rangle + \mu_n| > \frac{1}{2} (1 + |k|)^{-C} \text{ for } |k| < \varepsilon^{-1/2C}. \quad (91)$$

Thus, for

$$N < \varepsilon^{-1/2C} \quad (92)$$

one may invert $T_N = D_N + \varepsilon \mathcal{S}_N$ by a Neumann series, i.e.

$$T_N^{-1} = D_N^{-1} + \sum_{j=1}^{\infty} (-1)^j \varepsilon^j (S_N D_N^{-1})^j \quad (93)$$

since $\|S_N D_N^{-1}\| < \sqrt{\varepsilon}$, and (78), (89) are clearly satisfied.

In order to fulfill (78), (89) at later scales, requires to impose further conditions on (λ, λ', a) ; $a = (a_1, \dots, a_b)$, considered as initial parameters in the Newton iteration process (recall that at each stage, T depends on previous approximative solution). The effect of these further conditions is to excise from the (λ, λ', a) -parameter set exceptional subsets which measure tends rapidly to zero. Since λ' will be determined by solving the Q -equations (61) as

$$\lambda' = \lambda + \varepsilon(\lambda, a) \quad (94)$$

our aim is to remain with conditions on (λ, a) as in (32), (34); in particular in a model when a is fixed, λ will be restricted to a Cantor set of positive measure.

Essentially speaking, these conditions on (λ, λ', a) consist in keeping certain expressions $E(\lambda, \lambda', a)$ away from zero, where $E(\lambda, \lambda', a)$ is differentiable with more sensitivity on λ' than λ, a .

In order for $\hat{u}|_{(n,k)\notin\mathcal{S}}$ to solve the P -equations, previous conditions on (λ, λ', a) need to be fulfilled. We assume however, that $\hat{u}|_{(n,k)\notin\mathcal{S}}$ is smoothly defined on the entire parameter set (λ, λ', a) . Substitution in (61) allows then to solve in λ' invoking the standard implicit function theorem. One gets (94) or more precisely (assuming $a_j \in \mathbb{R}$)

$$\lambda'_j = \lambda_j + \varepsilon \frac{1}{a_j} \left[\frac{\partial H}{\partial \bar{u}} \left(\sum_{j=1}^b a_j e^{i\theta_j}, \sum_{j=1}^b a_j e^{-i\theta_j} \right) \right]^\wedge (n_j, e_j) + 0(\varepsilon^2) \quad (95)$$

where the second term depends on $a = (a_1, \dots, a_b)$ only.

In carrying out this program, there is a distinction between the case of time periodic solutions ($b = 1$) and quasi-periodic solutions ($b > 1$). The case $b = 1$ turns out to be significantly easier.

The arithmetic properties of the sequence $\{\mu_n\}$, more precisely the structure of clusters of the form

$$\{n \in \mathbb{Z}^d \mid |\mu_n - \mu| < 1\} \quad (96)$$

when $\mu \rightarrow \infty$ plays a basic role. In the PDE-context, the sequence $\{\mu_n\}$ is infinite and its properties depend on the space dimension d .

It turns out however that conditions of the form (55) are unnecessary and appear as an artifact of the KAM approach.

Remarks

- (i) The method described above is more flexible than KAM. In fact, in the main body of the analysis, which consists in solving the P -equations, the Hamiltonian structure plays essentially no role.

There are variants. For instance, one may use a truncated perturbation series to obtain an approximative solution of the P -equation up to ε^K , for any power K , and then apply the more rapidly converging Newton method and get an actual solution, ε^K -close to the approximative one.

(The Hamiltonian counterpart of this consists in achieving a normal form with nonresonant part $0(\varepsilon^K)$).

The condition (38)

$$\sum_{j=1}^b k_j \lambda_j + \mu_n \neq 0 \quad (n \notin \{n_1, \dots, n_b\}) \quad (97)$$

(1^e Melnikov condition) expresses nonresonance of the normal frequencies with the tangential ones. In the resonant case, one modifies the previous scheme as follows. Define now the resonant set $\tilde{\mathcal{S}} \supset \mathcal{S}$ as

$$\tilde{\mathcal{S}} = \{(n, k) \mid -\langle \lambda, k \rangle + \mu_n = 0\} \quad (98)$$

and consider for the Lyapunov-Schmidt decomposition

$$\begin{cases} (-\lambda'_j + \lambda_j) a_j + \varepsilon \frac{\partial \widehat{H}}{\partial u}(n_j, e_j) = 0 & (j = 1, \dots, b) \\ (-\langle \lambda', k \rangle + \mu_n) \hat{u}(n, k) + \varepsilon \frac{\partial \widehat{H}}{\partial u}(n, k) = 0 & \text{for } (n, k) \in \tilde{\mathcal{S}} \setminus \mathcal{S} \end{cases} \quad (99)$$

(Q -equations)

and

$$(-\langle \lambda', k \rangle + \mu_n) \hat{u}(n, k) + \varepsilon \frac{\partial \widehat{H}}{\partial u}(n, k) = 0 \text{ for } (n, k) \notin \tilde{\mathcal{S}} \quad (100)$$

(P -equations).

Fix

$$(\lambda'_j)_{j=1, \dots, b}, \{a_j\}_{j=1, \dots, b} \text{ and } \hat{u}|_{(n, k) \in \tilde{\mathcal{S}} \setminus \mathcal{S}}$$

and solve $\hat{u}|_{(n, k) \notin \tilde{\mathcal{S}}}$ from (100). One then uses (99) for the determination of λ' and $\hat{u}|_{\tilde{\mathcal{S}} \setminus \mathcal{S}}$. A typical example of this appears in Theorem 131 below.

- (ii) The Lyapunov-Schmidt approach to (1) yields a new method to solve stability problems in finite dimensional phase space as well. Observe that the Hamiltonian system

$$\dot{p}_n = \frac{\partial H}{\partial q_n} \quad \dot{q}_n = -\frac{\partial H}{\partial p_n} \quad (n = 1, \dots, N) \quad (101)$$

Hamiltonian

$$H(p, q) = H(p_1, \dots, p_N, q_1, \dots, q_N) \quad (102)$$

is equivalent to

$$i\dot{u} = 2 \frac{\partial H}{\partial \bar{u}} \quad (103)$$

where $u = p + iq$. Thus one gets a new proof of the KAM theorem (stability of N tori is $2N$ dimensional phase space) and Melnikov's theorem (more generally, stability of tori of dimension $n \leq N$) along the lines of Lyapunov's theorem (the periodic case $n = 1$). For Lyapunov's result, the nonresonance condition is

$$\mu_n \notin \lambda_1 \mathbb{Z} \quad (104)$$

corresponding to the case $b = 1$ in (38). Hence, in the finite dimensional case, we prove a Melnikov theorem for (1) under the nonresonance assumption (38),

without the need of the extra assumption (55). This result is new. From the previous remark, it appears that besides multiple normal frequencies one may investigate also the case of normal frequencies resonant with the tangential ones, thus when (104), (38) are partially violated. Presently there has been no systematic study of this.

(iii) Consider a NLW

$$\partial_{tt} - \Delta y + Vy + \varepsilon F'(y) = 0 \quad (105)$$

and denote

$$B = (-\Delta + V)^{1/2} \quad (106)$$

(assuming this makes sense).

Rewrite (105) in the following Hamiltonian form

$$\begin{cases} \dot{y} = Bz \\ \dot{z} = -By - \varepsilon B^{-1}F'(y) \end{cases} \quad (107)$$

considering instead of L^2 the Hilbert space $H^{1/2}$ with scalar product

$$\langle u, v \rangle_{\frac{1}{2}} = \langle u, Bv \rangle. \quad (108)$$

Denoting $u = y + iz$, (107) yields

$$i\dot{u} - Bu + \varepsilon B^{-1}F'(\operatorname{Re} u) = 0 \quad (109)$$

which is of the form (1) with A replaced by $-B$. Thus the spectrum in 1D under periodic (resp. Dirichlet) boundary conditions is given by $-\sqrt{\lambda_j}$ (resp. $-\sqrt{\mu_n}$) and behaves as

$$\sqrt{\lambda_j} = \frac{\pi j}{2} + o\left(\frac{1}{j}\right), \quad \sqrt{\mu_n} = \pi n + o\left(\frac{1}{n}\right). \quad (110)$$

In this particular case, $V = \rho$ (constant), we get for the periodic spectrum

$$\mu_n = (|n|^2 + \rho)^{1/2} \quad (111)$$

(cf example (iv) above).

Next, we state some concrete results.

Theorem 112. *Consider a 1D NLS [K1], [B1]*

$$iu_t - u_{xx} + Vu + \varepsilon \frac{\partial H}{\partial u} = 0 \quad (113)$$

or

$$iu_t = u_{xx} + M_\sigma u + \varepsilon \frac{\partial H}{\partial \bar{u}} = 0 \quad (114)$$

where $H = H(u, \bar{u})$ is polynomial or real analytic, V a real analytic periodic potential and M_σ a real Fourier multiplier (as discussed in the examples). Consider an unperturbed solution of (113), (114) with $\varepsilon = 0$

$$u_0(x, t) = \sum_{j=1}^b a_j e^{i\lambda_j t} \varphi_{n_j}(x) \quad (115)$$

where $\lambda_j = \mu_{n_j}$ ($j = 1, \dots, b$) and $\{\varphi_{n_j}\}$ are the corresponding eigenfunctions. Consider $\lambda = (\lambda_1, \dots, \lambda_b)$ as a b -parameter (for equation (113), this may be achieved by appropriate variation of V , cf. (35)–(36)). Assume a nonresonance condition (38)

$$|\mu_n + \langle k, \lambda \rangle| \geq c(1 + |k|)^{-C} \quad (116)$$

satisfied, for $|k| < N$ sufficiently large (depending on H).

Then, for $\lambda \in \Lambda_\varepsilon(|a|)$, a subset of the parameter set of small complementary measure when $\varepsilon \rightarrow 0$, there is a perturbed solution u_ε of (113), (114) with perturbed frequency λ'

$$u_\varepsilon(x, t) = \sum_{n,k} \hat{u}_\varepsilon(n, k) e^{i(\lambda', k)t} \varphi_n(x) \quad (117)$$

satisfying (26)–(28). Thus

$$\hat{u}_\varepsilon(n_j, e_j) = a_j \quad (j = 1, \dots, b) \quad (118)$$

$$\sum_{(n,k) \notin \mathcal{S}} e^{(|n|+|k|)^c} |\hat{u}_\varepsilon(n, k)| < \sqrt{\varepsilon} \quad (119)$$

$$|\lambda' - \lambda| < C\varepsilon \quad (120)$$

for some $c > 0$, and where $\mathcal{S} = \{(n_j, e_j) (j = 1, \dots, b)\}$.

In the case of (113), with Dirichlet boundary conditions, assume V even and $H(u, \bar{u})$ even (hence $\frac{\partial H}{\partial \bar{u}}$ odd), see discussion in example (ii) above.

Theorem 121. *The analogue of Theorem 112 holds for 2D Schrödinger equations, but in (113) the potential $V(x_1, x_2)$ should be assumed of the form $V_1(x_1) + V_2(x_2)$ (see discussion in example (iii) above). [B3]*

In 1D, a result similar to Theorem 112 may be stated for NLW equations [K1], [W]

$$u_{tt} - u_{xx} + V(x)u + \varepsilon f'(u) = 0 \quad (122)$$

and even derivative NLW equations [B1]

$$u_{tt} - u_{xx} + V(x)u + \varepsilon Bf'(u) = 0 \quad B = \left(\frac{-d^2}{dx^2} \right)^{1/2}. \quad (123)$$

In the next theorem, the role of outer parameters in the equation is replaced by amplitude-frequency modulation.

Theorem 124. *Consider a 1D NLW*

$$u_{tt} - u_{xx} + \rho u + (u^3 + \text{higher order terms}) = 0 \quad (125)$$

and assume $\rho \geq 0$ is a typical number in the sense of linear independence of the sequence

$$\mu_n = (n^2 + \rho)^{1/2}. \quad (126)$$

Fix a sequence

$$0 < n_1 < n_2 < \dots < n_b \in \mathbf{Z}_+ \quad (127)$$

and consider the solution

$$u_0(x, t) = \sum_{j=1}^b a_j \cos n_j x \cdot \cos \lambda_j t \quad (\lambda_j = \mu_{n_j}) \quad (128)$$

of the linear equation

$$u_{tt} - u_{xx} + \rho u = 0. \quad (129)$$

There is a Cantor set $C \subset \{a = (a_j)_{j=1, \dots, b} \mid a_j > 0\}$ of positive measure, in fact, of asymptotically full measure when $|a| \rightarrow 0$, such that for $a \in C$ the solution (128) of (129) persists for (125)

$$u(x, t) = \sum_{j=1}^b q_j \cos n_j x \cdot \cos \lambda'_j t + O(|a|^3). \quad (130)$$

The persistency problem for higher dimensional wave equations seems more difficult, due to the behavior of the frequencies $|n| = (n_1^2 + \dots + n_d^2)^{1/2}$ when $d \geq 2$. One may treat however, the special case of time periodic solutions in any dimension (this is also the case for NLS).

Theorem 131. *Consider the periodic wave equation in dimension d [B2]*

$$u_{tt} - \Delta u + \rho u + (u^3 + \text{higher order terms}) = 0 \quad (132)$$

where again $\rho > 0$ is a typical number. More precisely, we require ρ to satisfy a condition of the form

$$\left| \sum_{j=0}^r k_j \rho^j \right| > \left(\sum |k_j| \right)^{-C_r} \text{ for all } \{k_j\} \in \mathbb{Z}^{r+1} \setminus \{0\}. \quad (133)$$

Fix $n_0 \in \mathbb{Z}^d \setminus \{0\}$. There is a Cantor set C of positive measure in an interval $[0, \delta]$ and for $p_0 \in C$ a solution of (132) of the form

$$u(x, t) = p_0 \cos(\langle n_0, x \rangle + \lambda t) + O(p_0^3) \quad (134)$$

where

$$\lambda^2 = \lambda(p_0)^2 = |n_0|^2 + \rho + \left(\frac{3}{4} + O(1) \right) p_0^2. \quad (135)$$

The next two results are normal form reductions, bringing the problem back to perturbations of a linear problem with parameters. (cf. the discussion (39)-(45)).

Theorem 136. Consider a 1D NLS

$$iu_t - u_{xx} + mu + f(|u|^2)u = 0 \quad (137)$$

where f is a polynomial or real analytic and satisfying a nondegeneracy condition

$$f'(0) \neq 0. \quad (138)$$

Consider (137) with periodic boundary conditions say. Fix a sequence of positive integers

$$n_1 < n_2 < \dots < n_b. \quad (139)$$

Then for $a = \{a_j\}_{j=1, \dots, b}$ in a Cantor family C of positive measure, there is a quasi-periodic solution

$$u(x, t) = \sum_{j=1}^b q_j e^{i(n_j x + \lambda'_j t)} + O(|a|^3) \quad (140)$$

with frequencies $\lambda'_1, \dots, \lambda'_b$, where

$$\lambda'_j = n_j^2 + m + O(|a|^2) \quad (j = 1, \dots, b). \quad (141)$$

This result is due to S. Kuksin and J. Pöschel (under Dirichlet boundary conditions) [K-P].

Theorem 142. *Consider the 2D cubic NLS [B3]*

$$iu_t - \Delta u + cu|u|^2 = 0 \quad (c \neq 0) \quad (143)$$

or, more generally, an 2D NLS

$$iu_t - \Delta u + f(|u|^2)u = 0 \quad (144)$$

with f as above in Theorem 136, with periodic boundary conditions.

For the modes $n_1, \dots, n_b \in \mathbb{Z}^2$, we fix 2 lattice points n_1, n_2 on a same circle

$$|n_j| = R \quad (j = 1, 2), n_1 \neq -n_2 \quad (145)$$

(more complicated structures involving more than 2 points may be treated as well but this is the simplest case). There is a Cantor family C of positive measure such that for $a = \{a_j\}_{j=1,2} \in C$ (143) (144) has a quasi-periodic solution

$$u(x, t) = \sum_{j=1}^2 a_j e^{i((n_j, x) + \lambda'_j t)} + o(|a|^3) \quad (146)$$

with frequencies $\lambda' = (\lambda'_1, \lambda'_2)$

$$\lambda'_j = |n_j|^2 + o(|a|^2) \quad (j = 1, 2). \quad (147)$$

Further Comments.

It seems a natural program to extend the classical theory of smooth dynamical systems to the setting of infinite dimensional phase space, in particular in the context of Hamiltonian PDE's as discussed here. A subject closely related to persistency of invariant tori is that of Nekhoroshev stability [N]. This phenomenon may be roughly stated as follows. Consider in $2N$ -dimensional phase space a perturbed Hamiltonian

$$H(I_1, \dots, I_N, \varphi_1, \dots, \varphi_N) = H_0(I) + \varepsilon H_1(I, \varphi) \quad (148)$$

where (I, φ) are action-angle variables.

The unperturbed Hamiltonian is assumed to be of the form

$$H_0(I) = \sum_{j=1}^N \lambda_j I_j + c \sum I_j^2 + o(I^3) \quad (149)$$

where either the linear part is nonresonant or $c \neq 0$ (i.e. H_0 is strictly convex). Denote $I(t), \varphi(t)$ the evolution of I, φ for the H -flow

$$\begin{cases} \dot{I}_j = -\varepsilon \frac{\partial H_1}{\partial \varphi_j} \\ \dot{\varphi}_j = \frac{\partial H_0}{\partial I_j} + \varepsilon \frac{\partial H_1}{\partial I_j} \end{cases} \quad (j = 1, \dots, N). \quad (150)$$

Then there is stability

$$|I(t) - I(0)| < \varepsilon^a \quad (151)$$

for exponentially long time

$$|t| < \exp(\varepsilon^{-b}) \quad (152)$$

where the relation a, b depends on the dimension N . It is expected that typically for sufficiently long time, there will be a “drift” of the action variables, known as Arnold diffusion.

The question we address is to what extent such result may be formulated for Hamiltonian PDE's. There are 2 immediate problems arising here.

- (i) The fact that the phase space is infinite dimensional (in the proof of the previous result, at least in the convex case, the role of the dimension is significant).
- (ii) The choice of an appropriate topology on the phase space.

Some results may be proved in the nonresonant region [B4]. For instance, consider a 1D NL wave equation

$$u_{tt} - u_{xx} + \rho u + \varepsilon f'(u) = 0 \quad (153)$$

with typical ρ , i.e. $\{\sqrt{n^2 + \rho}\}$ have good diophantine properties. Assume $f'(u)$ odd and ε small. Then, for odd and smooth initial data $u(0), u_t(0)$, the corresponding solution to (153) will evolve close to a quasi-periodic function of time for times $t, |t| < T_\varepsilon$ where T_ε may be taken to be any power of ε^{-1} . The key facts on which the argument is based are the following.

- (i) The possibility to perform a perturbative analysis up to any order and obtain an approximative solution u_0 of (153).
- (ii) The fact that the equation linearized at u_0 has zero Lyapounov exponents.

Presently, there do not seem to be satisfactory PDE counterparts of the Nekhoroshev theorem under convexity assumption of the Hamiltonian, even endowing the phase space with the weak topology. On the other hand, let us make also following qualitative comment. Global smooth solutions u to (153) may be shown (in general) to satisfy estimates on higher derivatives of the form

$$\|u(t)\|_{H^s} < C^{|t|} \quad (154)$$

hence with exponential time dependence. It seems reasonable to expect improvements on this crude estimate, but essentially nothing more is known so far.

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