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CONSTRUCTION OF PERIODIC SOLUTIONS OF NONLINEAR WAVE EQUATIONS IN HIGHER DIMENSION

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1. INTRODUCTION

In this paper we deal with the problem of existence of space and time periodic solutions of nonlinear wave equations (NLWE)

$$u_{tt} - \Delta u + F'(u) = 0. \quad (1.1)$$

Here Δ stands for the Laplacian on the d -torus $\prod^d$ and $u = u(x, t)$ is a real function on $\prod^d \times \mathbb{R}$ which is λ -periodic in t . In the case $d = 1$, this problem was treated in [C-W1] (and subsequent papers). The main new difficulty in higher dimension compared with [C-W1] is due to a more complicated structure of the “singular sites”, which are here the pairs $(m, n) \in \mathbb{Z}^d \times \mathbb{Z}$ satisfying say

$$|n^2 \lambda^2 - |m|^2| < 1. \quad (1.2)$$

In [B2], we prove persistency of periodic and quasi periodic solutions of nonlinear Schrödinger equations (NLSE) of the form

$$iu_t + \Delta u + Vu + \varepsilon \frac{\partial F}{\partial \bar{u}}(u, \bar{u}; x) = 0 \quad (1.3)$$

thus obtained by Hamiltonian perturbation of the linear Schrödinger equation

$$iu_t + \Delta u + Vu = 0 \quad (1.4)$$

(the quasi-periodic case is a continuation of [B1]). In the periodic problem, the singular sites are now the pairs $(m, n) \in \mathbb{Z}^d \times \mathbb{Z}$ satisfying

$$|n\lambda - |m|^2| < 1. \quad (1.5)$$

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An important property of these singular sites are certain separation properties, which were established for the Schrödinger case in [B1] (see the appendix). In fact this result was observed earlier by A. Granville and T. Spencer. We will show here a similar property for (1.2), which is the main extra ingredient needed to carry out the analysis from [C-W1]. In contrast to (1.5), some conditions on λ need to be imposed in the case (1.2), due to a different geometry of the cone compared with a paraboloid.

In this Note, we concentrate on the following model case

$$u_{tt} - \Delta u + \rho u + u^3 = 0 \quad (1.6)$$

(other cases may be elaborated along the same lines). Replacing u by δu , (1.6) becomes

$$u_{tt} - \Delta u + \rho u + \delta^2 u^3 = 0 \quad (1.7)$$

which appears as a perturbation of the linear equation

$$u_{tt} - \Delta u + \rho u = 0. \quad (1.8)$$

Special solutions of (1.8) are given by

$$u = p \cos(\langle m, x \rangle + \lambda t) \text{ or } u = p \sin(\langle m, x \rangle + \lambda t) \quad (1.9)$$

with

$$\lambda = (|m|^2 + \rho)^{1/2}. \quad (1.10)$$

Fix $m_0 \in \mathbb{Z}^d \setminus \{0\}$. Our aim is to show the persistency of say the solution

$$u_0 = p_0 \cos(\langle m_0, x \rangle + \lambda_0 t) \quad (1.11)$$

$$\lambda_0 = (|m_0|^2 + \rho)^{1/2} \quad (1.12)$$

of (1.8) for the perturbed equation (1.7). The perturbed solution has the form

$$u(x, t) = \sum_{m \in \mathbb{Z}^d, n \in \mathbb{Z}_+} u(m, n) \cos(\langle m, x \rangle + n \lambda t) \quad (1.13)$$

where

$$u(m_0, 1) = p_0 \quad (1.14)$$

$$\sum_{(m, n) \neq (m_0, 1)} |u(m, n)| e^{c(|m|+|n|)^c} < O(\delta) \quad (1.15)$$

$$|\lambda - \lambda_0| < O(\delta). \quad (1.16)$$

In (1.15), $c > 0$ is some constant. The use of such norms rather than the analytic ones, i.e. $\sum |u(m, n)| e^{\rho(|m|+|n|)}$, $\rho > 0$, as in [C-W1], makes matters a bit easier when keeping track of bounds along the iteration procedure.

The construction of u is achieved using the Liapounov-Schmidt procedure, similarly to [C-W1], [B1, 2]. We only present those details of the method for which the specific structure of (1.7) plays a significant role. For a more complete discussion, the reader may consult [C-W1] and [B2] (especially Appendix 2).

The main idea of the Liapounov-Schmidt construction consists in splitting the problem in a finite dimensional piece, the Q -equation (corresponding to the exact resonance) and the remainder, the P -equation (containing small divisors). One first solves the P -equation, using Newton's iteration scheme, and then substitutes in the Q -equation in order to determine the remaining parameters. We proceed in this exposition as in [C-W1] and use amplitude-frequency modulation based on the specific nonlinearity u^3 . An alternative approach would consist in considering a parameter dependent equation, letting ρ be the parameter. In any case, we assume ρ has good diophantine properties, say

$$|a\rho + b| > |a|^{-c} \quad (a \in \mathbb{Z} \setminus \{0\}, b \in \mathbb{Z}). \quad (1.17)$$

This implies in particular the non-resonance property

$$|\mu_m - k\lambda_0| > |k|^{-c} \quad (k \neq 1, -1) \quad (1.18)$$

where λ_0 is given by (1.12) and $\mu_m = (|m|^2 + \rho)^{1/2}$ for $m \in \mathbb{Z}^d$ ($\lambda_0 = \mu_{m_0}$). (We use here and in the sequel the notation $0 < c, C < \infty$ for various constants that may depend on the dimension d).

Observe that if $|k| = 1$ in (1.18), resonances are obtained for $|m| = |m_0|$, i.e. the lattice points $m \in \mathbb{Z}^d$ on the sphere with radius $|m_0|$. Thus the “resonant set” of space-time Fourier frequencies is given by (cf. [C-W2])

$$S = \{(m, \pm 1) | m \in \mathbb{Z}^d, |m| = |m_0|\}. \quad (1.19)$$

Using the representation (cf. (1.13))

$$V(x, \xi) = \sum_{m \in \mathbb{Z}^d, n \in \mathbb{Z}_+} \hat{V}(m, n) \cos(\langle m, x \rangle + n\xi) \quad (1.20)$$

the equation (1.7) is equivalent with the following equation on the Fourier coefficients $\hat{u}(m, n)$ of u

$$(-n^2 \lambda^2 + \mu_m^2) \hat{u}(m, n) + \delta^2 \widehat{u^3}(m, n) = 0. \quad (1.21)$$

The P -equation (resp. Q -equation) is obtained restricting (1.21) to frequencies $(m, n) \notin S$ (resp. frequencies $(m, n) \in S$). Write u in the form

$$u(x, t) = \sum_{\substack{m \in \mathbb{Z}^d \\ |m| = |m_0|}} p_m \cos(\langle m, x \rangle + \lambda t) + \sum_{(m, n) \notin S} \hat{u}(m, n) \cos(\langle m, x \rangle + n\lambda t). \quad (1.22)$$

The strategy is as follows. Fix $\lambda = \lambda_0 + 0(\delta^2)$, $p_0 = p_{m_0} = 0(1)$, $(p_m)_{\substack{|m| = |m_0| \\ m \neq m_0}} = 0(\delta^2)$. Using the P -equation, we solve in $\hat{u}|_{S^c}$. This will require to restrict (λ, p) to a Cantor type set, constructed along the Newton iterative procedure. The function $\hat{u}|_{S^c}$ will however be smoothly defined on the entire (λ, p) parameter set. Next, substitute $\hat{u}|_{S^c}$ in the remaining Q -equations

$$(-\lambda^2 + |m_0|^2 + \rho) p_m + \delta^2 \widehat{u^3}(m, 1) = 0 \quad (|m| = |m_0|) \quad (1.23)$$

leading to equations in $(p_m)_{|m| = |m_0|}$ and λ . Parametrizing in p_0 one obtains a solution of the form

$$p_m = 0(\delta^2) \text{ for } |m| = |m_0|, m \neq m_0 \quad (1.24)$$

$$\lambda^2 = |m_0|^2 + \rho + \frac{3}{8} p_0^2 \delta^2 + 0(\delta^4) \quad (1.25)$$

provided an appropriate restriction of p_0 permits to fulfil the conditions on (λ, p) imposed in order to solve the P -equation.

When applying Newton's method, one is lead to consider the linearized operator corresponding to (1.21). This operator T has the form $T = D + T'$, where D is diagonal

$$D(m, n) = -n^2\lambda^2 + \mu_m^2 \quad (1.26)$$

and T' is given by

$$T' = 3\delta^2 S_{u^2}. \quad (1.27)$$

Here, we denote S_ϕ the Toeplitz-operator

$$S_\phi((m_1, n_1), (m_2, n_2)) = \phi(m_1 - m_2, n_1 - n_2) \quad (1.28)$$

which is the lattice representation of the multiplication operator by ϕ . Thus T' depends on the approximative solution obtained at a given stage. Its main properties are fast off-diagonal decay (as a consequence of (1.15)) and dependence on the difference of the sites (m_1, n_1) and (m_2, n_2) . To perform the Newton iteration step, one needs to bound the inverse of a restriction $T_M = T|_{\substack{|m| < M \\ |n| < M}}$ in a reasonable way, say

$$\|T_M^{-1}\| < M^C \quad (1.29)$$

and establish off-diagonal decay

$$|T_M^{-1}(x, y)| < e^{-|x-y|^c} \text{ for } |x - y| > M^c. \quad (1.30)$$

Observe that, as a consequence of (1.18), one has for $|m|, |n| < M$

$$|-n^2\lambda^2 + \mu_m^2| > |-n^2\lambda_0^2 + \mu_m^2| - n^2|\lambda^2 - \lambda_0^2| > M^{-C} - M^c\delta^2 > \frac{1}{2}M^{-C} \quad (1.31)$$

provided M is not too large. This fact permits to control the inverse operator T_M^{-1} for the initial steps in the iteration (first $\sim \log \frac{1}{\delta}$ steps). After that, further restrictions on λ need to be imposed. These restrictions appear as follows. One considers a certain set Ω of lattice elements (m, n) (corresponding to a singular island) and the restriction $T|_\Omega$. Since $T|_\Omega$ is a self-adjoint operator, one may invoke

first order eigenvalue variation to keep the eigenvalues $E_\Omega(\lambda, p)$ away from 0. The set Ω will be contained in a ball, i.e. one has in particular

$$|m| \sim |\bar{m}| \text{ and } |n| \sim |\bar{n}| \text{ for } (m, n) \in \Omega. \quad (1.32)$$

Hence, from (1.26), (1.27) and (1.25)

$$\frac{\partial E_\Omega(\lambda, p)}{\partial p_0} = \frac{\partial E}{\partial \lambda} \frac{\partial \lambda}{\partial p_0} + \frac{\partial E}{\partial p} \frac{\partial p}{\partial p_0}$$

where

$$\frac{\partial E}{\partial \lambda} \sim -|\bar{n}|^2, \quad \frac{\partial E}{\partial p} = 0(\delta^2), \quad \frac{\partial \lambda}{\partial p_0} \sim \delta^2, \quad \frac{\partial p}{\partial p_0} = 0(1) \quad (1.33)$$

which permits to get (away from the origin)

$$\left| \frac{\partial E_\Omega}{\partial p_0} \right| \gtrsim \delta^2 |\bar{n}|^2. \quad (1.34)$$

This yields the required bounds on $(T|_\Omega)^{-1}$ where Ω ranges over the different (neighborhoods of) singular islands. The control of T_M^{-1} (i.e. (1.29), (1.30)) results then from a Pöschel type lemma, as described in the appendix of [B2] for instance. The preceding requires thus resulting conditions on p_0 , leading to a solution $u = u_{[p_0]}$ of (1.7) for p_0 restricted to a certain Cantor set of positive measure (these considerations are similar to [C-W1]).

2. STRUCTURE OF THE SINGULAR SITES

Consider the “singular” region

$$\Omega = \{(m, n) \in \mathbb{Z}^{d+1} \mid |\lambda^2 n^2 - |m|^2| < 1\} \quad (2.1)$$

and its part Ω_M contained in a box B_M of size M . We will show that Ω may be partitioned in sets Ω_α satisfying

$$\text{diam } \Omega_\alpha < M^\delta \quad (2.2)$$

and

$$\text{dist}(\Omega_\alpha, \Omega_\beta) > M^\varepsilon \text{ for } \alpha \neq \beta \quad (2.3)$$

for some small $\varepsilon = \varepsilon(d) > 0$, $\delta = \delta(d) > 0$. This is the main technical ingredient needed for the problem considered here. We will require λ to satisfy a property of the form

$$\left| \sum_{j=0}^{10d} a_j \lambda^j \right| > \left(\sum |a_j| \right)^{-C} \text{ for all } \{a_j\} \in \mathbb{Z}^{10d+1} \setminus \{0\}. \quad (2.4)$$

Recall (1.25)

$$\lambda = [|m_0|^2 + \rho + cp_0^2 \delta^2 + o(\delta^4)]^{1/2}. \quad (2.5)$$

Thus (2.4) will be satisfied for $|a_j| < \delta^{-\varepsilon}$ by imposing on ρ a similar condition and afterwards may be ensured by variation and appropriate restriction of p_0 .

The existence of such partition $\{\Omega_\alpha\}$ will clearly result from the following fact

Lemma 2.6. *Given $x_0 \in \Omega$, there is a set $\Omega' \subset \Omega$ such that*

$$x_0 \in \Omega' \quad (2.7)$$

$$\text{diam } \Omega < M^\delta \quad (2.8)$$

$$\text{dist}(x, \Omega') > M^\varepsilon \text{ whenever } x \in \Omega \setminus \Omega'. \quad (2.9)$$

Lemma 2.6 is a consequence of

Lemma 2.10. *Let $B > 1$ and $\{x_j\}_{j=0,1,\dots,k}$ a sequence in Ω of distinct elements such that*

$$|x_j - x_{j+1}| < B. \quad (2.11)$$

Then

$$k < B^C \text{ for some fixed } C. \quad (2.12)$$

Proof of Lemma 2.10.

Writing $x_j = (m_j, n_j)$, we have by hypothesis

$$||m_j|^2 - \lambda^2 n_j^2| < 1 \quad (2.13)$$

$$|m_{j+1} - m_j| + |n_{j+1} - n_j| < B. \quad (2.14)$$

For each $j_0 = 0, 1, \dots, k$ and $J \leq k$, introduce the number

$$d+1 \geq s(j_0, J) = \dim\{(m_j - m_{j_0}, n_j - n_{j_0}) \mid |j - j_0| < J\}. \quad (2.15)$$

Consider a decreasing sequence $(J_t)_{t \leq d}$ of J -values to be specified later ($J_0 = k$).

Fix t satisfying

$$1 \leq s \equiv s(j_0, J_t) = \min_j s(j, J_t) = \min_j s(j, J_{t+1}). \quad (2.16)$$

Denote S the subspace of $\mathbb{Z}^{d+1}$

$$S = [(m_j - m_{j_0}, m_j - n_{j_0}) \mid |j - j_0| < J_{t+1}] = [(m_j - m_{j_0}, n_j - n_{j_0}) \mid |j - j_0| < J_t] \quad (2.17)$$

of dimension s . It follows from (2.13), (2.14) that for $|j_1 - j_2| < J_{t+1}$

$$|\langle m_{j_1}, m_{j_2} - m_{j_1} \rangle - \lambda^2 n_{j_1} (n_{j_2} - n_{j_1})| < J_{t+1}^2 B^2. \quad (2.18)$$

Thus one has a basis $\{e_\ell \mid \ell = 1, \dots, s\}$ for S , $e_\ell \in \mathbb{Z}^{d+1}$, satisfying for $\ell = 1, \dots, s$

$$|e_\ell| < J_{t+1} B \quad (2.19)$$

$$|\langle (m_{j_0}, -\lambda^2 n_{j_0}), e_\ell \rangle| < J_{t+1}^2 B^2. \quad (2.20)$$

Since

$$|\det((\langle e_{\ell'}, e_{\ell'} \rangle)_{\ell, \ell' = 1, \dots, s})| \geq 1 \quad (2.21)$$

(2.19), (2.20) easily imply that

$$|P_S((m_{j_0}, -\lambda^2 n_{j_0}))| \lesssim (J_{t+1} B)^{s+1}. \quad (2.22)$$

Take $|j_1 - j_0| < \frac{1}{2} J_t < \frac{1}{2} J_{t+1}$. Since by (2.16), (2.17), $[(m_j - m_{j_1}, n_j - n_{j_1}) \mid |j - j_1| < J_{t+1}] \subset S$ and has dimension at least s , it follows

$$[(m_j - m_{j_1}, n_j - n_{j_1}) \mid |j - j_1| < J_{t+1}] = S. \quad (2.23)$$

Consequently, one also has

$$|P_S((m_{j_1}, -\lambda^2 n_{j_1}))| \lesssim (J_{t+1} B)^{s+1} \quad (2.24)$$

and therefore, from (2.22)

$$P_S((m_{j_1} - m_{j_0}, -\lambda^2(n_{j_1} - n_{j_0})) \lesssim (J_{t+1}B)^{s+1}. \quad (2.25)$$

Writing $e_\ell = (\Delta_\ell m, \Delta_\ell n)$, $\ell = 1, \dots, s$ for the S -basis introduced above, we have thus

$$|\langle (m_{j_1} - m_{j_0}, n_{j_1} - n_{j_0}), (\Delta_\ell m, -\lambda^2 \Delta_\ell n) \rangle| \lesssim (J_{t+1}B)^{s+2}. \quad (2.26)$$

Denote $\tilde{e}_\ell = (\Delta_\ell m, -\lambda^2 \Delta_\ell n)$ for $\ell = 1, \dots, s$. Since $\xi \equiv (m_{j_1} - m_{j_0}, m_{j_1} - n_{j_0}) \in S$, we have $\xi = \sum_{\ell=1}^s a_\ell e_\ell$ and hence

$$\langle \xi, \tilde{e}_{\ell'} \rangle = \sum_{\ell=1}^s a_\ell \langle e_\ell, \tilde{e}_{\ell'} \rangle. \quad (2.27)$$

From (2.19), (2.26), the following bound on the coefficients a_ℓ is gotten

$$|a_\ell| \lesssim \frac{(J_{t+1}B)^{s+2}(J_{t+1}B)^{2(s-1)}}{|\det(\langle e_\ell, \tilde{e}_{\ell'} \rangle)_{\ell, \ell'=1, \dots, s}|} \quad (\ell = 1, \dots, s). \quad (2.28)$$

Hence

$$|m_{j_1} - m_{j_0}| + |n_{j_1} - n_{j_0}| \lesssim (J_{t+1}B)^{3s+1} |\det(\langle e_\ell, \tilde{e}_{\ell'} \rangle)|^{-1}. \quad (2.29)$$

Since we assumed all pairs $x_j = (m_j, n_j)$ distinct, one may clearly choose j_1 such that $|j_1 - j_0| < \frac{1}{2}J_t$ and

$$|x_{j_1} - x_{j_0}| \gtrsim J_t^{1/s} > J_t^{1/d+1}. \quad (2.30)$$

It remains to get a lower bound on

$$\det(\langle e_\ell, \tilde{e}_{\ell'} \rangle) = \det((\langle \Delta_\ell m, \Delta_{\ell'} m \rangle - \lambda^2 \Delta_\ell n, \Delta_{\ell'} n)_{\ell, \ell'=1, \dots, s}) \quad (2.31)$$

which is a polynomial in λ of degree $2s$ and integer coefficients bounded by $(J_{t+1}B)^{2s}$.

This polynomial does not vanish identically, since its value for $\lambda = i = \sqrt{-1}$ equals $\det(\langle e_\ell, e_{\ell'} \rangle) \neq 0$. Hence, from the assumption (2.4)

$$|(2.31)| \gtrsim (J_{t+1}B)^{-C}. \quad (2.32)$$

Collecting estimates (2.29), (2.30), (2.32), it follows that

$$J_t < (J_{t+1}B)^C \quad (2.33)$$

which will be a contradiction for appropriately defined $(J_t)_{t=0, 1, \dots, d}$.

Lemma 2.10 clearly follows. The constant C in (2.12) depends on the constant in (2.4) and on the dimension d .

3. SOLVING THE Q -EQUATIONS

We are coming back to system (1.23)

$$(-\lambda^2 + |m_0|^2 + \rho)p_m + \delta^2 \widehat{u^3}(m, 1) = 0 \quad (|m| = |m_0|). \quad (3.1)$$

Put

$$\sigma = \delta^{-2}(-\lambda^2 + |m_0|^2 + \rho) \quad (3.2)$$

to get

$$\sigma p_m + \widehat{u^3}(m, 1) = 0 \quad (|m| = |m_0|). \quad (3.3)$$

In particular, one has

$$u(x, t) = u_1(x, t) + 0(\delta^2) \text{ where } u_1(\alpha, t) \equiv \sum_{|m|=|m_0|} p_m \cos(\langle m, x \rangle + \lambda t). \quad (3.4)$$

Thus

$$\begin{aligned} \widehat{u^3}(m, 1) &= \widehat{u_1^3}(m, 1) + 0(\delta^2) \\ &= \int_{\Pi^d} \int_{\Pi} \left[\sum_{|m'|=|m_0|} p_{m'} \cos(\langle m', x \rangle + \xi) \right]^3 \cos(\langle m, x \rangle + \xi) dx d\xi + 0(\delta^2). \end{aligned} \quad (3.5)$$

Recall also that $p_0 \equiv p_{m_0} = 0(1)$, $p_m = 0(\delta^2)$ for $|m| = |m_0|$, $m \neq m_0$. Write

$$\begin{aligned} &\left[\sum_{|m'|=|m_0|} p_{m'} \cos(\langle m', x \rangle + \xi) \right]^3 \\ &= p_0^3 \cos^3(\langle m_0, x \rangle + \xi) + 3p_0^2 \cos^2(\langle m_0, x \rangle + \xi) \left[\sum_{\substack{|m'|=|m_0| \\ m' \neq m_0}} p_{m'} \cos(\langle m', x \rangle + \xi) \right] \\ &+ 0(|p_{m'}|^2; m' \neq m_0). \end{aligned} \quad (3.6)$$

In calculating the integral in (3.5), we will use the fact that the frequencies m' appear on the sphere $|m'| = |m_0|$.

Case $m = m_0$.

One gets for the first two terms in (3.6)

$$p_0^3 \iint \cos^4(\langle m_0, x \rangle + \xi) + 3p_0^2 \iint \cos^3(\langle m_0, x \rangle + \xi) \left[\sum_{m' \neq m_0} p_{m'} \cos(\langle m', x \rangle + \xi) \right] d\lambda d\xi. \quad (3.7)$$

The contribution of the first term in (3.7) is $\frac{3}{8}p_0^3$. Writing $\cos^3 \alpha = \frac{3}{4} \cos \alpha + \frac{1}{4} \cos 3\alpha$, the second clearly vanishes. Thus for $m = m_0$, equation (3.3) becomes

$$\sigma p_0 + \frac{3}{8}p_0^3 + 0(\delta^2) + 0(|p_{m'}|^2; m' \neq m_0) = 0. \quad (3.8)$$

Case $m \neq m_0$

One gets

$$p_0^3 \iint \cos^3(\langle m_0, x \rangle + \xi) \cos(\langle m, x \rangle + \xi) + 3p_0^2 \iint \cos^2(\langle m_0, x \rangle + \xi) \cos(\langle m, x \rangle + \xi) \left[\sum_{m' \neq m_0} p_{m'} \cos(\langle m', x \rangle + \xi) \right] d\lambda d\xi. \quad (3.9)$$

The first term vanishes and the second equals $\frac{3}{4}p_0^2 p_m$, using the fact that

$$\iint \cos(2\langle m_0, x \rangle + 2\xi) \cos(\langle m, x \rangle + \xi) \cos(\langle m', x \rangle + \xi) = 0 \quad (3.10)$$

since for $|m_0| = |m| = |m'|$

$$2m_0 \neq m + m' \quad (3.11)$$

unless $m = m' = m_0$.

Thus the corresponding equation (3.3) yields

$$\sigma p_m + \frac{3}{4}p_0^2 p_m + 0(\delta^2) + 0(|p_{m'}|^2; m' \neq m_0) = 0. \quad (3.12)$$

From (3.8), (3.12), statements (1.24), (1.25) are immediate.

Taking the scaling into account when passing from (1.6) to (1.7), the conclusion of the preceding is the following

Proposition. *Let $\rho \in \mathbb{R}$ satisfy a condition of the form*

$$\left| \sum_{j=0}^r a_j p^j \right| > \left(\sum |a_j| \right)^{-c(r)} \text{ for all } \{a_j\} \in \mathbb{Z}^{r+1} \setminus \{0\}. \quad (3.13)$$

Consider the periodic wave equation in dimension d

$$u_{tt} - \Delta u + \rho u + u^3 = 0 \quad (1.6)$$

Fix $m_0 \in \mathbb{Z}^d \setminus \{0\}$. There is a Cantor set C of positive measure in an interval $[0, \delta]$ and for $p_0 \in C$ a solution of (1.6) of the form

$$u = p_0 \cos(\langle m_0, x \rangle + \lambda t) + 0(p_0^3) \quad (3.14)$$

where

$$\lambda^2 = \lambda(p_0)^2 = |m_0|^2 + \rho + \frac{3}{8}p_0^2 + 0(p_0^4). \quad (3.15)$$

Remark. As mentioned in the introduction, persistency of quasi-periodic solutions for Hamiltonian perturbations of 1D and 2D Schrödinger equations of the form

$$iu_t + \Delta u + V u + \varepsilon \frac{\partial H}{\partial \bar{u}}(u, \bar{u}, x) = 0$$

under periodic boundary conditions was established in [B1, 2]. In the case of the wave equation

$$u_{tt} - \Delta u + V u + \varepsilon f'(u, x) = 0$$

this result was established in [B1] for 1D.

The present paper deals with the time periodic problem in higher dimension. In space dimension $D \geq 2$, the methods used in [B1, 2] and here seem to fail when considering the quasi-periodic case, due to absence of separation of the singular sites. Thus this problem for the wave equation remains presently unsettled.

The lack of separation of singular sites is easily seen as follows. Choose $\lambda = (\lambda_1, \lambda_2)$ and assume that $0 < \lambda_1 < 1$ say. We show the existence of $m, m', n, n' \in \mathbb{Z}^2$ such that

$$|m|^2 - \langle n, \lambda \rangle^2 \text{ and } |m'|^2 - \langle n', \lambda \rangle^2 \quad (1)$$

are both small.

Put

$$m' = m + (1, 0) \text{ and } n' = n + (1, 0). \quad (2)$$

If $\frac{\lambda_2}{\lambda_1} \notin Q$. The set $\{n_1\lambda_1 + n_2\lambda_2 | n_1, n_2 \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ and hence, we may choose $n = (n_1, n_2)$ such that

$$|m| - \langle n, \lambda \rangle = o(|m|^{-1}), |m|^2 - \langle n, \lambda \rangle^2 = o(1) \quad (3)$$

for any given $m \in \mathbb{Z}^2$.

It follows then from (2), (3) that

$$|m'|^2 - \langle n', \lambda \rangle^2 = |m|^2 + 2m_1 + 1 - \langle n, \lambda \rangle^2 - 2\langle n, \lambda \rangle\lambda_1 - \lambda_1^2 = 2m_1 - 2|m|\lambda_1 + 1 - \lambda_1^2 + o(1). \quad (4)$$

Thus it suffices to find $m = (m_1, m_2) \in \mathbb{Z}^2$ satisfying

$$1 - \lambda_1^2 + 2 \left(m_1 - \lambda_1 \sqrt{m_1^2 + m_2^2} \right) = o(1). \quad (5)$$

Let

$$\gamma = \frac{1}{2} \sqrt{\lambda_1^{-2} - 1} \quad (6)$$

and assume $\gamma \notin \mathbb{Q}$. One may then find $m_1 \in \mathbb{Z}_+$ such that

$$|(2m_1 + 1)\gamma| = o(1) \quad (7)$$

and hence $m_2 \in \mathbb{Z}, \theta \in \mathbb{R}$ satisfying

$$(2m_1 + 1)\gamma = m_2 + \theta \quad (8)$$

$$|\theta| = o(1). \quad (9)$$

Thus, from (8), (9)

$$m_2^2 = 4\gamma^2 m_1^2 + 4\gamma(\gamma - \theta)m_1 + (\gamma - \theta)^2 \quad (10)$$

$$\begin{aligned} (m_1^2 + m_2^2)^{1/2} &= (1 + 4\gamma^2)^{1/2} m_1 \left[1 + \frac{4\gamma(\gamma - \theta)}{m_1(1 + 4\gamma^2)} + o\left(\frac{1}{m_1^2}\right) \right]^{1/2} \\ &= (1 + 4\gamma^2)^{1/2} m_1 + \frac{2\gamma^2}{(1 + 4\gamma^2)^{1/2}} + o(1) \end{aligned} \quad (11)$$

and (5) holds, by the choice (6) of γ .

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