Quartic Anharmonic Oscillator
And Random Matrix Theory

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Abstract
In this paper the relationship between the problem of constructing the
ground state energy for the quantum quartic oscillator and the problem of
computing mean eigenvalue of large positively definite random hermitean
matrices is established. This relationship enables one to present several
more or less closed expressions for the oscillator energy. One of such
expressions is given in the form of simple recurrence relations derived by
means of the method of orthogonal polynomials which is one of the basic
tools in the theory of random matrices.

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1 Introduction

In recent years there has been an interest in quantum mechanical quasi-exactly solvable problems since they seem related to many recent developments in theoretical physics [1, 2].

It was also shown that the Schrödinger equation for the energy eigenvalues for generic, that is non-quasi-exactly solvable potentials, may be approximated by an infinite sequence of potentials each of which is quasi-exactly solvable. More specifically, the solutions of the bound state equation for the quartic anharmonic oscillator are related to a sequence where the potential is a even polynomial of degree six. This fact enabled one to formulate a regular iterative method for constructing the ground state energy of quartic oscillator as an analytic function of coupling constant [2].

In the present paper, it will be shown that the above mentioned sequence of approximations is naturally rewritten in terms of the large $N$ limit for a random matrix\footnote{This is a particular case of a more general result, according to which, any quasi-exactly solvable model is equivalent to a certain random matrix model [3].}. The analysis presented here still is not a practical approach to the quartic anharmonic oscillator. However the techniques to analyze large random matrices had impressive developments in the past few years, so this approach seems promising.

After rewriting the problem in terms of random matrix theory, it appears that the ground state energy of the quartic anharmonic oscillator is exactly provided by the first few terms in the topological expansion of a different "potential", in a zero-dimensional field theory. In other words, the exact sum of Feynman graphs of a one-dimensional model, which provide ground state energy of the quartic anharmonic oscillator, are exactly given by the Feynman graphs of low genus, of a different zero-dimensional model. This feature is quite unexpected and is the second motivation of the present paper.

2 Quasi-exactly solvable sextic oscillator model

Consider a one-dimensional quantum sextic oscillator model with hamiltonian

$$H_M = -\frac{\partial^2}{\partial x^2} + \left[b^2 - a(4M + 3)\right]x^2 + 2abx^4 + a^2x^6$$

(2.1)

in which $a > 0$ and $b$ are real parameters and $M$ is an arbitrarily chosen non-negative integer. It is known that model (2.1) is quasi-exactly solvable. This means that for any given $M$ the Schrödinger equation for it admits algebraic solutions only for some limited part of the spectrum. The corresponding (alge-
braically calculable) energy levels are given by the formula

\[ E_M = (4M + 1)b + 8a \sum_{i=1}^{M} \xi_i, \]  

(2.2)

in which \( \xi_1, \ldots, \xi_M \) are real numbers satisfying the system of \( M \) algebraic equations

\[ \sum_{k=1, k \neq i}^{M} \frac{1}{\xi_i - \xi_k} + \frac{1}{4\xi_i} - b - 2a\xi_i = 0, \]  

(2.3)

with \( i = 1, \ldots, M \). It can be shown that system (2.3) has \( M + 1 \) solutions describing energy levels with numbers 0, 2, \ldots, 2\( M \). If the number of a level is 2\( K \) then \( K \) of quantities \( \xi_i \) are positive and \( M - K \) of them are negative. In particular, for the ground state all the quantities \( \xi_i \) should be negative.

### 3 Quartic oscillator as a limiting case of quasi-exactly solvable sextic oscillators

Let us now demonstrate that if \( M \) tends to infinity, the model (2.1) is reduced to the model with quartic anharmonicity. In order to demonstrate this let us assume that parameters \( a \) and \( b \) are functions of \( M \). We choose this dependence in the form

\[ a = \frac{1}{2} g^{\frac{3}{2}} f^{-1}(g, M), \quad b = g^{\frac{1}{2}} f(g, M), \]  

(3.1)

where \( f(g, M) \) is a function of two variables \( g \) and \( M \) implicitly determined from the equation

\[ f^3(g, M) - g^{-\frac{3}{2}} f(g, M) = 2(M + \frac{1}{4}). \]  

(3.2)

In this case we will have

\[ b^2 - a(4M + 1) = 1, \quad 2ab = g, \]  

(3.3)

\[ H_M = -\frac{\partial^2}{\partial x^2} + (1 - 2a)x^2 + gx^4 + a^2 x^6 \]  

(3.4)

with \( a \) being the root of the cubic equation

\[ g^2 = 4a^2[1 + a(1 + 4M)] \]  

(3.5)
in the interval $0 < a < g/2$. Since
\[
a \approx \frac{1}{2} \frac{g^{3/2}}{(2M)^{1/3}} \quad \text{for} \quad M \to \infty, \quad g \text{ fixed}
\] (3.6)
in the limit $M \to \infty$ the model (2.1) reduces to the quartic oscillator model
with hamiltonian
\[
H = \lim_{M \to \infty} H_M = -\frac{\partial^2}{\partial x^2} + x^2 + gx^4.
\] (3.7)

Now what about the solutions of the model (3.7)? It is quite obvious that at least all its even solutions can be recovered from formulas (2.2) – (2.3) if we substitute in them the expressions (3.1) for $a$ and $b$ and then take the limit $M \to \infty$. For the sake of simplicity we can restrict ourselves by discussing only the ground state solution for which all the numbers $\xi_1, \ldots, \xi_M$ should be negative. But before going over to the limit $M \to \infty$, it is desirable to make (for the sake of further convenience) some appropriate rescaling of quantities $\xi_1, \ldots, \xi_M$, introducing instead of them the new quantities $\eta_1, \ldots, \eta_M$ by the formulas
\[
\xi_i = -g^{-\frac{1}{2}} f^2(g, M) \eta_i, \quad i = 1, \ldots, M.
\] (3.8)
After this the expression for the ground state energy in the model (3.7) takes the form
\[
E = \lim_{M \to \infty} E_M = \lim_{M \to \infty} (4M + 1)g^{\frac{1}{2}} f(g, M) \cdot [1 - S_M(\eta_1, \ldots, \eta_M)],
\] (3.9)
where
\[
S_M(\eta_1, \ldots, \eta_M) = \sum_{i=1}^{M} \frac{\eta_i}{M + 1/4}
\] (3.10)
and $\eta_1, \ldots, \eta_M$ are positive solutions of the system
\[
\sum_{k=1, k \neq i}^{M} \frac{1}{\eta_i - \eta_k} + \frac{1}{4 \eta_i} + f^3(g, M)(1 - \eta_i) = 0,
\] (3.11)
with $i = 1, \ldots, M$.
Taking into account the facts that the large $M$ asymptotics of function $f(g, M)$ is
\[
f(g, M) = 2^{\frac{1}{2}}(M + 1/4)^{\frac{1}{2}} \left\{ 1 + \frac{1}{3[2g(M + 1/4)]^{\frac{1}{2}}} + \cdots \right\},
\] (3.12)
and the ground state energy $E$ of the quartic anharmonic oscillator is a finite number, we can conclude that for large values of $M$ the quantity $S_M(\eta_1, \ldots, \eta_M)$ should behave as

$$S_M(\eta_1, \ldots, \eta_M) = 1 - \frac{E}{4(2g)^{\frac{1}{2}}} \cdot \frac{1}{M^{\frac{3}{4}}} + \ldots \quad (3.13)$$

In other words, if $S_M(\eta_1, \ldots, \eta_M)$ is already known and we would like to find the ground state energy of the quartic anharmonic oscillator, it is sufficient to find the first non-vanishing correction to unity in the expansion of $S_M(\eta_1, \ldots, \eta_M)$ in negative powers of $M$. All higher corrections will be irrelevant to us.

4 Integral representation for the ground state energy in quartic oscillator model

The result of the previous section enables one to replace $S_M(\eta_1, \ldots, \eta_M)$ in formula (3.9) by another quantity which we denote by $S_M$ and define as

$$S_M = \sum_{i=1}^{M} \frac{1}{M + 1/4} = \frac{D\eta S_M(\eta_1, \ldots, \eta_M) \exp[-V(\eta_1, \ldots, \eta_M)]}{\int_{0}^{\infty} D\eta \exp[-V(\eta_1, \ldots, \eta_M)]} \quad (4.1)$$

where

$$D\eta = \prod_{i=1}^{M} d\eta_i \quad (4.2)$$

and

$$V(\eta_1, \ldots, \eta_M) = -\sum_{i \neq k}^{M} \ln |\eta_i - \eta_k| - \frac{1}{2} \sum_{i=1}^{M} \ln |\eta_i| + f^9(g, M) \sum_{i=1}^{M} (\eta_i - 1)^2 \quad (4.3)$$

The reason for such a replacement can be explained as follows. It is clear that the exponent (4.3) has a minimum which is saturated by numbers $\eta$ satisfying the system of equations (3.11). As follows from these equations, all the numbers $\eta$ should be of order unity, and therefore, the most relevant values of exponent (4.3) should be of order $M^2$. This means that for large values of $M$ the multiple integrals in (4.1) can be evaluated by the saddle point method and the correction to the zero order approximation should be of order $M^{-2}$. Now note that the zero order approximation for (4.1) is nothing else than $S_M(\eta_1, \ldots, \eta_M)$. The fact that the difference between $S_M(\eta_1, \ldots, \eta_M)$ and $S_M$ is of order $M^{-2}$ (i.e. less than $M^{-\frac{3}{2}}$) makes it possible to replace $S_M(\eta_1, \ldots, \eta_M)$ by $S_M$ in the expression for $E$. This enables one to write the following simple formula for the ground state energy in the quartic oscillator model (3.7):

$$E = \lim_{M \to -\infty} (4M + 1)g^{\frac{1}{2}}f(g, M) \cdot [1 - S_M], \quad (4.4)$$

where $S_M$ is defined by formulas (4.1) - (4.3).
5 Random matrix model

In this section we recall that the expression (4.1) for $S_M$ naturally arises in the theory of random matrix models.

Consider a zero-dimensional "field-theoretical" model in which the role of a field is played by the random $M \times M$ positively definite hermitean matrix $A$. Let us choose the action of the model in the form

$$W_M[A] = \text{Tr}[-\frac{1}{2} \ln A + f^A(g, M)(A - 1)^2] \quad (5.1)$$

which is obviously invariant under unitary transformations $A \rightarrow UAU^{-1}$. The simplest invariant quantities in this model are the eigenvalues of the matrix $A$. Consider the problem of computing $\bar{A} = \langle \text{Tr} A \rangle / (M + 1/4)$. The result can be represented in the form of a multiple integral

$$\bar{A} = \frac{1}{M + 1/4} \frac{\int DA \text{Tr} A \exp\{-W_M[A]\}}{\int DA \exp\{-W_M[A]\}} \quad (5.2)$$

which can be simplified if one uses the well-known Faddeev-Popov method for extracting from (5.2) the volume of the unitary group. Using formulas

$$A = U \cdot \text{diag}\{\eta_1, \ldots, \eta_M\} \cdot U^{-1} \quad (5.3)$$

and

$$DA = \prod_{i=1}^{M} d\eta_i \prod_{i < k} (\eta_i - \eta_k)^2 DU \quad (5.4)$$

in which $\eta_k$ are the eigenvalues of $A$ and $DU$ denotes the measure on the unitary group, we can easily reduce the expression (5.2) to the form

$$\bar{A} = \frac{\int_0^\infty D\eta S_M(\eta_1, \ldots, \eta_M) \exp[-V(\eta_1, \ldots, \eta_M)]}{\int_0^\infty D\eta \exp[-V(\eta_1, \ldots, \eta_M)]} \quad (5.5)$$

in which $D\eta$, $S_M(\eta_1, \ldots, \eta_M)$ and $V(\eta_1, \ldots, \eta_M)$ are given by formulas (4.2), (3.10) and (4.3), respectively. We see that formulas (5.5) and (4.1) exactly coincide! This means that the problem of finding the ground state energy in the quartic oscillator model (3.7) is equivalent to the problem of finding the mean eigenvalue of the random matrix $A$ in the theory with the action (5.1).

6 Orthogonal polynomials and recurrence relations

The equivalence of models (3.7) and (5.1) enables one to simplify the expression (4.4) by using the method of orthogonal polynomials which is one of the basic tools in the theory of random matrix models [4].
The idea of this method can be formulated as follows. Let us rewrite the expression (4.1) in the form

\[
S_M = \frac{\int_0^\infty \prod_{i=1}^M d\eta_i S_M(\eta_1, \ldots, \eta_M) \prod_{i<k}^M (\eta_i - \eta_k)^2 \exp[-V_0(\eta_1, \ldots, \eta_M)]}{\int_0^\infty \prod_{i=1}^M d\eta_i \prod_{i<k}^M (\eta_i - \eta_k)^2 \exp[-V_0(\eta_1, \ldots, \eta_M)]} \tag{6.1}
\]

with

\[
V_0(\eta_1, \ldots, \eta_M) = -\frac{1}{2} \sum_{i=1}^M \ln |\eta_i| + f^2(g, M) \sum_{i=1}^M (\eta_i - 1)^2 \tag{6.2}
\]

and note that

\[
\prod_{i<k}^M (\eta_i - \eta_k)^2 = [\det ||P_{k-1}(\eta)||_{i, k=1, \ldots, M}]^2 = [\det ||P_{k-1}(\eta)||_{i, k=1, \ldots, M}]^2 \tag{6.3}
\]

where \(P_n(\eta)\) denote arbitrary linearly independent polynomials of degrees \(n\) normalized as \(P_n(\eta) = \eta^n + \ldots\). Because of the arbitrariness of these polynomials we can consider them (without any loss of generality) as polynomials orthogonal with the weight \(|\eta|^{1/2} \exp[-f^2(g, M)(\eta - 1)^2]\), i.e. requiring the condition

\[
\int_0^\infty d\eta P_n(\eta)P_m(\eta)|\eta|^{1/2} \exp[-f^2(g, M)(\eta - 1)^2] = h_n \delta_{mn} \tag{6.4}
\]

This immediately leads us to the formula

\[
S_M = \frac{1}{M + 1/4} \sum_{m=0}^{M-1} \int_0^\infty d\eta \frac{P_m^2(\eta)|\eta|^{1/2} \exp[-f^2(g, M)(\eta - 1)^2]}{h_m} \tag{6.5}
\]

By using well known methods, [5], recurrence relations are derived for the orthogonal polynomials, which allow the evaluation of the expression (6.5). The polynomials are given by a three terms equation

\[
\eta P_n(\eta) = P_{n+1}(\eta) + A_n(\eta) P_n(\eta) + R_n(\eta) P_{n-1}(\eta) \tag{6.6}
\]

with \(R_n(\eta) = \frac{h_n}{h_{n-1}}.\) It follows that

\[
S_M = \frac{1}{M + 1/4} \sum_{k=0}^{M-1} A_k \tag{6.7}
\]

It is also customary to introduce the states \(|n>\)

\[
<\eta|n> = P_n(\eta), \quad \text{where} \quad <m|n> = \delta_{mn} \tag{6.8}
\]
One obtains [6] a local string equation
\[ <n|\eta W'(\eta)|n> = 2n + 1 \] (6.9)
that is
\[ 2f^3(g, M)[A_n^2 - A_n + R_{n+1} + R_n] = 2n + \frac{3}{2} \] (6.10)
and a non-local equation
\[
\sum_{k=0}^{n} A_k = \sqrt{R_{n+1}} <n+1|\eta W'(\eta)|n> = R_{n+1} 2 f^3(g, M)(A_{n+1} + A_n - 1) \] (6.11)
which leads to a further expression for \( S_M \)
\[ S_M = \frac{2 f^3(g, M)}{M + 1,4} R_M [A_M + A_{M-1} - 1] \] (6.12)

If the coefficients \( A_k, R_k \) are known up to \( k = n \), eq. (6.10) allows the determination of \( R_{n+1} \), next \( A_{n+1} \) may be evaluated by eq. (6.11). The lowest values are
\[
A_0 = \frac{\mu_1}{\mu_0} , \quad R_0 = 0 \\
A_1 = 1 - \frac{\mu_1}{\mu_0} + \frac{1}{2f^3(g, M)[(\mu_2 - \mu_1)]} , \quad R_1 = \frac{\mu_2}{\mu_0} - A_0^2 \] (6.13)
where \( \mu_n \) are the moments
\[ \mu_n(g, M) = \int_{0}^{\infty} dx x^{n+1/2} e^{-f^3(g, M)(x-1)^2} \] (6.14)
Since \( S_M \) in eq. (6.12) determines the ground energy \( E(g) \) of the quartic anharmonic oscillator (4.4) for large values of \( M \), the above iterations do not seem very useful. It might seem more promising to derive a recurrence relation immediately for the quantities
\[ q_n = \sum_{k=0}^{n-1} A_k \] (6.15)
One easily obtains:
\[
(q_{n+1} - q_n)(q_{n+1} - q_n - 1) + \frac{1}{2f^3(g, M)} \left[ \frac{q_{n+1}}{q_{n+2} - q_n - 1} \right] - \frac{q_n}{q_{n+1} - q_{n-1} - 1} - 2n \frac{3}{2} = 0 \] (6.16)
where \( q_{n+2} \) is determined in terms of \( q_{n+1}, q_n, q_{n-1} \). The lowest values are \( q_0 = 0, q_1 = A_0, q_2 = A_0 + A_1 \).

Thus the final formula for the ground state energy of the quartic oscillator model (3.7) has the form

\[
E = \lim_{M \to \infty} E_M = \lim_{M \to \infty} 4(2M^4g)^{1/2} \left( 1 - \frac{q_M}{M + 1/4} \right) .
\] (6.17)

where \( q_M \) can be found from the recurrence relation (6.16).

7 Conclusion

The equations (6.16) and (6.17) obtained in the above Section are exact and, in principle, provide an exact evaluation of \( E(g) \) (i.e. with an arbitrarily high accuracy). It seems, however, that these equations have yet a purely theoretical significance, because the practical use of these formulas encounters serious difficulties.

The first difficulty follows from the fact that the deviation of the exact value of \( E \) from its approximations \( E_M \) obtained for finite \( M \)'s is of order \( M^{-1/3} \). Indeed, remember, that the energy \( E \) appeared as a coefficient in front of \( M^{-1/2} \) in the expansion of \( S_M(\eta_1, \ldots, \eta_M) \) (see formula (3.13) ), and we replaced \( S_M(\eta_1, \ldots, \eta_M) \) by \( S_M \) after neglecting corrections of order \( M^{-2} \) (see Section 4). This means that the sequence of \( E_M \) converges to \( E \) very slowly. For example, if we want to compute \( E \) with 1 percent accuracy, we should take \( M = 1000 \), and this means that we need 1000 steps of the recurrence procedure (6.16). The recurrence relations (6.16) are, however, rather unstable, and the iteration of \( q_n \) from low to high values of \( n \) does not seem practical, because of the rapid accumulation of errors.

The second difficulty lies in the fact that the direct analysis of the asymptotic behaviour of the difference equation (6.16) is non trivial. It is rather easy to obtain the leading behaviour of numbers \( q_n \) for large \( n \), considering \( q_n \) as an analytic function of \( n \), \( q_n = q(n) \) and solving the associated differential equation for \( q(n) \). The result, \( q_n \sim n + 1/4 \), does not contradict formula (6.17). However, a naive attempt of finding in such a way the non-leading behaviour of \( q(n) \), which is relevant for eq. (6.17), leads to a priori false result.

It seems that both above mentioned difficulties are closely related to each other. It is possible that function \( q(n) \) is continuous, but not differentiable. In other words, the first (or, perhaps, the second) derivative of \( q(n) \) might be a random function of \( n \). In this case, the approximation of the recurrence relations (6.16) by a differential equation is, generally, rather rough, and cannot be used for studying any subtle properties of function \( q_n \). If the behaviour of the differences \( q_n - q_{n-1} \) (the derivatives of \( q(n) \)) is really stochastic, then this fact would easily explain the rapid accumulation of errors in numerical computations of the ground state energy of quartic oscillator by formulas (6.16) and (6.17).
In conclusion of this Section we demonstrate that the problem of determination of a non-leading behaviour of numbers $q_n$ is related to the problem of finding the form of the second local string equation for the recurrence relations. In order to show this, let us first provide an approximate evaluation of $S_M$ for large $M$ by means of the saddle point method. At leading order, for large $M$, one may replace the discrete eigenvalues $\eta_i$ with the continuous function $\eta(i/M) = \eta$ and the algebraic system (3.11) with the singular integral equation

$$
\int_0^1 dy \frac{1}{\eta(x) - \eta(y)} + \frac{1}{4M\eta(x)} + \frac{f^3(g, M)}{M} [1 - \eta(x)] = 0 \quad (7.1)
$$

which, after the introduction of the eigenvalue density $\rho(\eta) = \frac{d}{d\eta}$, becomes

$$
\int_A^B d\mu \frac{\rho(\mu)}{\lambda - \mu} = \frac{f^3(g, M)}{M} [\lambda - 1] - \frac{1}{4M\lambda} \quad (7.2)
$$

This is the saddle-point approximation for the large-$M$ limit of the matrix model (5.1). Its solution with $0 < A < B$ was described in [7], together with the evaluation of expectation values. One obtains, for large $M$,

$$
\sum_{i=1}^M <\eta_i> \sim M \int_A^B \mu \rho(\mu) d\mu = (B - A)^3 (A + B - 1) \frac{f^3(g, M)}{8} \quad (7.3)
$$

where the extrema of the support are solution of the algebraic equations

$$
2(A + B)^3 + (B - A)^3 - 4(A + B) = \frac{4M}{f^3(g, M)} (2 + \frac{1}{2M})
$$

$$
2\sqrt{AB} (A + B - 2) f^3(g, M) = 1 \quad (7.4)
$$

From the system (7.4) one easily evaluates the large $M$ expansions

$$
A = \frac{a_1}{M^{2/3}} + \frac{a_2}{M^{4/3}} + ..
$$

$$
B = 2 + \frac{b_1}{M^{2/3}} + \frac{b_2}{M^{4/3}} + .. \quad (7.5)
$$

To estimate the present saddle-point approximation, we quote the first coefficients in the limit of large values of $g$

$$
a_1 = \frac{1}{2^{2/3}} [1 + O(g^{-2/3})] \quad , \quad b_1 = \frac{-1}{2(2g)^{2/3}}
$$

$$
a_2 = \frac{1}{32} \frac{1}{2^{1/3}} [1 + O(g^{-2/3})] \quad , \quad b_2 = \frac{-3}{64} \frac{1}{2^{1/3}} [1 + O(g^{-2/3})] \quad (7.6)
$$

Then

$$
S_M = \sum_{i=1}^M <\eta_i> M + 1/M^2 \sim 1 - \frac{1}{M^{4/3}} \frac{3}{16} \frac{1}{2^{1/3}} + O(g^{-2/3})
$$

$$
E(g) \sim g^{1/3} \frac{3}{4} [1 + O(g^{-2/3})] \quad (7.7)
$$

10
The comparison of the approximate result (7.7) with the known first term of the strong coupling expansion [9]

\[ E(g) = 2^{3/5}g^{1/3}[0.667 + O(g^{-2/3})] \]  

(7.8)

shows that the fractional powers of the coupling constant \( g \) are correctly reproduced, but the relative error for the first coefficient in the strong coupling limit of the ground state energy is about 32%. This fact is not surprising because we started with a continuous version of the saddle point equation, which is only an approximation to the true discrete one.

Let us now consider the equivalent approximation by recurrence relation. If we neglect the contributions in integration by parts at \( \eta = 0 \), discussed in [6], a second local string equation is obtained [5]:

\[ \langle n | W(\eta) | n \rangle = 0 \]  

(7.9)

and evaluating it in the continuum approximation (that is \( A_n = A(\frac{n}{M}) \), \( R_n = R(\frac{n}{M}) \)) and neglecting the derivatives, it reads

\[ 4f^3(g, M)[A(x) - 1] = \frac{1}{\sqrt{A^3(x) - 4R(x)}} \]  

(7.10)

The same approximation on the first local string eq.(6.10) is

\[ \frac{2f^3(g, M)}{M}[A^3(x) - A(x) + 2R(x)] = 2x + \frac{3}{2M} \]  

(7.11)

This algebraic system, for large values \( n \sim M \) that is \( x \sim 1 \) reproduces the system (7.4) after the usual correspondence [8]

\[ R(1) = \frac{(B - A)^2}{16} \]
\[ A(1) = \frac{(B + A)}{2} \]  

(7.12)

From the above exercise we learn that to improve on the approximate evaluation (7.7) by use of the recurrence relations, it is likely to need an improved evaluation of the second string equation, by including the derivatives and the boundary contributions at \( \eta = 0 \). We intend to consider this question in forthcoming publications.

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References


[8] The non-leading difference of the term \((2 + \frac{1}{2M})\) on the right side of eq.(7.4) versus the term \((2 + \frac{3}{2M})\) on the right side of eq.(7.11) is irrelevant in this context. Indeed both the approximate methods in this Section are equivalent at leading order for large \(M\) and fail to reproduce the relevant non leading behaviour.