Novel Solitonic Evolutions in a Coupled Integrable, Dispersionless System

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Abstract

A coupled integrable, dispersionless system is solved by the inverse scattering method. Three types of solitons are uncovered. Novel solitonic phenomena are recorded.

Key Words

soliton, integrable equation, dispersionless equation, inverse method

1 Introduction

In a recent paper, Oono and one of the authors (K.K.)¹⁾ introduced the following coupled integrable, dispersionless equations:

$$\begin{aligned} q_{xt} - 2\alpha qr_x - 2\beta qs_x + \gamma(rs)_x &= 0, \\ r_{xt} - 2\alpha rr_x + 2\beta(2qq_x + r_xs) - 2\gamma q_x r &= 0, \\ s_{xt} - 2\beta ss_x + 2\alpha(2qq_x + r_xs) - 2\gamma q_x s &= 0. \end{aligned} \tag{1}$$

Changing variables, we obtain a simple coupled system²⁾:

$$\begin{aligned} q_{xt} + (rs)_x &= 0, \\ r_{xt} - 2q_x r &= 0, \\ s_{xt} - 2q_x s &= 0. \end{aligned} \tag{2}$$

Here we consider solutions of the above system with the boundary conditions:

$$\left. \begin{aligned} q_x &\rightarrow q_0 \\ r &\rightarrow 0 \\ s &\rightarrow 0 \end{aligned} \right\} \quad \text{for } |x| \rightarrow \infty, \tag{3}$$

where q_0 is constant. Special cases of $r = s^{1)}$ and $r = s^{*3)}$ were solved by the inverse scattering method.

Eq.(2) has an important conserved quantity

$$q_x^2 + r_x s_x = q_0^2. \tag{4}$$

If $r = s$, (4) is rewritten as

$$q_x^2 + r_x^2 = q_0^2. \tag{5}$$

then (5) is related to the Euclidian group in two dimensions E_2 . Eq.(2) was found to be connected to the sine-Gordon equation.⁴⁾⁵⁾ If $r = s^*$, (4) is rewritten as

$$q_x^2 + \text{Re}(r_x)^2 + \text{Im}(r_x)^2 = q_0^2, \tag{6}$$

then (6) is related to the symmetry of $O(3) \sim SU(2)$. Eq.(2) was shown to be equivalent to the Pohlmeyer-Lund-Regge equation.⁶⁾ By changing variables $r = \rho + \eta$ and $s = \rho - \eta$, (4) yields

$$q_x^2 + \rho_x^2 - \eta_x^2 = q_0^2. \tag{7}$$

(7) is related to the symmetry of $O(2, 1) \sim SL(2, R)$. (7) implies that it is possible that amplitude of r or s is not bounded asymptotically. The group theoretical consideration will be continued in a separate paper. Here we solve (2) with the boundary conditions (3) by using the inverse scattering method.

We call soliton solutions for q_x , r and s , in turn, Q_x solitons, R solitons and S solitons. R solitons and S solitons are of bright type and Q_x is dark type. Usually, the number of solitons is given by the corresponding number of zeros of the scattering data, typically, in the upper half plane of complex eigenvalues. The most remarkable feature of system (2) is that, since there are no dependent relations between the upper and lower half planes of the Jost functions, the number of R solitons does not restrict to that of S solitons. Thus, the soliton solutions are provided independently, in turn, by zeros of scattering data in the upper half plane and of scattering data in the lower half plane. The number of R solitons and S solitons is not necessary the same. Indeed, there exists a soliton solution consisting of one R soliton and two S solitons. Q_x solitons are determined once the R solitons and S solitons are known. Interesting properties of one soliton solution are shortly reported in Ref.2.

In § 2, we consider the inverse scattering method of (2) and obtain a complicated set of the Gel'fand-Levitan equations. One soliton solution is discussed in § 3. In § 4 we give explicit two soliton solution and study interaction between solitons. The last section is devoted the concluding remarks.

2 The Inverse Problem

The inverse scattering scheme of (2) is given by

$$\begin{aligned} V_x &= UV, \\ V_t &= WV, \end{aligned} \tag{8}$$

where

$$\begin{aligned} U &= -i\lambda \begin{pmatrix} q_x & r_x \\ s_x & -q_x \end{pmatrix}, \\ W &= \begin{pmatrix} 0 & -r \\ s & 0 \end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{9}$$

We solve (8) with the boundary conditions (3) on the same way as Refs.7 and 8.

For real λ we define the Jost functions to be

$$\left. \begin{aligned} \phi &\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\lambda q_0 x) \\ \bar{\phi} &\rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} \exp(i\lambda q_0 x) \end{aligned} \right\} \quad \text{for } x \rightarrow \infty, \quad (10)$$

$$\left. \begin{aligned} \psi &\rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(i\lambda q_0 x) \\ \bar{\psi} &\rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\lambda q_0 x) \end{aligned} \right\} \quad \text{for } x \rightarrow -\infty, \quad (11)$$

and the scattering coefficients to be

$$\begin{aligned} \phi &= a \bar{\psi} + b \psi, \\ \bar{\phi} &= -\bar{a} \psi + \bar{b} \bar{\psi}. \end{aligned} \quad (12)$$

where

$$a \bar{a} + b \bar{b} = 1. \quad (13)$$

In order to examine the analytic properties of the Jost functions we introduce

$$\phi_1 = \exp \left\{ -i\lambda q_0 x + \int_{-\infty}^x \sigma(y) dy \right\} \quad (14)$$

Substitution of (14) into (8) yields

$$\frac{\partial \sigma}{\partial t} = \frac{\partial}{\partial x} \left[\frac{r(q_x - q_0)}{r_x} - \frac{ir\sigma}{\lambda r_x} + \frac{i}{2\lambda} \right]. \quad (15)$$

If we expand σ as a power series in λ of the form

$$\sigma = \sum_{n=0}^{\infty} \frac{\sigma_n}{(i\lambda)^n}. \quad (16)$$

we obtain an infinite number of conserved quantities by substituting (16) into (15). On the same way, introducing

$$\bar{\psi}_2 = \exp \left\{ i\lambda q_0 x + \int_{-\infty}^x \hat{\sigma}(y) dy \right\}, \quad (17)$$

we get

$$\frac{\partial \hat{\sigma}}{\partial t} = \frac{\partial}{\partial x} \left[\frac{s(q_x - q_0)}{s_x} + \frac{is\hat{\sigma}}{\lambda s_x} - \frac{i}{2\lambda} \right]. \quad (18)$$

Expansion of $\hat{\sigma}$ as

$$\hat{\sigma} = \sum_{n=0}^{\infty} \frac{\hat{\sigma}_n}{(i\lambda)^n}, \quad (19)$$

yields conserved quantities. Some of them are shown as

$$\sigma_{-1} = q_x^2 + r_x s_x = q_0^2, \quad (20a)$$

$$\sigma_0 = \frac{r_x}{2q_0} \left(\frac{q_x - q_0}{r_x} \right)_x, \quad (20b)$$

$$\hat{\sigma}_0 = \frac{s_x}{2q_0} \left(\frac{q_x - q_0}{s_x} \right)_x, \quad (20c)$$

$$\sigma_1 = \frac{r_x}{2q_0} \left[\frac{\sigma_0^2}{r_x} + \left(\frac{\sigma_0}{r_x} \right)_x \right], \quad (20d)$$

$$\hat{\sigma}_1 = \frac{s_x}{2q_0} \left[\frac{\hat{\sigma}_0^2}{s_x} + \left(\frac{\hat{\sigma}_0}{s_x} \right)_x \right]. \quad (20e)$$

We solve (10) and (11) and obtain

$$\begin{aligned} \phi_1 &= \exp(-i\lambda q_0 x) - i\lambda \int_{-\infty}^x [(q_y - q_0)\phi_1 + r_y \phi_2] \exp(-i\lambda q_0(x - y)) dy, \\ \phi_2 &= -i\lambda \int_{-\infty}^x [s_y \phi_1 - (q_y - q_0)\phi_2] \exp(i\lambda q_0(x - y)) dy, \\ \bar{\phi}_1 &= -i\lambda \int_{-\infty}^x [(q_y - q_0)\bar{\phi}_1 + r_y \bar{\phi}_2] \exp(-i\lambda q_0(x - y)) dy, \\ \bar{\phi}_2 &= -\exp(i\lambda q_0 x) - i\lambda \int_{-\infty}^x [s_y \bar{\phi}_1 - (q_y - q_0)\bar{\phi}_2] \exp(i\lambda q_0(x - y)) dy, \end{aligned} \quad (21)$$

and

$$\begin{aligned} \psi_1 &= i\lambda \int_x^{\infty} [(q_y - q_0)\psi_1 + r_y \psi_2] \exp(-i\lambda q_0(x - y)) dy, \\ \psi_2 &= \exp(i\lambda q_0 x) + i\lambda \int_x^{\infty} [s_y \psi_1 - (q_y - q_0)\psi_2] \exp(i\lambda q_0(x - y)) dy, \\ \bar{\psi}_1 &= \exp(-i\lambda q_0 x) + i\lambda \int_x^{\infty} [(q_y - q_0)\bar{\psi}_1 + r_y \bar{\psi}_2] \exp(-i\lambda q_0(x - y)) dy, \\ \bar{\psi}_2 &= i\lambda \int_x^{\infty} [s_y \bar{\psi}_1 - (q_y - q_0)\bar{\psi}_2] \exp(i\lambda q_0(x - y)) dy. \end{aligned} \quad (22)$$

We find that the relations

$$\begin{aligned}
\bar{\phi}_1(\lambda) &= \phi_2^*(\lambda^*), \\
\bar{\phi}_2(\lambda) &= -\phi_1^*(\lambda^*), \\
\bar{\psi}_1(\lambda) &= \psi_2^*(\lambda^*), \\
\bar{\psi}_2(\lambda) &= -\psi_1^*(\lambda^*), \\
\bar{a}(\lambda) &= a^*(\lambda^*), \\
\bar{b}(\lambda) &= b^*(\lambda^*),
\end{aligned} \tag{23}$$

are not satisfied because of no dependent relations between r and s such as $r = s$.

Asymptotic behavior of the Jost functions and the scattering coefficients is expressed by

$$\begin{aligned}
\phi &= \begin{pmatrix} 1 \\ -(q_x - q_0)/r_x \end{pmatrix} \exp(-i\lambda q_0 x + \mu_-) + O\left(\frac{1}{\lambda}\right), \\
\bar{\phi} &= \begin{pmatrix} -(q_x - q_0)/s_x \\ -1 \end{pmatrix} \exp(i\lambda q_0 x + \hat{\mu}_-) + O\left(\frac{1}{\lambda}\right), \\
\psi &= \begin{pmatrix} (q_x - q_0)/s_x \\ 1 \end{pmatrix} \exp(i\lambda q_0 x - \hat{\mu}_+) + O\left(\frac{1}{\lambda}\right), \\
\bar{\psi} &= \begin{pmatrix} 1 \\ -(q_x - q_0)/r_x \end{pmatrix} \exp(-i\lambda q_0 x - \mu_+) + O\left(\frac{1}{\lambda}\right), \\
a &= \exp(\mu_0) + O\left(\frac{1}{\lambda}\right), \\
\bar{a} &= \exp(\hat{\mu}_0) + O\left(\frac{1}{\lambda}\right),
\end{aligned} \tag{24}$$

where and μ_+ , μ_- , μ_0 , $\hat{\mu}_+$, $\hat{\mu}_-$ and $\hat{\mu}_0$ are defined with (20b) and (20c) as

$$\begin{aligned}
\mu_+(x) &= \int_x^\infty \sigma_0 dy, \\
\mu_-(x) &= \int_{-\infty}^x \sigma_0 dy, \\
\mu_0(x) &= \mu_-(x) + \mu_+(x), \\
\hat{\mu}_+(x) &= \int_x^\infty \hat{\sigma}_0 dy, \\
\hat{\mu}_-(x) &= \int_{-\infty}^x \hat{\sigma}_0 dy, \\
\hat{\mu}_0(x) &= \hat{\mu}_-(x) + \hat{\mu}_+(x).
\end{aligned} \tag{25}$$

Then, when $(q_x - q_0)/r_x$ and $(q_x - q_0)/s_x$ have compact support, $\phi \exp(i\lambda q_0 x)$, $\bar{\phi} \exp(-i\lambda q_0 x)$, $\psi \exp(-i\lambda q_0 x)$, $\bar{\psi} \exp(i\lambda q_0 x)$, a and $\bar{a}$ are entire functions of λ .

Considering the integral

$$\int_C \frac{d\lambda'}{\lambda' - \lambda} \frac{1}{a(\lambda')} \begin{pmatrix} \phi_1(\lambda') \\ \phi_2(\lambda') \end{pmatrix} \exp(i\lambda' q_0 x), \tag{26}$$

where the contour C is defined to be the contour in the complex λ' plane, starting from $\lambda' = -\infty + i0^+$, passing over all zeros of $a(\lambda')$ and ending at $\lambda' = +\infty + i0^+$ for λ below C , we shall derive the Gel'fand-Levitan equations. Since there are no dependent relations between the upper half plane and the lower half plane of the Jost function, we must also consider the integral

$$\int_{C'} \frac{d\lambda'}{\lambda' - \lambda} \frac{1}{\bar{a}(\lambda')} \begin{pmatrix} \bar{\phi}_1(\lambda') \\ \bar{\phi}_2(\lambda') \end{pmatrix} \exp(-i\lambda' q_0 x), \tag{27}$$

where the contour C' is defined to be the contour in the complex λ' plane, starting from $\lambda' = -\infty + i0^-$, passing over all zeros of $\bar{a}(\lambda')$ and ending at $\lambda' = +\infty + i0^-$ for λ above C' . To obtain the Gel'fand-Levitan equations, we must introduce two sets of the kernels $K_1(x, z)$, $K_2(x, z)$ and $\widehat{K}_1(x, z)$, $\widehat{K}_2(x, z)$ as

$$\begin{aligned}
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(i\lambda q_0 x - \hat{\mu}_+(x)) + \int_x^\infty \begin{pmatrix} K_1(x, z) \\ i\lambda K_2(x, z) \end{pmatrix} \exp(i\lambda q_0 z - \hat{\mu}_+(x)) dz, \\
\begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\lambda q_0 x - \mu_+(x)) + \int_x^\infty \begin{pmatrix} i\lambda \widehat{K}_1(x, z) \\ \widehat{K}_2(x, z) \end{pmatrix} \exp(-i\lambda q_0 z - \mu_+(x)) dz,
\end{aligned} \tag{28}$$

where we assume

$$\begin{aligned}
\lim_{z \rightarrow \infty} K_1(x, z) &= 0, \\
\lim_{z \rightarrow \infty} \frac{\partial K_1(x, z)}{\partial z} &= 0, \\
\lim_{z \rightarrow \infty} K_2(x, z) &= 0, \\
\lim_{z \rightarrow \infty} \widehat{K}_1(x, z) &= 0, \\
\lim_{z \rightarrow \infty} \frac{\partial \widehat{K}_1(x, z)}{\partial z} &= 0, \\
\lim_{z \rightarrow \infty} \widehat{K}_2(x, z) &= 0.
\end{aligned} \tag{29}$$

Solutions are expressed with the kernels by

$$q_x = q_0 \frac{q_0^2 - K_1(x, x) \widehat{K}_2(x, x)}{q_0^2 + K_1(x, x) \widehat{K}_2(x, x)}, \tag{30a}$$

$$r_x = \frac{2q_0^2 K_1(x, x)}{q_0^2 + K_1(x, x) \widehat{K}_2(x, x)}, \tag{30b}$$

$$s_x = \frac{2q_0^2 \widehat{K}_2(x, x)}{q_0^2 + K_1(x, x) \widehat{K}_2(x, x)}. \tag{30c}$$

The Gel'fand-Levitan equations are obtained by

$$\begin{aligned}
G_1(x, y) + q_0^2 \widehat{F}(x + y) + q_0^2 \int_x^\infty \widehat{G}_1(x, z) \widehat{F}(x + z) dz &= 0, \\
G_2(x, y) + \int_x^\infty \widehat{G}_2(x, z) \widehat{F}''(x + z) dz &= 0, \\
\widehat{G}_1(x, y) + \int_x^\infty G_1(x, z) F''(x + z) dz &= 0, \\
\widehat{G}_2(x, y) + q_0^2 F(x + y) + q_0^2 \int_x^\infty G_2(x, z) F(x + z) dz &= 0,
\end{aligned} \tag{31}$$

where $G_1, G_2, \widehat{G}_1$ and $\widehat{G}_2$ are defined as

$$\begin{aligned}
G_1(x, z) &= K_1(x, z) \exp(\mu_+(x) - \widehat{\mu}_+(x)), \\
G_2(x, z) &= K_2(x, z), \\
\widehat{G}_1(x, z) &= \widehat{K}_1(x, z), \\
\widehat{G}_2(x, z) &= \widehat{K}_2(x, z) \exp(\widehat{\mu}_+(x) - \mu_+(x)).
\end{aligned} \tag{32}$$

$F(z)$ and $\hat{F}(z)$ are given by

$$\begin{aligned}
F(z) &= \frac{1}{2\pi} \int_C \frac{d\lambda}{i\lambda q_0} \frac{b(\lambda)}{a(\lambda)} \exp(i\lambda q_0 z), \\
F''(z) &= \frac{d^2 F(z)}{dz^2} = \frac{q_0}{2\pi} \int_C i\lambda d\lambda \frac{b(\lambda)}{a(\lambda)} \exp(i\lambda q_0 z), \\
\hat{F}(z) &= -\frac{1}{2\pi} \int_{C'} \frac{d\lambda}{i\lambda q_0} \frac{\bar{b}(\lambda)}{\bar{a}(\lambda)} \exp(-i\lambda q_0 z), \\
\hat{F}''(z) &= \frac{d^2 \hat{F}(z)}{dz^2} = -\frac{q_0}{2\pi} \int_{C'} i\lambda d\lambda \frac{\bar{b}(\lambda)}{\bar{a}(\lambda)} \exp(-i\lambda q_0 z).
\end{aligned} \tag{33}$$

Time dependence of the scattering data is found from (8) to be

$$\begin{aligned}
a(\lambda, t) &= a(\lambda, 0), \\
b(\lambda, t) &= b(\lambda, 0) \exp\left(-\frac{i}{\lambda} t\right), \\
\bar{a}(\lambda, t) &= \bar{a}(\lambda, 0), \\
\bar{b}(\lambda, t) &= \bar{b}(\lambda, 0) \exp\left(\frac{i}{\lambda} t\right).
\end{aligned} \tag{34}$$

The bound states are given by the zeros of $a(\lambda)$ in the upper half plane and the zeros of $\bar{a}(\lambda)$ in the lower half plane. It should be noted that, since there are no dependent relations between the upper and the lower half plane of the Jost functions, the number of the zeros in the upper half plane does not restrict that of the zeros in the lower half plane. When all the zeros are simple $F(z)$ and $\hat{F}(z)$ are expressed by

$$\begin{aligned}
F(z) &= \sum_{k=1}^N \frac{iC_k(t)}{\lambda_k q_0} \exp(\lambda_k q_0 z) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\rho(\lambda, t)}{\lambda q_0} \exp(i\lambda q_0 z), \\
\hat{F}(z) &= \sum_{l=1}^M -\frac{i\hat{C}_l(t)}{\lambda_l q_0} \exp(-\lambda_l q_0 z) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{\rho}(\lambda, t)}{\lambda q_0} \exp(-i\lambda q_0 z),
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
C_k(t) &= C_k(0) \exp\left(-\frac{i}{\lambda_k}t\right), \\
\hat{C}_l(t) &= \hat{C}_l(0) \exp\left(\frac{i}{\lambda_l}t\right), \\
\rho(\lambda, t) &= \rho(\lambda, 0) \exp\left(-\frac{i}{\lambda}t\right), \\
\hat{\rho}(\lambda, t) &= \hat{\rho}(\lambda, 0) \exp\left(\frac{i}{\lambda}t\right).
\end{aligned} \tag{36}$$

With $G_1(x, x)$ and $\hat{G}_2(x, x)$, $K_1(x, x)$ and $\hat{K}_2(x, x)$ are expressed by

$$\begin{aligned}
K_1 &= K_0 \exp\left\{-\int_x^\infty \left(\frac{q_0^2 \hat{G}_{1y} - G_1^2 \hat{G}_{2y}}{G_1(q_0^2 - G_1 \hat{G}_2)}\right) dy\right\}, \\
\hat{K}_2 &= \hat{K}_0 \exp\left\{-\int_x^\infty \left(\frac{q_0^2 \hat{G}_{2y} - G_{1y} \hat{G}_2^2}{\hat{G}_2(q_0^2 - G_1 \hat{G}_2)}\right) dy\right\}.
\end{aligned} \tag{37}$$

with $K_0 \hat{K}_0 = q_0^2$.

Giving the scattering data $\{\rho(\lambda, 0), \hat{\rho}(\lambda, 0), \lambda; C_k(0), \lambda_k, k = 1, 2, \dots, N, \hat{C}_l, \lambda_l, l = 1, 2, \dots, M\}$, we can determine $F(z)$ and $\hat{F}(z)$ and $K_1(x, x)$ and $\hat{K}_2(x, x)$ through $G_1(x, x)$ and $\hat{G}_2(x, x)$ from the Gel'fand-Levitan equations. We then obtain the solutions by using the relations (30).

3 One Soliton Solution

One soliton solution is obtained under the conditions:

1. $\rho(\lambda, t) = 0$,
2. $\hat{\rho}(\lambda, t) = 0$,
3. one simple zero of $a(\lambda)$ is pure imaginary $\lambda = i\eta$ in the upper half plane,
4. one simple zero of $\bar{a}(\lambda)$ is pure imaginary $\lambda = -i\xi$ in the lower half plane.

Then $F(z)$ and $\hat{F}(z)$ reduce to

$$\begin{aligned} F(z) &= \frac{C(t)}{\eta q_0} \exp(-\eta q_0 z), \\ \hat{F}(z) &= \frac{\hat{C}(t)}{\xi q_0} \exp(-\xi q_0 z), \end{aligned} \tag{38}$$

where

$$\begin{aligned} C(t) &= C(0) \exp\left(-\frac{t}{\eta}\right), \\ \hat{C}(t) &= \hat{C}(0) \exp\left(-\frac{t}{\xi}\right). \end{aligned} \tag{39}$$

In order to solve the Gel'fand-Levitan equations, we introduce the representations:

$$\begin{aligned} G_1(x, z) &= A(x) \exp(-\xi q_0 x - \xi q_0 z), \\ G_2(x, z) &= B(x) \exp(-\eta q_0 x - \xi q_0 z), \\ \hat{G}_1(x, z) &= \hat{A}(x) \exp(-\xi q_0 x - \eta q_0 z), \\ \hat{G}_2(x, z) &= \hat{B}(x) \exp(-\eta q_0 x - \eta q_0 z). \end{aligned} \tag{40}$$

Substituting (40) into (31), we solve $A(x)$, $B(x)$, $\hat{A}(x)$, $\hat{B}(x)$ and give explicit form of $G_1(x, x)$ and $\hat{G}_2(x, x)$ as

$$\begin{aligned} G_1(x, x) &= -\frac{q_0 \frac{\hat{C}(t)}{\xi} \exp(-2\xi q_0 x)}{1 - \frac{\eta C(t) \hat{C}(t)}{\xi(\eta + \xi)^2} \exp(-2(\eta + \xi)q_0 x)}, \\ \hat{G}_2(x, x) &= -\frac{q_0 \frac{\hat{C}(t)}{\eta} \exp(-2\eta q_0 x)}{1 - \frac{\xi C(t) \hat{C}(t)}{\eta(\eta + \xi)^2} \exp(-2(\eta + \xi)q_0 x)}. \end{aligned} \tag{41}$$

From (37), $K_1(x, x)$ and $\widehat{K}_2(x, x)$ are given as

$$\begin{aligned} K_1(x, x) &= \frac{K_0}{\xi} \frac{\widehat{C}(t) \exp(-2\xi q_0 x)}{1 - \frac{\xi C(t) \widehat{C}(t)}{\eta(\eta + \xi)^2} \exp(-2(\eta + \xi) q_0 x)}, \\ \widehat{K}_2(x, x) &= \frac{\widehat{K}_0}{\eta} \frac{\widehat{C}(t) \exp(-2\eta q_0 x)}{1 - \frac{\eta C(t) \widehat{C}(t)}{\xi(\eta + \xi)^2} \exp(-2(\eta + \xi) q_0 x)}. \end{aligned} \quad (42)$$

One soliton solution is obtained by

$$\begin{aligned} q_x &= q_0 \left(1 - 2 \frac{\frac{C(t) \widehat{C}(t)}{\eta \xi} \exp(-2(\eta + \xi) q_0 x)}{T^2} \right), \\ r &= -\frac{K_0}{q_0} \frac{\frac{\widehat{C}(t)}{\xi^2} \exp(-2\xi q_0 x)}{T}, \\ s &= -\frac{\widehat{K}_0}{q_0} \frac{\frac{C(t)}{\eta^2} \exp(-2\eta q_0 x)}{T}. \end{aligned} \quad (43)$$

where

$$T = 1 + \frac{C(t) \widehat{C}(t)}{(\eta + \xi)^2} \exp(-2(\eta + \xi) q_0 x). \quad (44)$$

Profile of the R and S solitons is not symmetric with respect to their peak position in contrast with that of the Q_x soliton. Velocities of three kinds of soliton are the same

$$V = -\frac{1}{2\eta\xi q_0}. \quad (45)$$

At a given time, the maximum or the minimum positions of the Q_x , R and S solitons are not coincident. As these solitons evolve, however, the distance between the peaks of the

R and S solitons and the minimum point of the Q_x soliton remarkably remains the same throughout the motion in such that their trajectories are given by

$$\begin{aligned}
Q_x \text{ soliton} \quad & 2(\eta + \xi)q_0 x + \left(\frac{1}{\eta} + \frac{1}{\xi}\right) t + \ln(\eta + \xi)^2 = 0, \\
R \text{ soliton} \quad & 2(\eta + \xi)q_0 x + \left(\frac{1}{\eta} + \frac{1}{\xi}\right) t + \ln \frac{\xi}{\eta}(\eta + \xi)^2 = 0, \\
S \text{ soliton} \quad & 2(\eta + \xi)q_0 x + \left(\frac{1}{\eta} + \frac{1}{\xi}\right) t + \ln \frac{\eta}{\xi}(\eta + \xi)^2 = 0.
\end{aligned} \tag{46}$$

We illustrate in Fig.1 Q_x , R and S solitons with $\eta = 0.5$ and $\xi = 1.5$ where we can see that profiles of R and S solitons are asymmetric with respect to their peak position and that positions of the maximum or minimum amplitudes of Q_x , R and S solitons are not coincident. Peak height of the solitons is given by

$$\begin{aligned}
Q_x \text{ soliton} \quad & -\frac{\eta^2 + \xi^2}{2\eta\xi}, \\
R \text{ soliton} \quad & -\frac{\widehat{K}_0}{2\xi^2} \left[\frac{\xi}{\eta}(\eta + \xi)^2 \right]^{\frac{\eta}{\eta+\xi}} \exp\{-2(\xi - \eta)q_0 x_0\}, \\
S \text{ soliton} \quad & -\frac{K_0}{2\eta^2} \left[\frac{\eta}{\xi}(\eta + \xi)^2 \right]^{\frac{\xi}{\eta+\xi}} \exp\{2(\xi - \eta)q_0 x_0\}.
\end{aligned} \tag{47}$$

By depending on relative magnitude of the eigenvalues, amplitude of the R (or S) soliton increases and that of the S (or R) soliton decreases as time evolves. In Figs.2 and 3 we show time evolution of the R and S solitons with $\eta = 1$ and $\xi = 1.5$ at $t = -5, -2$ and 1 . If we take $\widehat{C}_0 = 0$, shape of R soliton is of an exponential type and Q_x soliton is constant and S soliton vanishes.

Taking $\eta = \xi$, we can reproduce the results of the case of $r = s^{(1)}$ where R soliton as well as Q_x soliton are stationary.

4 Two Soliton Solution

Two soliton solution is obtained under the conditions of $\rho(\lambda, t) = 0$, $\widehat{\rho}(\lambda, t) = 0$, two simple zeros of $a(\lambda)$ of pure imaginary $i\eta_1$ and $i\eta_2$ in the upper half plane and two simple

zeros of $\hat{a}(\lambda)$ of pure imaginary $-i\xi_1$ and $-i\xi_2$ in the lower half plane. Then $F(z)$ and $\hat{F}(z)$ become

$$\begin{aligned} F(z) &= \frac{C_1(t)}{\eta_1 q_0} \exp(-\eta_1 q_0 z) + \frac{C_2(t)}{\eta_2 q_0} \exp(-\eta_2 q_0 z), \\ \hat{F}(z) &= \frac{\hat{C}_1(t)}{\xi_1 q_0} \exp(-\xi_1 q_0 z) + \frac{\hat{C}_2(t)}{\xi_2 q_0} \exp(-\xi_2 q_0 z), \end{aligned} \quad (48)$$

where

$$\begin{aligned} C_1(t) &= C_1(0) \exp\left(-\frac{t}{\eta_1}\right), \\ C_2(t) &= C_2(0) \exp\left(-\frac{t}{\eta_2}\right), \\ \hat{C}_1(t) &= \hat{C}_1(0) \exp\left(-\frac{t}{\xi_1}\right), \\ \hat{C}_2(t) &= \hat{C}_2(0) \exp\left(-\frac{t}{\xi_2}\right). \end{aligned} \quad (49)$$

We introduce representations:

$$\begin{aligned} G_1(x, z) &= A_1(x) \exp(-\xi_1 q_0 x - \xi_1 q_0 z) + A_2(x) \exp(-\xi_2 q_0 x - \xi_2 q_0 z), \\ G_2(x, z) &= B_1(x) \exp(-\eta_1 q_0 x - \eta_1 q_0 z) + B_2(x) \exp(-\eta_2 q_0 x - \eta_2 q_0 z), \\ \hat{G}_1(x, z) &= \hat{A}_1(x) \exp(-\xi_1 q_0 x - \eta_1 q_0 z) + \hat{A}_2(x) \exp(-\xi_2 q_0 x - \eta_2 q_0 z), \\ \hat{G}_2(x, z) &= \hat{B}_1(x) \exp(-\eta_1 q_0 x - \xi_1 q_0 z) + \hat{B}_2(x) \exp(-\eta_2 q_0 x - \xi_2 q_0 z). \end{aligned} \quad (50)$$

Solving the Gelfand-Levitan equations and integrate (37) with (50), we obtain two soliton solution

$$\begin{aligned} q_x &= q_0 \left(1 - 2 \frac{U_1 U_2}{T^2}\right), \\ r &= -\frac{K_0}{q_0} \frac{W_1}{T}, \\ s &= -\frac{\hat{K}_0}{q_0} \frac{W_2}{T}. \end{aligned} \quad (51)$$

where

$$\begin{aligned}
T &= 1 + \frac{X_1 Y_1}{(\eta_1 + \xi_1)^2} + \frac{X_1 Y_2}{(\eta_1 + \xi_2)^2} + \frac{X_2 Y_1}{(\eta_2 + \xi_1)^2} + \frac{X_2 Y_2}{(\eta_2 + \xi_2)^2} \\
&\quad + \left[\frac{(\eta_1 - \eta_2)(\xi_1 - \xi_2)}{(\eta_1 + \xi_1)(\eta_1 + \xi_2)(\eta_2 + \xi_1)(\eta_2 + \xi_2)} \right]^2 X_1 X_2 Y_1 Y_2, \\
U_1 &= \frac{Y_1}{\xi_1} + \frac{Y_2}{\xi_2} - \frac{\eta_1(\xi_1 - \xi_2)^2}{\xi_1 \xi_2 (\eta_1 + \xi_1)^2 (\eta_1 + \xi_2)^2} X_1 Y_1 Y_2 - \frac{\eta_2(\xi_1 - \xi_2)^2}{\xi_1 \xi_2 (\eta_2 + \xi_1)^2 (\eta_2 + \xi_2)^2} X_2 Y_1 Y_2, \\
U_2 &= \frac{X_1}{\eta_1} + \frac{X_2}{\eta_2} - \frac{\xi_1(\eta_1 - \eta_2)^2}{\eta_1 \eta_2 (\eta_1 + \xi_1)^2 (\eta_2 + \xi_1)^2} X_1 X_2 Y_1 - \frac{\xi_2(\eta_1 - \eta_2)^2}{\eta_1 \eta_2 (\eta_1 + \xi_2)^2 (\eta_2 + \xi_2)^2} X_1 X_2 Y_2, \\
W_1 &= \frac{Y_1}{\xi_1^2} + \frac{Y_2}{\xi_2^2} + \frac{\eta_1^2(\xi_1 - \xi_2)^2}{\xi_1^2 \xi_2^2 (\eta_1 + \xi_1)^2 (\eta_1 + \xi_2)^2} X_1 Y_1 Y_2 + \frac{\eta_2^2(\xi_1 - \xi_2)^2}{\xi_1^2 \xi_2^2 (\eta_2 + \xi_1)^2 (\eta_2 + \xi_2)^2} X_2 Y_1 Y_2, \\
W_2 &= \frac{X_1}{\eta_1^2} + \frac{X_2}{\eta_2^2} + \frac{\xi_1^2(\eta_1 - \eta_2)^2}{\eta_1^2 \eta_2^2 (\eta_1 + \xi_1)^2 (\eta_2 + \xi_1)^2} X_1 X_2 Y_1 + \frac{\xi_2^2(\eta_1 - \eta_2)^2}{\eta_1^2 \eta_2^2 (\eta_1 + \xi_2)^2 (\eta_2 + \xi_2)^2} X_1 X_2 Y_2,
\end{aligned} \tag{52}$$

where

$$\begin{aligned}
X_1(t) &= C_1(t) \exp(-2\eta_1 q_0 x), \\
X_2(t) &= C_2(t) \exp(-2\eta_2 q_0 x), \\
Y_1(t) &= \hat{C}_1(t) \exp(-2\xi_1 q_0 x), \\
Y_2(t) &= \hat{C}_2(t) \exp(-2\xi_2 q_0 x).
\end{aligned} \tag{53}$$

Here we concentrate our consideration only R solitons because we can image Q_x solitons and S solitons easily. In Fig.4 we illustrate two R solitons which decrease as time evolves and seem to interact each other as like as the KdV solitons with the nearly same eigenvalues. Next we show profiles of two R solitons, one of which increases its amplitude and another decreases its amplitudes as time evolves in Fig.5. We can observe up and down motion between two solitons during two solitons exchanging their positions. Lastly, we illustrate in Fig.6 a special case of two R solitons, which consists of an exponential type of soliton ($C_{20} = 0$) and an ordinary soliton propagating negative direction. We observe that the exponential type of soliton remains in the left hand of the ordinary soliton even after collision occurs.

5 Concluding Remarks

We have discussed a set of integrable, dispersionless equations (2). We have solved it by using the inverse scattering method and found new solitonic phenomena both in one soliton solution and two soliton solution. Remarkable fact is that the shape of the R and S solitons is asymmetric with respect to their peak position and that the number of R solitons and S solitons is not necessarily the same.

Extending simply our system in such a way as

$$\begin{aligned} q_{xt} + \left(\sum_{i=1}^N \rho_i^2 - \sum_{j=1}^M \eta_j^2 \right)_x &= 0, \\ \rho_{ixt} - 2q_x \rho_i &= 0, \quad (i = 1, 2, \dots, N), \\ \eta_{jxt} - 2q_x \eta_j &= 0, \quad (j = 1, 2, \dots, M). \end{aligned} \tag{54}$$

which has a conserved quantity

$$q_x^2 + \sum_{i=1}^N \rho_{ix}^2 - \sum_{j=1}^M \eta_{jx}^2 = q_0^2. \tag{55}$$

(55) is related to $SO(N+1, M)$. For the cases of $N = 1$ and $M = 0$, $N = 2$ and $M = 0$, and $N = 1$ and $M = 1$ the systems are integrable. Are general $SO(N+1, M)$ cases also integrable?

We shall discuss relationship our system and equivalent systems considering the symmetry $O(2, 1) \sim SL(2, R)$ in a separate paper.

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Figure Captions

Fig.1 Profile of Q_x , R and S solitons with $\eta = 0.5$, $\xi = 1.5$, $K_0 = -1$, $\widehat{K}_0 = -1$ and $q_0 = 1$ at $t = 0.5$.

Fig.2 Time evolution of R soliton propagating negative direction with $\eta = 1$, $\xi = 1.5$, $K_0 = -1$ and $q_0 = 1$ at $t = -5$, -2 and 1 .

Fig.3 Time evolution of S soliton propagating negative direction with $\eta = 1$, $\xi = 1.5$, $\widehat{K}_0 = -1$ and $q_0 = 1$ at $t = -5$, -2 and 1 .

Figs.4 Time evolution of two R solitons propagating negative direction with $\eta_1 = 1$, $\eta_2 = 1.5$, $\xi_1 = 0.75$, $\xi_2 = 1.25$, $K_0 = -1$, $\widehat{K}_0 = -1$ and $q_0 = 1$ at $t = -5$, 0 , 5 and 10 .

Figs.5 Time evolution of two R solitons propagating negative direction with $\eta_1 = 0.5$, $\eta_2 = 1.25$, $\xi_1 = 1$, $\xi_2 = 0.75$, $K_0 = -1$, $\widehat{K}_0 = -1$ and $q_0 = 1$ at $t = -4$, -2 , -0 and 2 .

Figs.6 Time evolution of two R solitons which consists of an exponential type of soliton and an ordinary soliton propagating negative direction with $C_{20} = 0$, $\eta_1 = 0.5$, $\eta_2 = 1.25$, $\xi_1 = 1$, $\xi_2 = 0.75$, $K_0 = -1$, $\widehat{K}_0 = -1$ and $q_0 = 1$ at $t = -10$, -0 and 10 .

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.

Fig. 6.