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Lagrangian, Hamiltonian and Conserved Quantities
for
Coupled Integrable, Dispersionless Equations

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abstract

Starting from the Lagrangian of coupled integrable, dispersionless equations, we derive the Hamiltonian. Conserved quantities associated with the symmetries of the Lagrangian are obtained, which cannot be derive from the inverse scattering method of the equations.

Key Words: Integrable equation, dispersionless equation, Lagrangian, Hamiltonian, Noether theorem

Recently, present authors¹⁾ proposed general coupled integrable, dispersionless equations:

$$q_{xt} - (rs)_x = 0$$

$$r_{xt} + 2 q_x r = 0 \tag{1}$$

$$s_{xt} + 2 q_x s = 0$$

and solved by the inverse scattering method. This system includes the special cases of $s = r$ given by Konno-Oono²⁾ and $s = r^*$ by Konno³⁾ and Kotlyarov⁴⁾ and these cases are equivalent to the sine-Gordon equation⁵⁾⁶⁾ and the Pohlmeyer-Lund-Regge system⁴⁾, respectively. Kotlyarov pointed out that these equivalence suggests that both cases are of Hamiltonian integrable system. It is natural that eqs. (1) are also of Hamiltonian system. In this short note, we give an explicit Hamiltonian for eqs. (1) and derive conserved quantities through Noether theorem.

Here, we start from the Lagrangian:

$$L = \int dx [1/4 r_t s_x + 1/4 s_t r_x + 1/2 u_t u_x + (u_0 - u_x)rs], \tag{2}$$

where u_0 is a constant. The variational principle leads us the Euler-Lagrange equations,

$$u_{xt} + (rs)_x = 0$$

$$r_{xt} + 2(u_0 - u_x)r = 0 \tag{3}$$

$$s_{xt} + 2(u_0 - u_x)s = 0.$$

Letting $q_x = u_0 - u_x$ with the boundary conditions:

$$u_x \rightarrow 0, r \rightarrow 0 \text{ and } s \rightarrow 0 \quad (4)$$

for $|x| \rightarrow \infty$, eqs. (3) are just eqs. (1). The canonical momenta are defined by

$$\pi_r \equiv \partial L / \partial r_t = 1/4 s_x$$

$$\pi_s \equiv \partial L / \partial s_t = 1/4 r_x \quad (5)$$

$$\pi_u \equiv \partial L / \partial u_t = 1/2 u_x.$$

These momenta are not independent on the fields r , s and u , because of satisfying such constraints of the second class⁷⁾ as,

$$\chi_1 = \pi_r - 1/4 s_x = 0$$

$$\chi_2 = \pi_s - 1/4 r_x = 0 \quad (6)$$

$$\chi_3 = \pi_u - 1/2 u_x = 0.$$

For such systems, it is convenient to define the Dirac bracket

$$\{F(x), G(y)\}_{D.B.} \equiv \{F(x), G(y)\}_{P.B.} - \int dz dz' \{F(x), \chi_i(z)\}_{P.B.} \Delta_{ij}^{-1}(z - z') \{\chi_j(z'), G(y)\}_{P.B.} \quad (7)$$

where $\{, \}_{P.B.}$ stands for usual Poisson bracket and $\Delta_{ij}^{-1}(x - y)$ is given by

$$\Delta_{ij}^{-1}(x - y) = \begin{pmatrix} 1/2 \operatorname{sgn}(x - y) & 0 & 0 \\ 0 & 0 & \operatorname{sgn}(x - y) \\ 0 & \operatorname{sgn}(x - y) & 0 \end{pmatrix}, \quad (8)$$

as an inverse of matrix $\Delta_{ij}(x - y) = \{\chi_i(x), \chi_j(y)\}_{P.B.}$ and $\operatorname{sgn}(x)$

is the sign function. Now, we can calculate the fundamental Dirac brackets,

$$\{r(x), s(y)\}_{D.B.} = \text{sgn}(x - y) \quad (9)$$

$$\{u(x), u(y)\}_{D.B.} = 1/2 \text{sgn}(x - y).$$

The Dirac bracket of other combinations of the fields vanishes. The Hamiltonian is given by,

$$H = - \int dx (u_0 - u_x) r s, \quad (10)$$

via the Legendre transformation and reflects invariance of the Lagrangian (2) under the time displacement. The Hamilton equations can be written by the Dirac bracket in the form:

$$\phi_{xt} = \{H, \phi_x\}_{D.B.} \quad (11)$$

where ϕ stands for r , s and u , which coincides with equations (2).

Since the Lagrangian (2) is invariant under the time displacement $t \rightarrow t + \varepsilon_t$ and space displacement $x \rightarrow x + \varepsilon_x$ with constants ε_t and ε_x , we obtain the Hamiltonian (10) and the momentum:

$$P = 1/2 \int dx (u_x^2 + r_x s_x), \quad (12)$$

respectively. The Lagrangian (2) is also invariant under the translation of the field u :

$$u \rightarrow u + \varepsilon_u \quad (13)$$

and the scale transformation:

$$\begin{aligned} r &\rightarrow \theta r, \\ s &\rightarrow s/\theta \end{aligned} \quad (14)$$

where ϵ_u and θ are constants, and the scale transformation is the phase transformation in the case of $s = r^*$. According to the Noether theorem, we obtain conserved quantities:

$$Q_T = \int dx \ u_X \quad (15)$$

i.e. the topological charge from the former transformation (13), for which the quantity (15) with the Noether theorem can be derived from the first equation of eqs. (3), and

$$Q_S = \int dx \ (1/4 \ s_X \ r - 1/4 \ r_X \ s), \quad (16)$$

from the latter one (14). The conserved quantities from the inverse scattering method does not include eqs. (10), (12), (15) and (16).

The inverse scattering scheme leads to conserved quantity

$$\sigma_{-1} = (u_0 - u_X)^2 + r_X s_X = u_0^2. \quad (17)$$

On the other hand, the momentum is given by eq. (12). Combining eqs. (12), (15) and (17), we find a relation,

$$u_0 \ Q_T = P. \quad (18)$$

Therefore, Q_T is non-vanishing, that is, solitons must be topological.

Finally, we make a comment on boundary conditions. Since conserved quantity σ_{-1} can be written

$$\sigma_{-1} = (u_0 - u_X)^2 + 1/4 \ (r_X + s_X)^2 - 1/4 \ (r_X - s_X)^2, \quad (19)$$

σ_{-1} may take not only positive value but also negative or zero, for real fields r , s and u . This suggests that there exists other boundary conditions different from eqs. (4) for eqs. (1).

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