Approximate spectral functions in thermal field theory

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Abstract

Causality requires, that the (anti-) commutator of two interacting field operators vanishes for space-like coordinate differences. This implies, that the Fourier transform of the spectral function of this quantum field should vanish in the space-like domain. We find, that this requirement is violated by the gauge boson propagators commonly used in high temperature gauge theory.
In the past decade the interest in quantum field theory at nonzero temperature has grown considerably [1–3]. In part this is due to experimental and theoretical efforts to understand various hot quantum systems, like e.g. ultrarelativistic heavy-ion collisions or the early universe.

A particularly interesting problem when investigating such systems is the question of their single particle spectrum. Although for an interacting quantum field this spectrum in general has a very rich structure, we may also count calculations of particle masses and spectral width parameters (i.e., damping rates) in this category. Consequently we find, that a large number of research papers is dealing with the question of the single particle spectrum on an approximate level.

With the present paper we address the question, whether the approximations used in many of these papers are consistent with basic requirements of quantum field theory. To this end we investigate some common approximations made to the quantity which summarizes the spectral properties of a quantum field, i.e., its spectral function \(\mathcal{A}(E, p)\). Up to a factor this function is the imaginary part of the full retarded two-point function, propagating an excitation with energy \(E\) and momentum \(p\). We show that a common model for spectral functions in “hot gauge theory” indeed violates one of the most stringent requirements of quantum field theory.

The paper is organized as follows: First we present a brief introduction dealing with free boson and fermion quantum fields. We then investigate the properties of simple approximate spectral functions for interacting boson and fermion fields, and finally we turn to the physical problem of gauge theory at finite temperature.

The spectral function has two features which are intimately related to fundamental requirements of quantum field theory. To begin with, the quantization rules for boson and fermion fields (which we will use in the free as well as in the interacting case),

\[
\begin{align*}
[\phi(t, x), \partial_t \phi(t, y)] &= i\delta^3(x - y), \\
\{\psi(t, x), \psi^\dagger(t, y)\} &= \delta^3(x - y),
\end{align*}
\]

require that the spectral function is normalized. Although the normalization may be difficult to achieve numerically [4], we do not consider this a serious principal problem.

The second important feature of the spectral function is that its four dimensional Fourier transform into coordinate space must vanish for space-like arguments. This is equivalent to the Wightman axiom of locality, i.e., field operators must (anti-)commute for space-like separations in Minkowski space [5].

In an interacting many-body system, we may very well expect non-locality in a causal sense: Wiggling the system at one side will certainly influence the other side after some time. The locality axiom ensures that this influence does not occur over space-like separations, i.e., faster than a physical signal can propagate. Thus, to distinguish between the causal non-locality and the violation of the locality axiom, we will henceforth denote the latter a violation of causality.

In the following we will furthermore distinguish fermionic and bosonic quantities by a lower index. For two field operators the locality axiom then amounts to the requirement, that the (anti-) commutator function of two field operators and also its expectation value fulfills...
In terms of the spectral function, these expectation values are

\[ C_B(x, y) = \langle [\phi(x), \phi(y)] \rangle = 0 \text{ if } (x - y) \text{ space-like}. \tag{2} \]
\[ C_F(x, y) = \langle \{ \psi(x), \psi^\dagger(y) \} \rangle \]

We first consider the case of free quantum fields, for completeness we quote the free spectral functions found in any textbook on field theory:

\[ A^0_B(E, k) = \text{sign}(E) \delta(E^2 - k^2 - m_B^2) \]
\[ A^0_F(E, p) = \left( E\gamma^0 + p\gamma + m_F \right) \text{sign}(E) \delta(E^2 - p^2 - m_F^2). \tag{4} \]

For these spectral functions one obtains as the (anti-) commutator expectation value \( C_B(x, y) = C_0(x - y) \) and \( C_F(x, y) = (i\gamma^0 \partial_{x^0} + m_F) C_0(x - y) \), where

\[ C_0(x) = \frac{-i}{4\pi} \left( \delta(x^2 - \bar{x}^2) - \Theta(x^2 - \bar{x}^2) \frac{m_{B,F}}{2\sqrt{x_0^2 - \bar{x}^2}} J_1(m_{B,F}\sqrt{x_0^2 - \bar{x}^2}) \right). \tag{5} \]

Clearly this is zero for space-like arguments, i.e., for \( \bar{x} = |\mathbf{x}| > |x_0| \). This free commutator function has support only in the unshaded area of Fig. 1, and it is singular at its boundaries (but zero outside).

We now turn to nontrivial spectral functions, which are more appropriate for a thermal system. The physical reason is, that at finite temperature particles are subject to collisions,
hence their state of motion will change after a certain time. In a hot quantum system therefore the off-shell propagation of particles plays an important role. This off-shellness is contained in a continuous spectral function, which must not have an isolated $\delta$-function like pole. In principle this means that at nonzero temperature every quantum system must be described on the same footing as a gas of resonances. However, this does not imply that thermal particles may decay - they are merely scattered thermally by the other components of the system.

Apart from this physically motivated use of continuous spectral functions at finite temperature, one may also adopt a mathematically rigorous stance. We do not elaborate on this, but rather quote the Narnhofer-Thirring theorem [6]. It states, that interacting systems at finite temperature cannot be described by particles with a sharp dispersion law, only non-interacting “hot” systems may have a $\delta$-like spectral function. Ignoring this mathematical fact one finds as an echo serious infrared divergences in high temperature perturbative quantum chromodynamics (QCD). Consequently, these unphysical singularities are naturally removed within an approach of finite temperature field theory with continuous mass spectrum [7,8].

Thus, for a mathematical as well as a physical reason, finite temperature spectral functions are more complicated than those given in eq. (4). The question then arises, how much more complicated they have to be in order to be consistent with the requirements we have discussed above: Fully self consistent calculations of the corresponding spectral functions are very rare due to the numerical difficulties involved [8-10]. More often one uses an ansatz for such a function which involves only a small number of parameters which are then determined in a more or less “self”-consistent scheme.

As an example we consider two seemingly simplistic generalizations of the spectral functions in eq. (4), which involve only one additional parameter:

\[
A^1_B(E, k) = \frac{1}{\pi} \frac{2E\gamma_B}{(E^2 - k^2 - m_B^2 - \gamma_B^2)^2 + 4E^2\gamma_B^2},
\]

\[
A^1_F(E, p) = \frac{\gamma_F}{\pi} \frac{\omega(p)\gamma^0 + \omega(p)\gamma^0 + 2E\gamma^0 + 2Em_F}{(E - \omega(p))^2 + 4\gamma_F^2}
\]

\[
= \frac{1}{4\pi i\omega(p)} \left( \frac{\omega(p)\gamma^0 + \omega(p)\gamma^0 + \omega(p)\gamma^0 + \omega(p)\gamma^0 + m}{E - \omega(p) - i\gamma_F} - \frac{-\omega(p)\gamma^0 + \omega(p)\gamma^0 + m}{E + \omega(p) - i\gamma_F} \right)
\]

\[
- \frac{\omega(p)\gamma^0 + \omega(p)\gamma^0 + m}{E - \omega(p) + i\gamma_F} + \frac{-\omega(p)\gamma^0 + \omega(p)\gamma^0 + m}{E + \omega(p) + i\gamma_F} \right) \tag{6}
\]

where $\omega(p)^2 = p^2 + m_F^2$. It is a matter of a few lines to show, that the (anti-) commutator functions for the quantum fields defined by these spectral functions are

\[
C_B^1(x, y) = e^{-\gamma_B |x_0 - y_0|} C_0(x - y)
\]

\[
C_F^1(x, y) = \left( i\gamma^0 \partial_{x^0} + m_F \right) e^{-\gamma_F |x_0 - y_0|} C_0(x - y). \tag{7}
\]
These simple generalizations of the free spectral function therefore have the important property to preserve causality: Their Fourier transform vanishes for space-like arguments, i.e., the quantum fields constructed with these spectral functions obey the locality axiom. The most general form of such spectral functions has been given in ref. [11].

We now turn to the subject of hot gauge theory, as discussed in the current literature [4,12,13]. Naturally we cannot possibly check all the existing calculations of spectral functions, and therefore restrict ourselves to the most basic picture obtained in high-temperature QED. Remarkably this exactly comprises the gauge boson spectral functions obtained in the hard thermal loop resummation scheme of QCD [4].

In this formalism, the full two-point function of a gauge boson quantum field propagates longitudinal as well as transverse degrees of freedom, with propagators

\[
\Delta_t(E, p) = \left[ E^2 - p^2 - q_D^2 \left( \frac{E^2}{2p^2} + \frac{E(p^2 - E^2)}{4p^2} \log \left( \frac{E + p}{E - p} \right) \right) \right]^{-1}
\]

\[
\Delta_t(E, p) = \left( \frac{p^2}{E^2 - p^2} \right) \left[ p^2 + q_D^2 \left( 1 - \frac{E}{2p} \log \left( \frac{E + p}{E - p} \right) \right) \right]^{-1}.
\]

Here \( q_D \) is the Debye screening “mass”, which is directly related to the plasma frequency. It sets the only scale inherent to these propagators.

Both of them have a continuous imaginary part (= spectral function) in the regime \( |E| < |p| \), as well as a \( \delta \)-function pole at some energy \( > |p| \), and they are combined to find the canonical components by means of transverse and longitudinal projection operators [12]. In particular, the canonical 33-component is
\[ \Delta_{33}(E, \vec{p}) = - \left( 1 - \frac{p_3 p_3}{\vec{p}^2} \right) \Delta_t(E, p) - \frac{E^2 p_3 p_3}{\vec{p}^2(E^2 - \vec{p}^2)} \Delta_t(E, p) + \text{gauge pieces} \]  

We will henceforth concentrate on the first (transverse) piece, since its Fourier transform is numerically easier to obtain: We are sure, that our calculation (which was done in several different ways) will be checked by other authors. However, the following results apply equally well to the second (longitudinal) part, and also to the combination of these to the 33-component of the gauge boson propagator.

The four dimensional Fourier transform is a linear functional of the imaginary part of the propagator. Thus, each contribution to the spectral function may be transformed separately, and their sum then constitutes the full Fourier transform.

In Fig. 2 we show the Fourier transform of the continuous part of the transverse piece of \( \text{Im}(\Delta_{33}) \), for several values of \( \tilde{x} = |\vec{x}| \) as function of \( t \) (See Fig. 1 for the location of the displayed curves in the \( x-t \) plane). To each curve in the figure, we have added a thin vertical line separating physical from unphysical region. Clearly, this piece of the propagator violates the locality axiom: The commutator function for two gauge bosons has support for space-like separations.

In Fig. 3 we show the Fourier transform of the pole part of the transverse piece of \( \text{Im}(\Delta_{33}) \), similar to the representation of Fig. 2. This piece of the commutator has a discontinuity at the "lightcone", i.e., at the curve \( |\vec{x}| = |t| \) separating the physical from the unphysical region in coordinate space. Relevant for our purpose however is the fact that it is not zero in the unphysical region, and that it does not cancel the contribution depicted in Fig. 2.

For brevity we do not display the corresponding parts of the longitudinal piece in \( \Delta_{33} \).
However, our calculation clearly shows, that the total Fourier transform of \( \Delta_{33} \) is not identically zero in the unphysical region of Fig. 1.

Let us now discuss, whether this is of any relevance for measurable quantities: Following ref. [4] one may argue, that the gauge field itself has no physical meaning and non-causal propagation of gauge fields occurs in any physical gauge. However, the commutator of two magnetic field components in our example, where the commutator expectation value of different spacelike components is zero, reads

\[
\langle [B_i(x), B_j(y)] \rangle = \varepsilon^{ijk} \frac{\partial^2}{\partial x_i \partial x_j} \langle [A_k(x), A_k(y)] \rangle .
\]  

This implies, that in the present example a non-vanishing commutator function of the gauge field outside the lightcone leads to a non-vanishing commutator function for observable quantities. Consequently the violation we are discussing here has the physical effect that the magnetic field cannot be “measured” independently at two points with a space-like separation.

A second question arising then is, whether the violation of causality inherent to the high temperature “hot QED” propagators (8) is severe or not. From Fig. 2 we learn, that non-causal influence (i.e., over space-like distances in Minkowski space) occurs for short distances, on the order of the Debye screening length.

This can be understood when recalling that in the method of “hard thermal loops” the above propagators may be used only for soft momenta which are smaller than \( q_D \). For high momenta, within this method one is supposed to use free propagators. However, our numerical calculation shows that such a separation of scales does not solve the causality problem: Even when combining the effective spectral function with the free one in such a way, the contribution to the commutator function does not vanish outside the physical region.

We may now draw two conclusions from the present work. Firstly we find, that seemingly simplistic ansatz spectral functions as given in eqs. (6) obey the important axiom of locality, i.e., they allow only causal non-locality. This makes them a good starting point for any nonperturbative treatment of matter at high temperature.

The second conclusion is, that for an explicitly causal propagation of gauge bosons one needs a spectral function which interpolates smoothly between the high-momentum and the soft-momentum region. As noted before, the most general form for such a function at finite temperature has been given in ref. [11]. From the propagators (8) of “hot QED”, such a spectral function may be obtained in the following way: In coordinate space, multiply the commutator function by \( \Theta(t^2 - \bar{x}^2) \), then transform it back into momentum space. Equivalently, one may convolute the old momentum space propagators with the Fourier transform of such a \( \Theta \)-function.

We are currently exploring, how such a prescription would affect the results obtained in the hard thermal loop resummation technique.

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