Covariant double-null dynamics: \((2 + 2)\)-splitting of the Einstein equations.

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Abstract

The paper develops a \((2 + 2)\)-imbedding formalism adapted to a double foliation of spacetime by a net of two intersecting families of lightlike hypersurfaces. The formalism is two-dimensionally covariant, and leads to simple, geometrically transparent and tractable expressions for the Einstein field equations and the Einstein-Hilbert action, and it should find a variety of applications. It is applied here to elucidate the structure of the characteristic initial-value problem of general relativity.
1 Introduction

The classic analysis of Arnowitt, Deser and Misner (ADM) [1] formulates gravitational dynamics in terms of the evolution of a spatial 3-geometry. The geometrical framework is the imbedding formalism of Gauss and Codazzi for the foliation of spacetime by spacelike hypersurfaces [2].

Quite often, however, one encounters circumstances where a lightlike foliation is especially suitable. Because of the degeneracies that arise in the lightlike case the imbedding relations are very different and the situation not quite so familiar and under control. To bypass the degeneracies, one is forced to fall back to a foliation of codimension 2, by spacelike 2-surfaces. It is our aim in this paper to develop a simple \((2 + 2)\)-imbedding formalism of this kind.

Several \((2 + 2)\)-formalisms are extant [3], the earliest and best known being the generalized spin-coefficient formalism of Geroch, Held and Penrose (GHP) [4]. Basically, of course, all such formalisms have the same content, but they take very different forms.

The essential feature of the present approach is that it maintains manifest two-dimensional covariance while operating with objects having direct geometrical meaning. Two-dimensional covariance permits reduction of the Einstein field equations to an especially concise and transparent form: the ten Ricci components are embraced in a set of just three compact, two-dimensionally covariant expressions.

There is a limitation, at least in the version presented here. (It applies to most of the formalisms we have listed [3].) The two independent normals to an imbedded 2-surface—conveniently taken as a pair of lightlike vectors, since their directions are uniquely defined—are assumed from the beginning to be hypersurface-orthogonal. This precludes choosing them as principal null vectors of the Weyl tensor for a twisting geometry like Kerr. In this respect, the formalism is less flexible than GHP, and not as well tailored for the study of algebraically special metrics.

\((2 + 2)\) formalisms have a wide range of applications: to the analysis of the characteristic initial-value problem [5], the dynamics of strings [6] and of real and apparent horizons [7] and light-cone quantization [8] and gravitational interactions in ultra-high energy collisions [9]. In a separate publication [10], we shall use the present formalism to study the nature of the singularity at the Cauchy horizon in a generic black hole.

We conclude this Introduction by briefly outlining the contents of the paper. The basic metrical notions (adapted co-ordinates, basis vectors and form of the metric) are defined in Sec. 2. In Sec. 3, we introduce in two-dimensionally covariant form the geometrical information encoded in first derivatives of the metric: the extrinsic curvatures and “twist,” as well as the invariant operators which perform differentiation along the two lightlike normals. This comprises the basic formal machinery needed in Sec. 4, which presents the central result of the paper, the tetrad components of the Ricci tensor as three concise equations (27)–(29). (To make direct access to these results easy, their derivation is deferred to the second half of the paper (Secs. 9–12), which also provides (Sec. 13) the tetrad components of the full Riemann tensor.)

The contracted Bianchi identities (Sec. 5) are applied in Sec. 7 to analyze the structure of the characteristic initial-value problem. In Sec. 8 we sketch the Lagrangian formulation of covariant double-null dynamics.

The Ricci and Riemann components result from the commutation relations for four-dimensional covariant differentiation. Their most efficient derivation calls for a formalism that is both
four- and two-dimensionally covariant. Unfortunately, these two requirements do not mesh easily. Four-dimensional covariance tends to clutter the formulae by treating subsidiary two-dimensional quantities like shift vectors and the two-dimensional connection as 4-scalars, on a par with the primary geometrical properties, extrinsic curvature and twist. Those properties, for their part, are correlated, not with four-dimensional covariant derivatives, but with Lie derivatives, which are non-metric and have no direct link to curvature. To patch up these differences, and thus streamline the derivations, seems to need a certain degree of artifice. In Sec. 10 we address this (purely technical) problem by temporarily working with a “rationalized” covariant derivative which exhibits both four-dimensional and restricted (“rigid”) two-dimensional covariance.

Some brief remarks (Sec. 14) conclude the paper.

2 \((2 + 2)-split \) of the metric

We shall suppose that we are given a foliation of spacetime by lightlike hypersurfaces \(\Sigma^0\) with normal generators \(\ell_0^{(0)}\), and a second, independent foliation by lightlike hypersurfaces \(\Sigma^1\) with generators \(\ell_0^{(1)}\) nowhere parallel to \(\ell_0^{(0)}\). The intersections of \(\{\Sigma^0\}\) and \(\{\Sigma^1\}\) define a foliation of codimension 2 by spacelike 2-surfaces \(S\). (The topology of \(S\) is unspecified. All our considerations are local.) \(S\) has exactly two lightlike normals at each of its points, co-directed with \(\ell^{(0)}\) and \(\ell^{(1)}\).

In terms of local charts, the foliation is described by the imbedding relations

\[
x^\alpha = x^\alpha(u^A, \theta^a).
\]

Here, \(x^\alpha\) are four-dimensional spacetime co-ordinates (assumed admissible in the sense of Lichnerowicz [11]); \(u^0\) and \(u^1\) are a pair of scalar fields constant over each of the hypersurfaces \(\Sigma^0\) and \(\Sigma^1\) respectively; and \(\theta^2, \theta^3\) are intrinsic co-ordinates of the 2-spaces \(S\), each characterized by a fixed pair of values \((u^0, u^1)\).

**Notation:** Our conventions are: Greek indices \(\alpha, \beta, \ldots\) run from 0 to 3; upper-case Latin indices \(A, B, \ldots\) take values \((0, 1)\); and lower-case Latin indices \(a, b, \ldots\) take values \((2, 3)\). We adopt MTW curvature conventions [2] with signature \((- + + +)\) for the spacetime metric \(g_{\alpha\beta}\). When there is no risk of confusion we shall often omit the Greek indices on 4-vectors like \(\ell_\alpha^{(A)}\) and \(e_\alpha^{(a)}\): they are easily identifiable as 4-vectors by their parenthesized labels. Four-dimensional covariant differentiation is indicated either by \(\nabla_\alpha\) or a vertical stroke: \(\nabla_\beta A_\alpha \equiv A_{\alpha\beta}\). Four-dimensional scalar products are often indicated by a dot: thus, \(\ell_\alpha^{(A)} \cdot \ell_\beta^{(B)} \equiv g_{\alpha\beta} \ell_\alpha^{(A)} \ell_\beta^{(B)}\). Further conventions will be introduced as the need arises.

Without essential loss of generality we may assume the functions \(x^\alpha(u^A, \theta^a)\) to be smooth (at least thrice differentiable). (We are always free to make the co-ordinate choice \(x^A = u^A, x^a = \theta^a\), but at the cost of losing manifest four-dimensional and two-dimensional covariance.)

The lightlike character of the hypersurfaces \(\Sigma^A\) is encoded in

\[
\nabla u^A \cdot \nabla u^B \equiv g^{\alpha\beta}(\partial_\alpha u^A)(\partial_\beta u^B) = e^{-\lambda(x)} \eta^{AB}
\]

for some scalar field \(\lambda(x^\alpha)\), where

\[
\eta^{AB} = \text{anti-diag}(-1, -1) = \eta_{AB};
\]

2
$\eta^{AB}$ and its inverse $\eta_{AB}$ are employed to raise and lower upper-case Latin indices, e.g., $\ell(0) = -\ell(1)$.

The generators $\ell(A)$ of $\Sigma^A$ are parallel to the gradients of $u^A(x^{\alpha})$. It is symmetrical and convenient to define $\ell(A) = e^A \nabla u^A$, i.e.,

$$\ell^{(A)} = e^\lambda \partial_\alpha u^A.$$  \hfill (4)

Then

$$\ell(A) \cdot \ell(B) = e^\lambda \delta^A_B.$$  \hfill (5)

The pair of vectors $e_{(a)}$, defined from (1) by

$$e^\alpha_{(a)} = \partial x^\alpha/\partial \theta^a,$$

are holonomic basis vectors tangent to $S$. The intrinsic metric $g_{ab} d\theta^a d\theta^b$ of $S$ is determined by their scalar products:

$$g_{ab} = e_{(a)} \cdot e_{(b)}.$$  \hfill (7)

Lower-case Latin indices are lowered and raised with $g_{ab}$ and its inverse $g^{ab}$; thus $e^{(a)} \equiv g^{ab} e_{(b)}$ are the dual basis vectors tangent to $S$, with $e_{(a)} \cdot e_{(b)} = \delta^a_b$. Since $\ell(A)$ is normal to every vector in $\Sigma^A$, we have

$$\ell^{(A)} \cdot e_{(a)} = 0.$$  \hfill (8)

Figure 1: The 2 + 2 splitting of the four dimensional space time into a foliation of intersecting null surfaces $u^0 = \text{const.}$ and $u^1 = \text{const.}$.

In general, $\theta^a$ cannot be chosen so as to remain constant along both sets of generators $\ell^{(A)}$. They are convected (Lie-transported) along the pair of vector fields $\partial x^a/\partial u^A$ (in general, non-lightlike).
From (4) and (5) one finds that $\partial x^\alpha / \partial u^A - \ell^{(A)}_\alpha$ is orthogonal to $\ell^{(B)}$, i.e., tangent to $S$. This validates the decomposition
\[ \frac{\partial x^\alpha}{\partial u^A} = \ell^{(A)}_\alpha + s^\alpha_A e^{(a)}, \]
thus defining a pair of "shift vectors" $s^\alpha_A$ tangent to $S$ (see Fig. 1).

An arbitrary displacement $dx^\alpha$ in spacetime is, according to (6) and (9), decomposable as
\[ dx^\alpha = \ell^{(A)}_\alpha du^A + e^{(a)}(d\theta^a + s^a_A du^A). \]

From (5), (7) and (8) we read off the completeness relation
\[ g_{\alpha\beta} = e^{-\lambda} \eta_{(A)} e^{(a)}(e^{(a)}_\alpha e^{(b)}_\beta + g^{ab}) \]
Combining (10) and (11) shows that the spacetime metric is decomposable as
\[ g_{\alpha\beta} dx^\alpha dx^\beta = e^{\lambda} \eta_{(A)} du^A du^B + g^{ab}(d\theta^a + s^a_A du^A)(d\theta^b + s^b_B du^B). \]

3 Two-dimensionally covariant objects embodying first derivatives of the metric: extrinsic curvatures $K_{Aab}$, twist $\omega^a$ and normal Lie derivatives $D_A$

Absolute derivatives of four-dimensional tensor fields with respect to $u^A$ and $\theta^a$ are projections of the four-dimensional covariant derivative $\nabla_\alpha$, and denoted by
\[ \frac{\delta}{\delta u^A} = \frac{\partial x^\alpha}{\partial u^A} \nabla_\alpha, \quad \frac{\delta}{\delta \theta^a} = e_{(a)} \cdot \nabla. \]

From (6) and the symmetry of the mixed partial derivatives and the affine connection,
\[ \frac{\delta e^{(a)}_\alpha}{\delta u^A} = \frac{\delta}{\delta \theta^a} \left( \frac{\partial x^\alpha}{\partial u^A} \right), \quad \frac{\delta e^{(a)}_\alpha}{\delta \theta^b} = \frac{\delta e^{(b)}_\alpha}{\delta \theta^a}. \]

The object
\[ \Gamma^c_{ab} = e^{(c)} \cdot \delta e_{(a)}/\delta \theta^b \]
is, as the notation suggests, the Christoffel symbol associated with $g_{ab}$, as is easily verified by forming $\partial_c g_{ab}$, recalling (7) and applying Leibnitz's rule.

Associated with its two normals $\ell^{(A)}$, $S$ has two extrinsic curvatures $K_{Aab}$, defined by
\[ K_{Aab} = e_{(a)} \cdot \delta \ell^{(A)}_{\alpha}/\delta \theta^b = \ell^{(A)\alpha}_{\alpha\beta} e^{(a)}_{(a)} e^{(b)}_{(b)} \]
(Since we are free to rescale the null vectors $\ell^{(A)}$, a certain scale-arbitrariness is inherent in this definition.) Because of (8), we can rewrite
\[ K_{Aab} = -\ell^{(A)} \cdot \delta e_{(a)}/\delta \theta^b, \]
which exhibits the symmetry in $a, b$.  

A further basic geometrical property of the double foliation is given by the Lie bracket of \(\ell(\text{B})\) and \(\ell(\text{A})\), i.e., the 4-vector

\[
[\ell(\text{B}), \ell(\text{A})] = 2(\ell(\text{B}) \cdot \nabla)\ell(\text{A}).
\]

(18)

Noting (9) and the fact that the Lie bracket of the vectors \(\partial x^\alpha/\partial u^\text{B}\) and \(\partial x^\alpha/\partial u^\text{A}\) vanishes identically, and recalling (14), we find

\[
[\ell(\text{B}), \ell(\text{A})] = \epsilon_{AB} \omega^a e(a),
\]

(19)

where the 2-vector \(\omega^a\) is given by

\[
\omega^a = \epsilon^{AB}(\partial_{\text{B}} s_A^a - s_B^b s_A^a),
\]

(20)

the semicolon indicates two-dimensional covariant differentiation associated with metric \(g_{ab}\), and \(\epsilon_{AB}\) is the two-dimensional permutation symbol, with \(\epsilon_{01} = +1\). (Note that raising indices with \(\eta^{AB}\) to form \(\epsilon^{AB}\) yields \(\epsilon^{10} = +1\).)

The geometrical significance of the “twist” \(\omega^a\) can be read off from (19): the curves tangent to the generators \(\ell(0), \ell(1)\) mesh together to form 2-surfaces (orthogonal to the surfaces \(S\)) if and only if \(\omega^a = 0\). In this case, it would be consistent to allow the co-ordinates \(\theta^a\) to be dragged along both sets of generators, and thus to gauge both shift vectors to zero.

We denote by \(D_A\) the two-dimensionally invariant operator associated with differentiation along the normal direction \(\ell(\text{A})\). Acting on any two-dimensional geometrical object \(X_{b...}^a\), \(D_A\) is formally defined by

\[
D_A X_{b...}^a = (\partial_A - \mathcal{L}_{s_A^a}) X_{b...}^a.
\]

(21)

Here, \(\partial_A\) is the partial derivative with respect to \(u^\text{A}\) and \(\mathcal{L}_{s_A^a}\) the Lie derivative with respect to the 2-vector \(s_A^a\).

As examples of (21), we have for a 2-scalar \(f\) (this includes any object bearing upper-case, but no lower-case, Latin indices):

\[
D_A f = (\partial_A - s_A^a \partial_a) f = \ell(\text{A}) \partial_a f
\]

(22)

(in which the second equality follows at once from (9)); and for the 2-metric \(g_{ab}\):

\[
D_A g_{ab} = \partial_A g_{ab} - 2 s_{A(a;b)} = 2 K_{Aab},
\]

(23)

in which the second equality is derivable from (7), (14) and (9). (For the detailed derivation, see (75) below, or Appendix B.)

The geometrical meaning of \(D_A\) is quite generally the following (see Appendix B):

\(D_A X_{b...}^a\) is the projection onto \(S\) of the Lie derivative of the equivalent tangential 4-tensor

\[
X_{b...}^a \equiv X_{b...}^a e(a) e(b) ...$

with respect to the 4-vector \(\ell(\text{A})\):

\[
D_A X_{b...}^a = e(a) e(b) \cdot \mathcal{L}_{\ell(\text{A})} X_{b...}^a
\]

(24)
The objects $K_{Aab}$, $\omega^a$ and $D_A$ comprise all the geometrical structure that is needed for a succinct two-dimensionally covariant expression of the Riemann and Ricci curvatures of spacetime. According to (16), (19) and (24), all are simple projections onto $S$ of four-dimensional geometrical objects. Consequently, they transform very simply under two-dimensional co-ordinate transformations. Under the arbitrary reparametrization

$$\theta^{a} \to \theta^{a'} = f^{a}(\theta^{b}, u^{A})$$

(25)

(which leaves $u^{A}$ and hence the surfaces $\Sigma^{A}$ and $S$ unchanged), $\omega_a$ and $K_{Aab}$ transform cogrediently with

$$e_{(a)} \to e'_{(a)} = e_{(b)} \partial \theta^{b} / \partial \theta^{a'}$$

(26) (see (6)), and $D_A$ is invariant. By contrast, $\partial x^{\alpha} / \partial u^{A}$ and hence the shift vectors $s^a_A$ (see (9)) undergo a more complicated gauge-like transformation, arising from the $u$-dependence in (25).

4 Ricci tensor

The geometrical and notational groundwork laid in the previous sections allows us now to simply display the components of the Ricci tensor, deferring derivations to Secs. 9–12. Our notation for the tetrad components is typified by

$$(4) R_{ab} = R_{\alpha\beta} e^\alpha_{(a)} e_\beta^{(b)}, \quad R_{aA} = R_{\alpha\beta} e^\alpha_{(a)} f^\beta_A.$$  

The results are

$$(4) R_{ab} = \frac{1}{2} (2) R_{gab} - e^{-\lambda} (D_A + K_A) K^A_{ab}$$

$$+ 2 e^{-\lambda} K_{(a A^b) d} - \frac{1}{2} e^{-2\lambda} \omega_a \omega_b - \lambda_{ab} - \frac{1}{2} \lambda_a \lambda_b$$

$$R_{AB} = -D_{(A K_B)} - K_{Aab} K^{ab}_{B} + K_{(A D_B)} \lambda$$

$$- \frac{1}{2} \eta_{AB} \left[ (D^E + K^E) D_{E} \lambda - e^{-\lambda} \omega^a \omega_a + (e^\lambda)^{-a}_{a} \right]$$

$$R_{Aa} = K^b_{Aa;b} - \partial_a K_A - \frac{1}{2} \partial_a D_A \lambda + \frac{1}{2} K_A \partial_a \lambda$$

$$+ \frac{1}{2} \epsilon_{AB} e^{-\lambda} \left[ (D^B + K^B) \omega_a - \omega_a D^B \lambda \right]$$

(27)

(28)

where $(2) R$ is the curvature scalar associated with the 2-metric $g_{ab}$, and $K_A \equiv K_A^{a}_{Aa}$.

5 Bianchi identities. Bondi’s lemma

The Ricci components are linked by four differential identities, the contracted Bianchi identities

$$\nabla_\beta R^\beta_\alpha = \frac{1}{2} \partial_\alpha R,$$  

(30)
where the four-dimensional curvature scalar \( R = R^\alpha_\alpha \) is given by

\[
R = e^{-\lambda} R^A_A + R^a_a,
\]

according to (11).

As we show in Sec. 12, projecting (30) onto \( e_{(a)} \) leads to

\[
(D_A + K_A) R^A_a = \frac{1}{2} \partial_a R^A_A + \frac{1}{2} e^\lambda \partial_a \left( e^{(4)} R^{b}_b \right) - \left( e^{(4)} R^{b}_a \right)_{;b}.
\]

(32)

Projection of (30) onto \( \ell_{(A)} \) similarly yields

\[
D_B \left( R^B_A - \frac{1}{2} \delta^B_A R^D_D \right) - \frac{1}{2} e^\lambda D_A \left( e^{(4)} R^a_a \right) = e^\lambda \left( R_{ab} K^a_b - R^B_A K_B - \left( e^\lambda R^a_A \right)_{;a} + \epsilon_{AB} \omega^a R^B_a. \right.
\]

(33)

Equations (32) and (33) express the four Bianchi identities in terms of the tetrad components of the Ricci tensor.

We now look at the general structure of these equations.

For \( A = 0 \) in (33), \( R^0_0 \) does not contribute to the first (parenthesized) term, since

\[
-R_{01} = R^0_0 = R^1_1 = \frac{1}{2} R^A_A.
\]

(34)

This equation therefore takes the form

\[
D_1 R_{00} + \frac{1}{2} e^\lambda D_0 \left( e^{(4)} R^a_a \right) = -K_0 R_{01} + \mathcal{L}(e^{(4)} R_{ab}, R_{00}, R_{0a}, \partial_a),
\]

(35)

in which the schematic notation \( \mathcal{L} \) implies that the expression is linear homogeneous in the indicated Ricci components and their two-dimensional spatial derivatives \( \partial_a \).

The other \( A = 1 \) component of (33) has the analogous structure

\[
D_0 R_{11} + \frac{1}{2} e^\lambda D_1 \left( e^{(4)} R^a_a \right) = -K_1 R_{01} + \mathcal{L}(e^{(4)} R_{ab}, R_{11}, R_{1a}, \partial_a).
\]

(36)

The form of the remaining two Bianchi identities (32) is

\[
D_0 R_{1a} + D_1 R_{0a} = \mathcal{L}(e^{(4)} R_{ab}, R_{01}, R_{Aa}, \partial_a).
\]

(37)

It is noteworthy that the appearance of \( R_{01} \) in (35) and (36) is purely algebraic: its vanishing would be a direct consequence of the vanishing of just six of the other components. Bondi et al [13] and Sachs [5] therefore refer to the \( R_{01} \) field equation as the "trivial equation."

The structure of (35)–(37) provides insight into how the field equations propagate initial data given on a lightlike hypersurface. Let us (arbitrarily) single out \( u^0 \) as "time," and suppose that the six "evolutionary" vacuum equations

\[
(4) R_{ab} = 0, \quad R_{00} = R_{0a} = 0
\]

(38)
are satisfied everywhere in the neighbourhood of a hypersurface \( u^0 = \text{const} \). (Bondi and Sachs refer to \((4)^{R_{ab}}\) as the “main equations” and to \( R_{00}, R_{0a} \) as “hypersurface equations.” \( R_{00} \) is, in fact, the Raychaudhuri focusing equation \([12]\), governing the expansion of the lightlike normal \( \ell_{(0)} = -\ell_{(1)} \) to the \textit{transverse} hypersurface \( u^1 = \text{const.} \), and \( R_{0a} \) similarly governs its shear.)

Then (35) shows that the trivial equation \( R_{01} = 0 \) is satisfied automatically. From (36) and (37) it can be further inferred that if \( R_{11} \) and \( R_{1a} \) vanish on one hypersurface \( u^0 = \text{const.} \), then they will vanish everywhere as a consequence of the six evolutionary equations (38). This is the content of the Bondi-Sachs lemma \([13, 5]\), which identifies the three conditions \( R_{11} = R_{1a} = 0 \) as constraints—on the expansion and shear of the generators \( \ell_{(1)} \) of an initial hypersurface \( u^0 = \text{const.} \)—which are respected by the evolution.

## 6 Co-ordinate conditions and gauge-fixing

The characteristic initial-value problem \([5]\) involves specifying initial data on a given pair of lightlike hypersurfaces \( \Sigma^0, \Sigma^1 \) intersecting in a 2-surface \( S_0 \).

It is natural to choose our parameters \( u^A \) so that \( u^0 = 0 \) on \( \Sigma^0 \) and \( u^1 = 0 \) on \( \Sigma^1 \). The requirement (2) that \( u^A \) be \textit{globally} lightlike already imposes two co-ordinate conditions on \((u^A, \theta^a)\), considered as co-ordinates of spacetime. Two further global conditions may be imposed. We may, for instance, demand that \( \theta^a \) be convected (Lie-propagated) along the lightlike curves tangent to \( \ell_{(0)} \) from values assigned arbitrarily on \( \Sigma^0 \). According to (9), this means the corresponding shift vector is zero everywhere:

\[
s^a_{(0)} = -\ell_{(0)}^\alpha \partial_\alpha \theta^a = 0. \tag{39}
\]

In this case, (20) shows that

\[
\omega^a = \partial_0 s^a_1 \tag{40}
\]

is just the “time”-derivative of the single remaining shift vector.

These global co-ordinate conditions can still be supplemented by appropriate initial conditions. We are still free to require that \( \theta^a \) be convected along generators of \( \Sigma^0 \) from assigned values on \( S_0 \); then

\[
s^a_1 = 0 \quad (u^0 = 0) \tag{41}
\]

in addition to (39).

In addition to (or independently of) (39) and (41), we are free to choose \( u^1 \) along \( \Sigma^0 \) and \( u^0 \) along \( \Sigma^1 \) to be \textit{affine} parameters of their generators. On \( \Sigma^0 \), for instance, this means, by virtue of (9) and (4),

\[
\ell_{(1)}^\alpha = \left( \frac{dx^\alpha}{du^1} \right)_{\text{gen.}} = -g^{\alpha\beta} \partial_\beta u^0 = -e^{-\lambda} g^{\alpha\beta} \ell_{(0)}^\beta
\]

so that \( \lambda \) vanishes over \( \Sigma^0 \). There is a similar argument for \( \Sigma^1 \). Thus, we can arrange

\[
\lambda = 0 \quad (\Sigma^0 \text{ and } \Sigma^1). \tag{42}
\]

Alternatively, in place of (42), the co-ordinate condition

\[
D_1 \lambda = \frac{1}{2} K_1 \quad \text{on} \quad \Sigma^0 \tag{43}
\]
could be imposed to normalize $u^1$. (A corresponding condition on $\Sigma^1$ would normalize $u^0$.) The Raychaudhuri equation (28) for $R_{11}$ on $\Sigma^0$ would then become linear in the expansion rate $K_1 = \partial_t \ln g^{\frac{1}{2}}$, and that facilitates its integration (cf Hayward [3], Brady and Chambers [7]).

7 Characteristic initial-value problem

We are now ready to address the question of what initial data are needed to prescribe a unique vacuum solution of the Einstein equations in a neighbourhood of two lightlike hypersurfaces $\Sigma^0$ and $\Sigma^1$ intersecting in a 2-surface $S_0$ [5].

We arbitrarily designate $u^0$ as “time,” and shall refer to $\Sigma^0 (u^0 = 0)$ as the “initial” hypersurface and to $\Sigma^1 (u^1 = 0)$ as the “boundary.”

We impose the co-ordinate conditions (39), (41) and (42) to tie down $\theta^a$ and $u^A$. While (39) and (41) control the way $\theta^a$ are carried off $S_0$, onto $\Sigma^0$ and into spacetime, the choice of $\theta^a$ on $S_0$ itself is unrestricted. Thus, our procedure retains covariance under the group of two-dimensional transformations $\theta^a \rightarrow \theta^a' = f^a(\theta^b)$.

In the 4-metric $g_{\alpha\beta}$, given by (12), the following six functions of four variables are then left undetermined:

$$g_{ab}, \lambda, s_1^a. \quad (44)$$

(In place of $s_1^a$, it is completely equivalent to specify $\omega^a = \partial_0 s_1^a$, since the “initial” value of $s_1^a$ is pegged by (41).)

We shall formally verify that a vacuum 4-metric is uniquely determined by the following initial data:

(a). On $S_0$, seven functions of two variables $\theta^a$:

$$g_{ab}, \omega^a, K_A = \partial_A \ln g^{\frac{1}{2}} \quad (S_0); \quad (45)$$

(b). on $\Sigma^0$ and $\Sigma^1$, two independent functions of three variables which specify the intrinsic conformal 2-metric:

$$g^{-\frac{1}{2}} g_{ab} \quad (\Sigma^0 \text{ and } \Sigma^1). \quad (46)$$

Instead of (46), it is equivalent to give the shear rates of the respective generators,

$$\sigma_{1a}^b \text{ on } \Sigma^0, \quad \sigma_{0a}^b \text{ on } \Sigma^1, \quad (47)$$

defined as the trace-free extrinsic curvatures:

$$\sigma_{Aab} = K_{Aab} - \frac{1}{2} g_{ab} K_A = \frac{1}{2} g^{\frac{3}{2}} \partial_A (g^{-\frac{1}{2}} g_{ab}). \quad (48)$$

These two functions correspond to the physical degrees of freedom (“radiation modes”) of the gravitational field [5, 14].

To build a vacuum solution from the initial data (45), (47), we begin by noting that (39) implies that $D_0 = \partial_0$, $K_{0ab} = K_{0ab}$ everywhere. Hence the general expression (28) for $R_{00}$ reduces here to

$$-R_{00} = \left( \partial_0 + \frac{1}{2} K_0 - \lambda_0 \right) K_0 + \sigma_{0ab} \sigma_0^{ab}. \quad (49)$$
On $\Sigma^1$, we have $\lambda = \lambda_0 = 0$ by (42). Thus, (49) becomes an ordinary differential equation for

$$K_0 = \overline{K}_0 = \partial_0 \ln g^{\frac{1}{2}}$$

as a function of $u^0$. This can be integrated along the generators, using the given data for $\sigma^0_{ab}$ on $\Sigma^1$, and the initial value of $\overline{K}_0$ on $S_0$, to obtain $g^{\frac{1}{2}}$, hence the full 2-metric $g_{ab}$ (hence also $K_{0ab}$) over $\Sigma^1$.

Expression (29) for $R_{0a}$ reduces similarly to

$$R_{0a} = -\frac{1}{2}e^{-\lambda} (\partial_0 + K_0 - \lambda_0) \omega_a - \frac{1}{2}(\partial_0 - K_0) \lambda_a + K_{0ab} \partial_a \lambda$$

in a spacetime neighbourhood of $\Sigma^0$ and $\Sigma^1$. On $\Sigma^1$, since $K_{0ab}$ is now known, and $\lambda = \lambda_a = \lambda_0 = 0$, (50) is a linear ordinary differential equation for $\omega_a$ which may be integrated along generators, with initial condition (45), to find $\omega_a$ (hence $s^a_1$).

Thus, our knowledge of the six metric functions (44) has been extended to all of $\Sigma^1$ with the aid of the evolutionary equations $R_{00} = R_{0a} = 0$. A similar procedure, applied to the constraint equations $R_{11} = R_{1a} = 0$, determines the functions (44) (hence also $K_{1ab}$) over the initial hypersurface $\Sigma^0$. (Here we exploit (41)—implying $D_1 = \partial_1$—which holds on $\Sigma^0$ only. This limitation is of little practical consequence, since the Bianchi identities (Sec. 7) relieve us of the need to recheck the constraints off $\Sigma^0$.)

Thus, the data (44), together with their tangential derivatives $\partial_1, \partial_a$—which we denote schematically by

$$\mathcal{D} = \{g_{ab}, \lambda, \omega_a, s^a_1, \partial_1, \partial_a\}$$

—are now known all over the initial hypersurface $\Sigma^0 u^0 = 0$. (Note that $\mathcal{D}$ includes $K_{1ab}$.)

We now proceed recursively. Suppose that $\mathcal{D}$ is known over some hypersurface $\Sigma: u^0 = \text{const}$. We show that the six evolutionary equations $R_{ab} = 0$, $R_{00} = R_{0a} = 0$, together with the known boundary values of $g_{ab}$, $K_{0ab}$, $\omega_a$ and $s^a_1$ on $\Sigma^1$, determine all first-order time-derivatives $\partial_0$ of $\mathcal{D}$, and hence the complete evolution of $\mathcal{D}$.

Expression (27) for the evolutionary equations $R_{ab} = 0$ can be written more explicitly, with the aid of the identity

$$D_A K^A_{ab} - 2K_A^{d} K_{db} = -2D_1 K_{0ab} + 4K_{0(a}^{d} K_{1b)d} + \omega_{(a;b)}$$

which is rooted in the symmetry

$$\partial_B K_{A|ab} = 0, \quad K_{Aab} \equiv K_{Aab} + s_{A(a;b)}$$

(see (23) and Appendix B).

The equations $R_{ab} = 0$ are thus seen to reduce to a system of three linear ordinary differential equations for $K_{0ab}$ as functions of $u^1$ on $\Sigma$, whose coefficients are concomitants of the known data $\mathcal{D}$ on $\Sigma$. Together with the boundary conditions on $K_{0ab}$ at $u^1 = 0$ (i.e., the intersection of $\Sigma$ with $\Sigma^1$), they determine a unique solution for $K_{0ab}$ on $\Sigma$.

We next turn to (49) and (50) to read off the values of $\partial_0 \lambda$ and $\partial_0 \omega_a$ on $\Sigma$. Since the remaining time-derivatives are known trivially from

$$\partial_0 s^a_1 = \omega^a, \quad \frac{1}{2}\partial_0 g_{ab} = \overline{K}_{0ab} = K_{0ab},$$
we are now in possession of the first time-derivatives of all the data $\mathcal{D}$ on $\Sigma$.

This completes our formal demonstration that the initial conditions (45) and (46), or (45) and (47), determine (at least locally) a unique vacuum spacetime.

8 Lagrangian

According to (12) and (31), the Einstein-Hilbert Lagrangian density $\mathcal{L} = (-\frac{4}{\sqrt{g}})^{1/2} R_A^\alpha (e^{-\lambda} R^A_\alpha + (4) R^a_a)$ decomposes as

$$\mathcal{L} = g^\frac{1}{2} e^{\lambda} (e^{-\lambda} R^A_\alpha + (4) R^a_a),$$

in which $g^\frac{1}{2}$ refers to the determinant of $g_{ab}$. Substitution from (27) and (28) yields the explicit form

$$g^\frac{1}{2} L = e^{\lambda} (2 R - D_A (2 K^A + D^A \lambda) - K_A K^A - K^a_b K^A_a + \frac{1}{2} e^{-\lambda} \omega^a \omega_a - e^{\lambda} \left( 2 \lambda^a_a + \frac{3}{2} \lambda_a \lambda^a \right)).$$

Second derivatives of the metric in (55) can be isolated in the form of a pure divergence by calling on the identities

$$g^\frac{1}{2} D_A X^A = \partial_\alpha [(-\frac{4}{\sqrt{g}})^{1/2} e^{-\lambda} X^A \ell_\alpha] - g^\frac{1}{2} X^A K_A,$$

$$A^a_{;a} + A^a \lambda_a = \nabla_\alpha (A^a \ell_\alpha),$$

which follow from (102) below, and hold for any scalars $X^A$ and 2-vector $A^a$. We thus obtain

$$\mathcal{L} = -\partial_\alpha \left[ (-\frac{4}{\sqrt{g}})^{1/2} e^{\lambda} \left( 2 K^A + D^A \lambda \right) \ell_\alpha + 2 \left( -\frac{4}{\sqrt{g}} \right)^{1/2} \lambda^a \ell_\alpha \right]$$

$$+ g^\frac{1}{2} \left[ e^{\lambda} (2 R + K_A K^A - K^a_b K^A_a + \frac{1}{2} e^{-\lambda} \omega^a \omega_a + K^A D_A \lambda + \frac{1}{2} e^{\lambda} \lambda_a \lambda^a) \right].$$

The divergence term integrates as usual to a surface term in the action $S = \int \mathcal{L} d^4 x$, and has no influence on the classical equations of motion.

Variation of $S$ with respect to $-e^\lambda = g_{(0)(1)} = \ell_{(0)} \cdot \ell_{(1)}$

reproduces the expression obtained from (27) for $G^{01} = \frac{1}{2} e^\lambda R^a_a$. Similarly, variation with respect to $s^a_A$ yields the expression (29) for $G_{Aa} = R_{Aa}$, if we take account of the implicit dependence of $K_{Aab}$, $D_A$ and $\omega^a$ on $s^a_A$ through

$$K_{Aab} = \frac{1}{2} \partial_A g_{ab} - s_{A(a;b)},$$

(22) and (20). Finally, variation with respect to $g_{ab}$ yields $(4) G^{ab}$, if we note the identity

$$g^\frac{1}{2} \delta \int \varphi (2 R g^\frac{1}{2} d^2 \theta = g_{ab} \varphi_c^e c - \varphi_{;ab}. $$

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Thus, variation of the action (58) yields eight of the ten Einstein equations. The remaining two equations—the Raychaudhuri equations for $R_{00}$ and $R_{11}$—cannot be retrieved directly from (58), because the a priori conditions $\eta^{00} = \eta^{11} = 0$ (expressing the lightlike character of $u^0$ and $u^1$) which is built into (58), precludes us from varying with respect to these “variables.” The two Raychaudhuri equations can, however, be effectively recovered from the other eight equations via the Bianchi identities.

The Hamiltonian formulation of the dynamics has been discussed in detail by Torre [3]. We hope to pursue this topic elsewhere.

9 Gauss-Weingarten (first order) relations

In this second half of the paper, we return to the beginning and to the task of laying a more complete geometrical foundation for the Ricci and Bianchi formulas which we quoted without derivation in (27)–(29) and (32), (33). We begin with the first-order imbedding relations for the 2-surface $S$ as a subspace of spacetime.

(15) and (17) allow us to decompose the 4-vector $\delta e^{\alpha}_{(a)}/\delta \theta^b$ in terms of the basis $(\ell_{(A)}, e_{(a)})$. Recalling (5), we find

$$\delta e^{\alpha}_{(a)}/\delta \theta^b = -e^{-\lambda}K^A_{ab}\ell_{(A)} + \Gamma^c_{ab}e_{(c)}.$$  

Similarly, in view of (16), we may decompose

$$\delta \ell_{(A)}/\delta \theta^a = L_{ABa}\ell_{(B)} + K_{ab}e_{(b)}$$

where the first coefficient is given by

$$L_{ABa} = e^{-\lambda}\ell_{(B)} \cdot \delta \ell_{(A)}/\delta \theta^a.$$  

This coefficient can be reduced to a much simpler form. Its symmetric part is

$$L_{(AB)a} = \frac{1}{2}e^{-\lambda}\partial_a(\ell_{(A)} \cdot \ell_{(B)}) = \frac{1}{2}\eta_{AB}\partial_a \lambda.$$  

To obtain the skew part, we note first that

$$e^\beta_{(a)}\ell_{[(B)]}^{\alpha}\ell_{(A)(\alpha|\beta\alpha]} = 0.$$  

since $\ell_{(A)}$ is proportional to a lightlike gradient (see (4)). With the aid of (65) and (19) we now easily derive

$$L_{[AB]a} = e^{-\lambda}e^\beta_{[(B)]}e^{\alpha}_{(a)}\ell_{(A)(\alpha|\beta\alpha]} = \frac{1}{2}e^{-\lambda}[\ell_{(B)}, \ell_{(A)}]^{\alpha}e_{(a)\alpha} = \frac{1}{2}\eta_{AB}\omega_a e^{-\lambda},$$

Combining (64) and (66), we arrive at the simple expression

$$2L_{ABa} = \eta_{AB}\partial_a \lambda + e^{-\lambda}\epsilon_{AB}\omega_a$$
for the first coefficient in (62).

The Gauss-Weingarten equations (61) and (62) govern the variation of the 4-vectors $\ell_{(A)}$ and $e_{(a)}$ along directions tangent to $S$. We now turn to their variation along the two normals.

We have from (4),

$$\nabla_\beta \ell_{(A)\alpha} = 2\ell_{(A)[\alpha} \partial_\beta] + \nabla_\alpha \ell_{(A)\beta}. \tag{68}$$

Multiplying by $\ell_{(B)\beta}$ and symmetrizing in $A, B$ gives

$$(\ell_{(B)} \cdot \nabla) \ell_{(A)} + (\ell_{(A)} \cdot \nabla) \ell_{(B)} = 2\ell_{((A)D(B)}\lambda - \eta_{AB} e^\lambda \nabla \lambda \tag{69}$$

where $D_A \lambda$ is defined as in (22). It follows that

$$\nabla \lambda = e^{-\lambda} \ell^{(A)} D_A \lambda + e^{(a)} \partial_a \lambda. \tag{70}$$

On the other hand, the difference of the two terms on the left of (69) is given by (19) as $\epsilon_{AB} \omega_a e^{(a)}$.

Adding finally yields

$$((\ell_{(B)} \cdot \nabla) \ell_{(A)})\ell_{(C)} = N_{ABC} \ell_{(C)} - s_{A}^{b} e_{(a)} \ell_{(B)} \tag{71}$$

where

$$N_{ABC} = D_{(A} \eta_{B)C} - \frac{1}{2} \eta_{AB} D_C \lambda, \tag{72}$$

and $L$ was defined in (67).

Proceeding finally to the transverse variation of $e_{(a)}$, we have from (14) and (9),

$$\frac{\delta e_{(a)}}{\delta u^A} = \frac{\delta}{\delta \theta^a} \left( \ell_{(A)} + s_{A}^{b} e_{(b)} \right).$$

Substituting from (61) and (62), it is straightforward to reduce this to

$$\frac{\delta e_{(a)}}{\delta u^A} = (L_{AB a} - s_{A}^{b} e^{-(A)} K_{Bab}) \ell_{(B)} + \tilde{K}_{A a} e_{(b)} \tag{73}$$

where

$$\tilde{K}_{Aab} = K_{Aab} + s_{A}^{b} e_{(a)} \tag{74}$$

Applying Leibnitz’s rule to

$$\partial_A g_{ab} = \frac{\delta}{\delta u^A} (e_{(a)} \cdot e_{(b)}),$$

we read off from (73) the result

$$\frac{1}{2} \partial_A g_{ab} = K_{Aab} \equiv \tilde{K}_{A(ab)}, \tag{75}$$

which gives direct geometrical meaning to the extrinsic curvature in terms of transverse variation of the 2-metric.

The normal absolute derivatives of $e_{(a)}$ are given by (recalling (9) and (13))

$$(\ell_{(A)} \cdot \nabla) e_{(a)} = \frac{\delta e_{(a)}}{\delta u^A} - s_{A}^{b} \frac{\delta e_{(a)}}{\delta \theta^b}. \tag{76}$$
With the help of (73) and (61) this reduces to

\[(\ell (A) \cdot \nabla) e_{(a)} = L_{ABa} \ell (B) + (\bar{K}_{Aa}^b - s_A^c \Gamma_{ac}^b) e_{(b)}\]  

(76)

Correspondingly, the two normal derivatives of \(g_{ab}\) are

\[(\ell (A) \cdot \nabla) g_{ab} = 2 \bar{K}_{Aab} - s^a_A \partial_c g_{ab}.\]  

(77)

The two-dimensionally noncovariant terms which appear in (76) and (77) are not a mistake. They arise because the normal gradient \(\ell (A) \cdot \nabla\), applied to objects carrying lower-case Latin indices—let us say \(g_{ab}\)—does not preserve manifest two-dimensional covariance, since it contains (see (22)) a piece \(-s^a_A \partial_c g_{ab}\) involving ordinary (rather than two-dimensional covariant) derivatives with respect to \(\theta^c\). Although not incorrect, this is a formal impediment: it threatens to clutter our formulae with terms in the shift vectors \(s^a_A\) which are, to boot, noncovariant. In the following section, we explain how this can be remedied by introducing a “rationalized” gradient operator \(\bar{\nabla}\).

10 Rationalized operators \(\bar{\nabla}\), \(\bar{D}_A\), \(\nabla_a\)

The rationalized operator \(\bar{\nabla}_a\) avoids the two-dimensionally noncovariant terms which appear when \(\nabla_a\) is applied to objects bearing lower-case Latin indices, as in (61), (76) and (77).

Applied to scalar fields or to 4-tensors not bearing lower-case Latin indices, \(\bar{\nabla}\) is identical with \(\nabla\). If the object does carry such indices, there are supplementary terms involving the two-dimensional connection \(\Gamma^a_{bc}\).

Specifically, we define

\[\bar{\nabla}_a = \nabla_a + p^{(a)}_\alpha (\nabla_a - e_{(a)} \cdot \nabla)\]  

(78)

in which \(e_{(a)} \cdot \nabla \equiv \delta / \delta \theta^a\) is the absolute derivative introduced in (13), and the operator \(\nabla_a\) will be specified in a moment. We have introduced the pair of 4-vectors \(p^{(a)}_\alpha = \nabla \theta^a\), i.e.,

\[p^{(a)}_\alpha = \partial \theta^a / \partial x^\alpha.\]  

(79)

Their projections onto the basis vectors are, according to (6) and (9),

\[p^{(a)} \cdot e_{(b)} = \delta^a_b, \quad p^{(a)} \cdot \ell (A) = -s^a_A,\]  

(80)

from which follows the identity

\[\delta^\beta_\alpha - p^{(a)}_\alpha e_{(a)}^\beta = e^{-\lambda} \ell (A) \partial x^{\beta} / \partial u^A.\]  

(81)

Hence (78) can be recast in terms of the absolute derivative \(\delta / \delta u^A\):

\[\bar{\nabla} = e^{-\lambda} \ell (A) \partial / \partial u^A + p^{(a)}_\alpha \nabla_a.\]  

(82)

We next introduce the differential operator

\[\bar{D}_A \equiv \ell (A) \cdot \bar{\nabla} = \ell (A) \cdot \nabla - s^a_A (\nabla_a - \delta / \delta \theta^a).\]  

(83)

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An alternative form
\[ \tilde{D}_A = \delta / \delta u^A - s_A^a \nabla_a \] (84)
follows from (82).

Since \( e_{(a)} : \nabla = \nabla_a \), we can reconstruct \( \nabla \) from (83) in yet another form:
\[ \nabla = e^{-\lambda} \ell^{(a)} \tilde{D}_A + e^{(a)} \nabla_a. \] (85)

We now specify the operator \( \nabla_a \). It is defined so as to act as a two-dimensional covariant derivative on all lower-case Latin indices (including parenthesized ones), and at the same time as an absolute derivative \( \delta / \delta \theta^a \) on Greek indices. Upper-case Latin indices are treated as inert.

As an example,
\[ \nabla_b e_{(a)} = \delta e_{(a)} / \delta \theta^b - \Gamma^c_{ab} e_{(c)}. \] (86)

It is evident that, quite generally, the “correction” \( \nabla_a - \delta / \delta \theta^a \) in (83) and (78) is linear and homogeneous in the two-dimensional connection \( \Gamma^c_{ab} \).

Examples of how \( \nabla_a \) and \( \tilde{D}_A \) act on scalars and 2-tensors are
\[ \nabla_a f = \partial_a f, \quad \tilde{D}_A f = (\partial_A - s_A^a \partial_a) f = D_A f, \]
\[ \nabla_a X^b_{c...} = X^b_{c...|a}, \quad \tilde{D}_A X^b_{c...} = (\partial_A - s_A^a \nabla_a) X^b_{c...}. \] (87)

For the 2-metric \( g_{ab} \), we have from (84),
\[ \tilde{D}_A g_{ab} = \partial_A g_{ab}, \] (88)

since \( \nabla_c g_{ab} \equiv g_{abc} = 0 \). Thus, (75) can be expressed as
\[ \frac{1}{2} \tilde{D}_A g_{ab} = K_{Aab}, \] (89)

which should be contrasted with (77).

Similarly, with the aid of (86) and (83), the noncovariant expressions (61) and (76) become
\[ \nabla_b e_{(a)} = \nabla_a e_{(b)} = -e^{-\lambda} K_{aab} \ell^{(A)}, \] (90)
\[ \tilde{D}_A e_{(a)} = L_{ABa} \ell^{(B)} + K_{aab} e_{(b)}. \] (91)

Quite generally, \( \tilde{D}_A, \nabla_a \) and \( \nabla \) preserve both four-dimensional covariance and covariance under “rigid” two-dimensional co-ordinate transformations \( \theta^a \rightarrow \theta'^a = f^a(\theta^b) \), with no dependence on \( u^A \). (u-dependence of \( f^a \) would induce “gauge” transformations of the shift vectors \( s_A^a \), see the remarks following (26).)

With the aid of (85), the last two results can be put together to form the rationalized covariant derivative of \( e_{(a)} \):
\[ e^\lambda \nabla_a e_{(a)} = (L_{BAa} \ell^{(A)} + K_{bab} e_{(b)}) \ell^{(B)} - K_{aab} \ell^{(A)} e_{(b)}. \] (92)

Equations (62) and (71) similarly combine to produce
\[ \nabla_b \ell^{(A)} = (e^{-\lambda} N_{ABC} e_{(C)} - L_{BAa} e_{(a)} \ell^{(B)} + (L_{ABB} e_{(B)} + K_{aab} e_{(a)}) e_{(b)}. \] (93)

(No distinction here between \( \nabla \) and \( \nabla \), since \( \ell^{(A)} \) carries no lower-case Latin indices.)

To sum up: equations (92) and (93) encapsulate the full set of first-order (Gauss-Weingarten) equations, which control tangential and normal variations of the basis vectors \( e_{(a)}, \ell^{(a)}. \) The coefficients in these equations are given by (16), (67), (20), (72) and (74). Their geometrical meaning emerges from (75), (19) and remarks following those equations.
11 Rationalized Ricci commutation rules

The usual commutation relations need to be modified for $\tilde{\nabla}_\alpha$. To derive the modified form, consider the action of $\tilde{\nabla}$ on any field object $X_a$ bearing just one lower-case Latin and an arbitrary set of other indices.

From (78) and (86),
$$\tilde{\nabla}_\gamma \tilde{\nabla}_\beta X_a = (\delta^c_\gamma \nabla_\gamma - p^{(n)}_\gamma \Gamma^c_\gamma) (\delta^b_\beta \nabla_\beta - p^{(m)}_\beta \Gamma^b_\beta) X_b.$$  

Skew-symmetrizing with respect to $\beta$ and $\gamma$, and noting from (79) that $\nabla_{[\gamma}p^{(m)}_\beta] = 0$ leads to
$$\tilde{\nabla}_{[\gamma} \tilde{\nabla}_{\beta]} X_a = \nabla_{[\gamma} \nabla_{\beta]} X_a + p^{(m)}_\gamma \{ p^{(n)}_\beta \frac{1}{2} R^b_\gamma_{\beta m n} - e^{-\lambda\ell(A)} \partial_A \Gamma^b_\beta \} X_b,$$  \hspace{1cm} (94)

in which $\partial_\gamma \Gamma^b_{\gamma m a}$ has been expanded using
$$\partial_\gamma = e^{-\lambda\ell(A)} \partial_A + p^{(n)}_\gamma \partial_n,$$  \hspace{1cm} (95)

which is a special case of (82).

The right-hand side of (94) can be further reduced: $\partial_A \Gamma^b_{\gamma m a}$ is a 2-tensor, given by
$$\partial_A \Gamma^b_{\gamma m a} = 2K^b_{A(a;m)} - K^{Ama;b}$$  \hspace{1cm} (96)

according to (75); and in two dimensions we have
$$R^b_\gamma_{\beta m n} = (2)R^{b}_{[m}g_{n]a}.$$  \hspace{1cm} (97)

If $e^a_{(a)}$ is substituted for $X_a$, (94), (92) and (93) can be used to express the projection onto $e_{(a)}$ of the four-dimensional Riemann tensor in terms of the first-order Gauss-Weingarten variables $K$, $L$, $N$ and their derivatives. If our interest is primarily in the Ricci tensor, the contracted form ($\gamma = \alpha$) of (94) suffices:
$$\tilde{\nabla}_\alpha \tilde{\nabla}_\beta - \tilde{\nabla}_\beta \tilde{\nabla}_\alpha e^a_{(a)} = e^a_{(a)} R_{\alpha\beta} - \frac{1}{2} (2)R p_{(a}\beta + e^{-\lambda\ell(A)} \partial A \ell(A) \ell(\beta)$$  \hspace{1cm} (98)

where
$$K_A \equiv K_{Aa} = \partial_A \ln g^\frac{1}{2}$$  \hspace{1cm} (99)

and $g \equiv \det g_{ab}$.

12 Contracted Gauss-Codazzi (second order) relations. Ricci tensor

The Gauss-Codazzi relations are the integrability conditions of the system of first order (Gauss-Weingarten) differential equations (92), (93). As just noted, they express projections of the four-dimensional Riemann tensor in terms of $K$, $L$, $N$ and their first derivatives.
Contraction of these equations gives frame components of the Ricci tensor. The most concise way of deriving these components in practice is through recourse to a generalized form of Raychaudhuri’s equation [12].

Let $A^\alpha$ be an arbitrary 4-vector (which may bear arbitrary label indices) and $B^\alpha$ a second vector free of lower-case Latin indices, so that $\nabla_\beta B^\alpha = \nabla_\beta B^\alpha$ and the standard commutation rules apply. Then it is easy to check the identity

$$R_{\alpha\beta} A^\alpha B^\beta = \nabla_\beta (A^\alpha \nabla_\alpha B^\beta) - A^\alpha \nabla_\alpha (\nabla_\beta B^\beta) - (\nabla_\beta A^\alpha)(\nabla_\alpha B^\beta).$$ (100)

If, on the other hand, $B$ is replaced by $e^{(b)}$, then we call upon the commutation law (98) for $\nabla$, with the result

$$R_{\alpha\beta} A^\alpha e^{(b)} = \nabla_\beta (A^\alpha \nabla_\alpha e^{(b)}) - A^\alpha \nabla_\alpha (\nabla_\beta e^{(b)}) - (\nabla_\beta A^\alpha)(\nabla_\alpha e^{(b)})$$

$$+ \frac{1}{2} (2)^R(A \cdot p^{(b)}) - e^{-\lambda}(\partial_\beta K_B)(A \cdot e^{(B)}).$$ (101)

With the choices $A = \ell(A)$ and $e^{(a)}$, $B = \ell(B)$ we can recover all frame components of the Ricci tensor from these equations in tandem with (92) and (93).

Some details of these calculations are recorded in Appendix A. The final results have already been listed in (27)–(29).

We next turn to the contracted Bianchi identities. The projection of (30) onto $e^{(a)}$ yields

$$\nabla_\beta (R^\beta e^{(a)}) - R^\beta \nabla_\beta e^{(a)} = \frac{1}{2} \partial_a R.$$ (102)

The second term is evaluated with the aid of (92). In the first term, we expand

$$e^{(a)} R^\beta = R^b_a e^{(b)} + e^{-\lambda} R^A e^{(A)},$$

and note the (often used) results

$$\nabla_\beta e^{(b)} = \partial_b \lambda, \quad \nabla_\beta \ell^{(A)} = K_A + D_A \lambda,$$

which follow from (92) and (93). The result is (32), and (33) is obtained similarly.

## 13 Riemann tensor

We list here the tetrad components of the Riemann tensor, obtainable from the uncontracted Ricci commutation rules (see, e.g., (94)). The notation for the tetrad components is as in Sec. 4.

$$(^4)\mathbf{R}_{cd}^{ab} = (^2)\mathbf{R}_{cd}^{ab} - 2 e^{-\lambda} K_A a K_{\alpha} B_d$$

$$R_{ABCD} = \frac{1}{4} \varepsilon_{AB} \varepsilon_{CD} (2 e^{\lambda} D E D E \lambda - 3 \omega^a \omega_a + e^{2 \lambda} \omega^a \lambda_a)$$

$$R_{Aabc} = 2 K_{A[a \mid b,c]} - K_{A[a \mid b} \lambda_{c]} - e^{-\lambda} \varepsilon_{AB} K_{a[b} \omega_{c]}$$

$$R_{aABC} = \frac{1}{2} \varepsilon_{BC} \{ D_A \omega_a + K_{A[a \mid b} \omega_b - e^{\lambda} \varepsilon_{AE} (D E \partial_a \lambda - K_{ab} \lambda_b) - \omega_a D_A \lambda \}$$

$$R^A_{\ a} B^b = -D^A_{K_{ab}} + K_{b[a} K_{a]} + D^A_{\lambda} K_{ab} - \frac{1}{2} \eta^{AB} D E \lambda K^E_{ab}$$

$$- \frac{1}{4} \eta^{AB} (e^{-\lambda} \omega_a \omega_b + e^{\lambda} \lambda_a \lambda_b + 2 e^{\lambda} \lambda_{ab}) - \frac{1}{4} \varepsilon^{AB} g^{\tau} \varepsilon_{ab} \tau.$$
We have here defined
\[ e^{-\lambda \tau} = g^{-\frac{1}{2}} \epsilon^{ab} \partial_a (e^{-\lambda \omega_b}) \]

14 Concluding remarks

The \((2+2)\) double-null imbedding formalism developed in this paper leads to simple and geometrically transparent expressions for the Einstein field equations (27)–(29) and the Einstein-Hilbert action (96). It should find ready application in a variety of areas, as indicated in the Introduction. \((2+2)\) formalisms are certainly not new [3, 4], but they have languished on the relativist’s back-burner. We hope that this exposition will play a role in promoting these versatile methods from the realm of esoterica into an everyday working tool.

Acknowledgement

This work was supported by the Canadian Institute for Advanced Research and by NSERC of Canada.

A Computing Ricci components: some intermediate details

For the convenience of enterprising readers who wish to derive the Ricci components (64)–(66) for themselves, we record here some intermediate steps of the computations.

Computation of \((4)R_{ab}\) from (101) requires evaluation of
\[
\nabla_\beta (e^\alpha_{(a)} \nabla_\alpha e^\beta_{(b)}) = -e^{-\lambda} (\tilde{D}_A + K_A) K^A_{ab}
\]
\[
(\nabla_\beta e^\alpha_{(a)}) (\nabla_\alpha e^\beta_{(b)}) = \frac{1}{2} (\lambda_a \lambda_b + e^{-2\lambda} \omega_a \omega_b) - 2 e^{-\lambda} K_A d \tilde{K}_{A d}^A ,
\]
which can be verified from (92) and (67).

Computation of \(R_{AB}\) from (100) requires

\[
\ell_{(A)\alpha | \beta}^{eta\alpha}_{(B)} = (D_A \lambda)(D_B \lambda) - \frac{1}{2} \eta_{AB}(D_E \lambda)(D_E \lambda)
\]
\[
+ K_{a b} K^a_b - \frac{1}{2} e^\lambda \eta_{AB}(\lambda_a \lambda^a + e^{-2\lambda} \omega^a \omega_a)
\]
which follows from (93), (67) and (72).

Finally, computation of \(R_{Aa}\) requires

\[
(\nabla_\beta e^\alpha_{(a)}) \ell_{(A)\alpha}^\beta = \frac{1}{2} \lambda_a D_A \lambda + \left( K_{a b} + \frac{1}{2} \Delta K_{A a b} \right) \lambda^b
\]
\[
+ \frac{1}{2} \epsilon_{A b e} e^{-\lambda} (\omega_a D^B \lambda + \Delta K_{a b}^R \omega^b)
\]
in which
\[
\Delta K_{A a b} \equiv \tilde{K}_{A a b} - K_{A a b} = s_{A b a}.
\]
B The operator $D_A$: commutation rules and other properties

In Sec. 3 we gave two definitions—(21) and (24)—for the operator $D_A$. It is straightforward to show their equivalence. We have

$$[e_{(a)}, e_{(b)}] = 0, \quad [\partial x / \partial u^A, e_{(b)}] = 0,$$

since the Lie bracket of two holonomic vectors vanishes (cf (14)). In combination with (9), this yields

$$[\ell_{(A)}, e_{(b)}] = - [s^a_A e_{(a)}, e_{(b)}] = (\partial_b s^a_A) e_{(a)}.$$

Hence, for any 2-vector $X^b$,

$$\mathcal{L}_{\ell_{(A)}} (X^b e_{(b)}) = \left\{(\partial_{\alpha} - L_{s^a_A}) X^b \right\} e_{(b)}$$

which proves the equivalence of (24) and (21) when applied to $X^b$.

This argument is easily extended. For instance, for the 2-metric $g_{ab}$, the definition (24) gives

$$D_A g_{ab} = e^{\alpha}_{(a)} e^{\beta}_{(b)} \mathcal{L}_{\ell_{(A)}} g_{\alpha \beta} = 2 e^{\alpha}_{(a)} e^{\beta}_{(b)} \ell_{\alpha (A) \beta}$$

by (16), which agrees with the form (23) obtained from the definition (21). (Strictly speaking, (24) requires that the projector $\Delta_{\alpha \beta} \equiv e_{(a) \alpha} e_{(a) \beta}$ should replace $g_{\alpha \beta}$ in the first equality above. But, according to the completeness relation (11), the difference involves the Lie derivative of $\ell_{\alpha (A)} \ell_{(A) \beta}$, which is linear homogeneous in $\ell_{(A)}$ and projects to zero.)

Commutation relations for $D_A$ follow most easily from the definition (24). For a scalar field $f$,

$$2D_B D_A f = [\mathcal{L}_{\ell_{(B)}}, \mathcal{L}_{\ell_{(A)}}] f = \mathcal{L}_{[\ell_{(B)}, \ell_{(A)}]} f = \epsilon_{AB} \omega^a \partial_a f,$$

where we have recalled the well-known result that the commutator of two Lie derivatives is the Lie derivative of the commutator (i.e., Lie bracket), and made use of (19).

Consider next the operation on a 2-vector $X^a$. We have from (24),

$$D_B D_A X^a = e^{\alpha}_{(a)} \mathcal{L}_{\ell_{(B)}} (e^{\beta}_{(b)} D_A X^b)$$

$$= e^{\alpha}_{(a)} \mathcal{L}_{\ell_{(B)}} \{\Delta_{\alpha \beta} \mathcal{L}_{\ell_{(A)}} (e^{a}_{(a) X^a})\}.$$

The projection tensor $\Delta_{\alpha \beta}$ can be replaced by $\delta_{\alpha \beta}$, because the Lie derivative, operating on the difference, gives terms proportional to $\ell^{\beta}_{(E)}$ or $\ell_{(E) \alpha}$, which project to zero, noting (B2). Thus,

$$D_B D_A X^a = e^{\alpha}_{(a)} \mathcal{L}_{\ell_{(B)}} \mathcal{L}_{\ell_{(A)}} (e^{a}_{(a) X^a}).$$

We can now proceed exactly as for the scalar case to derive the commutator. The result (generalized to an arbitrary 2-tensor) is

$$[D_B, D_A] X^a_{b ...,} = \epsilon_{AB} \mathcal{L}_{\omega^a} X^a_{b ...}.$$

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In particular,

\[ [D_B, D_A]g_{ab} = 2\epsilon_{AB}\omega_{(a;b)}]. \]

Recalling (B3), this may be written

\[ D_BK_{A|ab} = \frac{1}{2} \epsilon_{AB} \omega_{(a;b)}, \]

which was used in (52). It contracts to

\[ D_BK_{A} = \frac{1}{2} \epsilon_{AB} \omega^a_{\ a}. \]

These last two identities also play a role in symmetrizing—or, more properly, recognizing the implicit symmetry of—the raw expressions for \( R_{AB} \) and \( R_{AaBb} \) that emerge from the Ricci commutation relations. The manifestly symmetric expressions listed in (28) and Sec. 13 have been symmetrized with the aid of these identities.

To conclude, we note the rule for commuting \( D_A \) and the two-dimensional covariant derivative \( \nabla_a \). The commutator \([D_A, \nabla_a]\), applied to any 2-tensor, is formed by a pattern similar to its two-dimensional covariant derivative, but with \( \Gamma^a_{bc} \) replaced by

\[ D_A\Gamma^a_{bc} = 2K^a_{(b;c)} - K_{A(b,c)}a. \]

As examples:

\[ [D_A, \nabla_a]X^b = X^dD_A\Gamma^b_{da}, \]
\[ [D_A, \nabla_ag_{bc}] = -2(D_A\Gamma^d_{a(b)}g_{cd}) = -2K_{A(cda}. \]

The justification for the rule is that the partial derivative \( \partial_a \) (applied to any two-dimensional geometrical object) commutes with both \( \partial_A \) and the two-dimensional Lie derivative \( L_{s_A} \), so that, by (21),

\[ [D_A, \partial_a]X^b_{...} = 0. \]

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