Projective Representations of the $1 + 1$ Dimensional Poincaré Group

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Abstract

The unitary irreducible representations of the central extension of the Poincaré group in $1 + 1$ dimensions are constructed by an application of the Kirillov theory. These are then lifted to projective unitary irreducible representations of the Poincaré group. The $1 + 1$ Galilean group is treated separately in an appendix.

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We continue a program initiated sometime ago to study the properties of symmetry groups of $d + 1$ dimensional spacetimes with $d < 3$. In earlier publications the Galilean group for the case $d = 2$ was considered, its central extension carried out¹ and the system of unitary irreducible representations, both faithful and projective, constructed.² The representations of the Galilean group have been constructed independently by Grigore³ via considerations of a certain group, which is the central extension of the universal covering group of the Galilean group. The conclusions of ref. 3 are in agreement with those of refs. 1 and 2. In this paper we consider the Poincaré group for $d = 1$ and construct its projective representations. As for the faithful representations of this group, these are well-understood and are discussed, for instance, by Barut and Raczka.⁴ The Galilean group for $d = 1$ is discussed separately in an appendix.

As is well-understood, the projective representations of a group can be obtained from the ordinary representations of its universal central extension.^{5,6} The $1 + 1$ dimensional Poincaré group $P(1,1)$ has a one parameter family of central extensions, as shown by Bargmann.⁵ This is true of central extension either by the circle group T or by the additive group R of reals. The latter choice has certain technical advantages since the corresponding centrally extended group is simply-connected. Here we focus attention on $\hat{P}(1,1)$, which is the central extension of $P(1,1)$ by R .

The group $\hat{P}(1,1)$ is a (non-nilpotent) solvable group (recall that a nilpotent group is solvable but a solvable group is not necessarily nilpotent). The representations of $\hat{P}(1,1)$ may be constructed by an application of the method of Kirillov,⁷ wherein the group representations are induced from one-dimensional representations of certain subgroups associated with the co-adjoint orbits of the group. An important question for a general (non-nilpotent) solvable group is if the Kirillov construction yields *all* the irreducible representations of the group. A broad criterion for this to happen is given by the Auslander-Konstant theorem.⁸ The latter states that for a connected and simply-connected solvable Lie group of type 1 (a Lie group whose representations generate type 1 Von Neumann algebras), the Kirillov method or possibly a generalization of the original method involving passage from real to complex extension groups, will yield all the irreducible

representations. A less general result was given earlier by Bernat,⁹ which states that for a solvable Lie group which is exponential (an exponential group is one for which the canonical map from the Lie algebra to the group is surjective) the Kirillov method will give all the irreducible representations. While it is true that $\hat{P}(1,1)$ is exponential and of type 1, as will be shown below, it is actually not necessary to appeal to results of such generality.^{8,9} A more restrictive result due to Kirillov,⁷ which states that for a *completely solvable* group his method will yield all the representations, is sufficient for our purpose. A completely solvable group, as defined in ref. 7, is a solvable Lie group G that contains an increasing chain of normal subgroups $\{e\} = G_0 \subset G_1 \subset \dots \subset G_n = G$ whose dimensions increase by one at each step so that $\dim G_k = k$. Our $\hat{P}(1,1)$ is completely solvable by this definition (see below).

This paper is organized as follows. In the next section, we study the groups $P(1,1)$ and $\hat{P}(1,1)$ and establish some of their properties. The co-adjoint orbits of $\hat{P}(1,1)$ are obtained in sections 3 and 4. The unitary irreducible linear representations of $\hat{P}(1,1)$ are obtained in section 5 and the corresponding projective ones of $P(1,1)$ in section 6. Concluding remarks are made in section 7.

The $1 + 1$ Galilean group is treated separately (from the main body of the paper) in an appendix. Here the only contribution that we claim as original is in the construction of the central extension of the group. Once this is done, the centrally-extended group is recognized to be isomorphic with a certain group whose representations are already known in the mathematical literature.

2. The Poincaré group and its central extension.

We denote by $P(1,1)$ the $1 + 1$ dimensional Poincaré group. Let x and t be the coordinates of space-time. The set of transformations $(x,t) \rightarrow (x', t')$, where

$$t' = t \cosh z + x \sinh z + \eta \quad (2.1)$$

$$x' = t \sinh z + x \cosh z + u$$

constitute $P(1,1)$ under composition. Here z, η and u are real parameters and $\tanh z$ the relative velocity connecting the two inertial frames. Thus the typical group element can be displayed in the fashion $(z, \mathbf{u})$, when $\mathbf{u}$ is a $1 + 1$ vector and the group multiplication law is

$$(z, \mathbf{u})(z', \mathbf{u}') = (z + z', \mathbf{u} + \delta(z) \mathbf{u}') \quad (2.2)$$

where

$$\mathbf{u} = \begin{pmatrix} \eta \\ u \end{pmatrix} \text{ and } \delta(z) = \begin{pmatrix} \cosh z & \sinh z \\ \sinh z & \cosh z \end{pmatrix} \quad (2.3)$$

$P(1, 1)$ has two distinguished subgroups. Elements of the form $(z, 0)$ generate the subgroup of homogeneous Lorentz transformations, isomorphic with the additive group $\mathbf{R}$ of reals. The normal subgroup of translations consists of elements $(0, \mathbf{u})$ and is a two-dimensional vector group $\mathbf{R}^2$. The group $P(1, 1)$ is a semi-direct product of these two subgroups. The group $P(1, 1)$ as defined above is connected and simply-connected and is a solvable (non-nilpotent) group of rank 2, as is easy to verify.

The central extension of $P(1, 1)$ is given by Bargmann.⁵ For any pair $\mathbf{u}, \mathbf{u}'$ of $1 + 1$ vectors let $\mathbf{u} \wedge \mathbf{u}' = \eta u' - u \eta'$. Let $r = (z, \mathbf{u})$, $r' = (z', \mathbf{u}')$; $r, r' \in P(1, 1)$. Then a $\mathbf{R}$ -valued representative (of an equivalence class with respect to co-boundaries) 2-cocycle on $P(1, 1)$ is⁵

$$m(r, r') = \mathbf{u} \wedge \delta(z) \mathbf{u}' \quad (2.4)$$

The above 2-cocycle gives the desired central extensions. We have the central extension of $P(1, 1)$ by T (circle group) given by $\{(r, \xi)\}$, $r \in P(1, 1)$, $\xi \in T$, with the composition law

$$(r, \xi)(r', \xi') = \left(r r', \xi \xi' \exp \frac{i\lambda}{2} m(r, r') \right) \quad (2.5)$$

On the other hand, the central extension of $P(1, 1)$ by $\mathbf{R}$ — the group $\hat{P}(1, 1)$ — consists of elements (r, ω) , $r \in P(1, 1)$, $\omega \in \mathbf{R}$ with the multiplication rule

$$(r, \omega)(r', \omega') = \left(r r', \omega + \omega' + \frac{1}{2} m(r, r') \right) \quad (2.6)$$

where r, r' is given by eq. (2.2) and $m(r, r')$ by eq. (2.4). We proceed to analyze the structure of $\widehat{P}(1, 1)$.

Elements of $\widehat{P}(1, 1)$ of the form $(r, 0)$ with $r \in P(1, 1)$ restricted to be of the form $r = (z, 0)$ constitute a subgroup isomorphic with the homogeneous Lorentz group. We call this subgroup L . Elements (r, ω) with r having the form $r \in P(1, 1) = (0, \mathbf{u}) \equiv (0, \eta, u)$ generate a subgroup isomorphic with the Heisenberg-Weyl group (the group of 3×3 triangular matrices under matrix multiplication); the required isomorphism is given by

$$(\eta, u, \omega) \equiv (0, \eta, u, \omega) \leftrightarrow \begin{pmatrix} 1 & \eta & \omega + \frac{1}{2} \eta u \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \quad (2.7)$$

Call the above subgroup N . First, it is easy to check using eq. (2.6) that the first derived group (commutator subgroup) of $\widehat{P}(1, 1)$ is N ; the second derived group is $R = \{(0, 0, \omega)\}$ and the third derived group is trivial, thus $\widehat{P}(1, 1)$ is a solvable group of rank 3. It is also easy to verify that $\widehat{P}(1, 1)$ is not nilpotent. Further, $\widehat{P}(1, 1)$ has the structure of a semi-direct product. This follows from the existence of a unique decomposition for every element $(r, \omega) \equiv (z, \eta, u, \omega) \in \widehat{P}(1, 1)$ given by

$$(z, \eta, u, \omega) = (0, \eta, u, \omega)(z, 0, 0, 0) \quad (2.8)$$

where $(0, \eta, u, \omega) \in N$ and $(z, 0, 0, 0) \in L$, plus the fact that the only element that N and L share in common is the identity of $\widehat{P}(1, 1)$. Thus $\widehat{P}(1, 1) = NL$ with N normal. The group $\widehat{P}(1, 1)$ is connected and simply-connected; in fact, the group manifold is just R^4 topologically. We note next that the semi-direct product is *regular*, in the sense of Mackey. This may be seen as follows. Let N^* denote the dual space of N (the space of equivalence classes of unitary, irreducible representations of N). The unitary irreducible representations of N are of two types:⁷ i) those labeled by a non-zero real number λ , which is essentially the eigenvalue of the central charge of the Lie algebra of N and ii) those labeled by a pair (p_0, p) of real numbers, which is the case when the central charge has eigenvalue zero, the group effectively abelian ($\approx R^2$) and (p_0, p) label the

characters of R^2 . Thus the space N^* may be visualized as a plane with a line perpendicular to it. Under the action of L the points of N^* respond as follows: the points on the line, labeled by λ , do not move, since the central charge of $\text{Lie}(N)$ commutes with the L -action; and the points on the plane $\{(p_0, p)\}$ behave as $1 + 1$ vectors. We thus see that the L -orbits in N^* are as follows: i) one point orbits for each value of λ , ii) the one point orbit given by $(0, 0)$ on the plane, iii) the sets $\{(p_0, p)\}$ with $p_0^2 - p^2 = \pm m^2$, $m \geq 0$; the corresponding orbits being hyperbolas and "light-cones." Now, the semi-direct product $\widehat{P}(1, 1) = N \ltimes L$ will be regular if there exists in N^* a set (a Borel set) that meets each L -orbit in N^* at exactly one point.¹⁰ The desired set is simply the union of the λ -line with S , where S is a Borel set that meets each planar orbit at one point. The set S may be constructed without difficulty. Thus the semi-direct product is regular. Consider now the stability groups (little groups) of the L -orbits in N^* . It is clear from the foregoing discussion that the stability group of each one-point orbit is L , and the stability group of each non one-point orbit is trivial (one element group). In every case, the stability group is of type 1. From the last conclusion and using the regularity of the semi-direct product, it now follows from a theorem of Mackey¹⁰ that $\widehat{P}(1, 1)$ is a group of type 1. The same conclusion may also be reached from an analysis of the co-adjoint orbits (see below).

We shall now show that the group $\widehat{P}(1, 1)$ is completely solvable, in the sense of Kirillov.⁷ For this purpose, the coordinate system on the group that we have used thus far is not convenient. Another set of (global) coordinates on $\widehat{P}(1, 1)$ will be introduced in section 4 (see eq. (4.2)). Using the latter, and the corresponding group multiplication law given by eq. (4.3), we may verify the existence of the following sequence of normal subgroups (normal in $\widehat{P}(1, 1)$)

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset G_3 = N \subset G_4 = \widehat{P}(1, 1) \quad (2.9)$$

where $\dim(G_k) = k$. Here $G_2 = \{(0, x, 0, \omega)\} = R^2$ and $G_1 = \{(0, 0, 0, \omega)\} = R$, in the notation of eq. (4.2). This proves that $\widehat{P}(1, 1)$ is completely solvable.

The sequence (2.9) gives rise to a corresponding sequence at the level of Lie algebras in which every entry G_k is replaced by its Lie algebra $\text{Lie}(G_k)$ and every $\text{Lie}(G_k)$ is an ideal in $\text{Lie}(\widehat{P}(1, 1))$, and $\dim \text{Lie}(G_k) = k$. Since $\widehat{P}(1, 1)$ is connected, simply-connected and solvable, the existence of such a sequence of ideals in its *real* Lie algebra implies, via a result due to Dixmier¹¹ (corollary to his theorem 3), that $\widehat{P}(1, 1)$ is an exponential group.

We pass on to compute the co-adjoint orbits of $\widehat{P}(1, 1)$.

3. Co-adjoint orbits.

The Lie algebra of $\widehat{P}(1, 1)$ is a 4-dimensional real vector space. We may choose basis vectors N, H, P and E that obey the commutation relations

$$[N, H] = P, \quad [N, P] = H, \quad [H, P] = E \quad (3.1)$$

N, P, H are the infinitesimal generators of Lorentz transformation, space translations and time translation, respectively; E is the central generator. The operators H, P and E generate the Heisenberg algebra, which is an ideal.

The typical element (z, η, u, ω) of the group $\widehat{P}(1, 1)$ is

$$(z, \eta, u, \omega) = \exp(\omega E + \eta H + u P) \exp(z N) \quad (3.2)$$

Several comments are in order. First, in writing eq. (3.2), we utilized the semi-direct product structure of $\widehat{P}(1, 1)$ plus the fact that the Heisenberg-Weyl group is nilpotent and hence exponential; here η, u and ω are the exponential coordinates. Thus (3.2) gives a globally valid coordinate system. The group multiplication law can be derived from eqs. (3.1) and (3.2) together with the use of the Baker-Hausdorff formula; one thus obtains eq. (2.6). An alternative route to the same conclusion will employ a faithful matrix representation of eq. (3.1). Such a representation is

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.3)$$

$$P = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is useful to note $H^2 = P^2 = E^2 = 0$ and $N^3 = N$ and further that

$$N^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.4)$$

A group acts on its Lie algebra with the aid of the adjoint representation. Precisely, this action is $\text{Ad}(g) W = g W g^{-1}$ for $g \in G$, $W \in \text{Lie}(G)$. Use of the above matrix realization together with eq. (3.2) thus allows us to explicitly compute the adjoint action. Let us write the generic element W of the Lie algebra as $W = a E + b H + c P + d N$, where a, b, c, d are real numbers. Then the adjoint action of g^{-1} (it will turn out more useful for us to consider the action of the inverse element; here, g is given by eq. (3.2)) is $\text{Ad}(g^{-1}) W = W' = a' E + b' H + c' P + d' N$. Where

$$\begin{aligned} a' &= a - c\eta + bu + \frac{d}{2}(u^2 - \eta^2) \\ b' &= (b + du) \cosh z - (c + d\eta) \sinh z \\ c' &= (c + d\eta) \cosh z - (b + du) \sinh z \\ d' &= d \end{aligned} \quad (3.5)$$

as a straightforward calculation shows.

A group acts on a vector space which is dual to its Lie algebra with the aid of the co-adjoint representation. Let E^*, H^*, P^* and N^* be a basis in the dual vector space to $\widehat{\text{Lie } \hat{P}}(1, 1)$. This means

$$\begin{aligned}
(E^*, E) &= 1, \quad (E^*, X) = 0 \quad \text{for } X = H, P, N \\
(H^*, H) &= 1, \quad (H^*, X) = 0 \quad \text{for } X = E, P, N \\
(P^*, P) &= 1, \quad (P^*, X) = 0 \quad \text{for } X = E, H, N \\
(N^*, N) &= 1, \quad (N^*, X) = 0 \quad \text{for } X = E, H, P
\end{aligned} \tag{3.6}$$

where $(,)$ denotes a scalar product. Let us consider the typical element ℓ of the dual space

$$\ell = \delta E^* + \beta H^* + \gamma P^* + \alpha N^* \tag{3.7}$$

and let W be, as before,

$$W = a E + b H + c P + d N \tag{3.8}$$

then

$$(\ell, W) = a \delta + b \beta + c \gamma + d \alpha \tag{3.9}$$

Let us write $\ell(W) = (\ell, W)$. Let K denote the co-adjoint action; $K(g)\ell$ that of g on ℓ . Then precisely

$$(K(g)\ell)(W) = \ell(Ad(g^{-1})W) \tag{3.10}$$

defines the co-adjoint action. Let $K(g)\ell = \ell' = \delta' E^* + \beta' H^* + \gamma' P^* + \alpha' N^*$. Then we compute from eqs. (3.5) and (3.10) that

$$\delta' = \delta$$

$$\beta' = u\delta + \beta \cosh z - \gamma \sinh z \quad (3.11)$$

$$\gamma' = -\eta\delta - \beta \sinh z + \gamma \cosh z$$

$$\alpha' = \alpha + \frac{\delta}{2}(u^2 - \eta^2) + \beta(u \cosh z - \eta \sinh z) + \gamma(\eta \cosh z - u \sinh z)$$

where g is given by eq. (3.2). It follows, from the above, that $\beta^2 - \gamma^2 - 2\delta\alpha$ remain invariant under the co-adjoint action. The co-adjoint orbits (K-orbits) are now easy to enumerate. Each K-orbit is two-dimensional; since $\{(\alpha, \beta, \gamma)\}$ generate an orbit and there is a constraint connecting these three (real) variables. Each K-orbit is characterized by two numbers: the value of δ and that of the invariant $\beta^2 - \gamma^2 - 2\delta\alpha$. The K-orbits are of two broad types.

The K-orbits of $\hat{P}(1, 1)$ for the case $\delta = 0$ plainly coincide with the K-orbits of $P(1, 1)$. A complete listing of the latter may be found in a recent contribution of Ali and Mueller.¹² The representations constructed from these orbits coincide with the known⁴ faithful representations of $P(1, 1)$. We do not discuss these orbits any further.

Consider now the K-orbits corresponding to $\delta \neq 0$. An orbit is characterized by δ and the constant $c = \beta^2 - \gamma^2 - 2\delta\alpha$. The point $(\alpha_0 = -c/2\delta, 0, 0, \delta) \equiv \delta E^* + \alpha_0 N^*$ belongs to the orbit. Indeed, we can reach the typical point $(\alpha = -(c + \gamma^2 - \beta^2)/2\delta, \beta, \gamma, \delta)$ on the orbit from the point $(\alpha_0, 0, 0, \delta)$ under the action (3.11) of the group element $(0, \eta, u, 0)$ with $\eta = -\gamma/\delta$ and $u = \beta/\delta$. Thus the constant c plays no essential role and we may simply characterize an orbit by the point $(\alpha, 0, 0, \delta)$ belonging to it. Consider now the stability group H of the orbit i.e., of the representative point $(\alpha, 0, 0, \delta)$. From eq. (3.11) we get $H = \{(z, 0, 0, \omega)\} \equiv \mathbb{R} \otimes \mathbb{R}$. From the group multiplication law (2.6), it follows that every element of $\hat{P}(1, 1)$ admits the decomposition

$$(z, \eta, u, \omega) = (z, 0, 0, \omega)(0, \eta', u', 0) \quad (3.12)$$

where $\eta' = \eta \cosh z - u \sinh z$ and $u' = u \cosh z - \eta \sinh z$. Thus η and u may be chosen to (globally) parametrize the homogeneous space of right cosets of H in $\widehat{P}(1, 1)$. Plainly, the coset space is $\mathbb{R}^2$. The K -orbit is homeomorphic to the coset space since $\widehat{P}(1, 1)$ is locally compact. Thus the K -orbit has trivial de-Rham cohomology. In particular, the Konstant-Kirillov symplectic two-form on the K -orbit is *exact* (the conclusion remains valid for $\delta = 0$ orbits).¹²

From the above it would follow, via an application of the Auslander-Konstant theorem, that $\widehat{P}(1, 1)$ is a group of type 1, provided the *space of orbits* can be shown to possess the T_0 separability property. As for the latter requirement, we note that each K -orbit is parametrized by the pair (α, δ) of real numbers (or equivalently by the pair (c, δ) of real numbers) and hence the space of orbits is the plane $\mathbb{R}^2$ which has T_0 separability property. The conclusion, once again, is that $\widehat{P}(1, 1)$ belongs to type 1.

In the Kirillov method, a vital role is played by the admissible subalgebras corresponding to an orbit. We proceed to construct these. Let G be a Lie group, $\text{Lie}(G)$ its Lie algebra, Ω a K -orbit of G and $\tilde{\ell}$ a point belong to Ω . A sub-Lie algebra A of $\text{Lie}(G)$ is said to be *admissible* (or *polarizing*) if the following conditions are satisfied:¹³

- (1) $\tilde{\ell}([X, Y]) = 0$ for arbitrary X and Y belonging to A ; here, as before, $\tilde{\ell}(X) \equiv (\tilde{\ell}, X)$. The algebra A is said to be subordinate to the point $\tilde{\ell}$;
- (2) the co-dimension of A in $\text{Lie}(G)$ (that is the difference $\dim \text{Lie}(G) - \dim A$) is one-half the dimension of Ω ;
- (3) $\tilde{\ell} + A^\perp \subset \Omega$; here $A^\perp$ is the set of all points F belonging to the dual space of $\text{Lie}(G)$ for which $F(X) = 0$ for all $X \in A$; this is Pukanszky's condition.

We select the point $\tilde{\ell} = (\alpha, 0, 0, \delta)$ on the orbit. Then we have two admissible subalgebras A_+ and A_- ; A_+ is generated by N, P_+ and E while A_- is generated by N, P_- and E ; here $P_\pm = H \pm P$. First, these are subalgebras since $[N, P_\pm] = \pm P_\pm$. The condition (1) above is checked easily.

Thus for A_+ , let $X = x_1 N + x_2 P_+ + x_3 E$ and $Y = y_1 N + y_2 P_+ + y_3 E$, then $[X, Y] = (x_1 y_2 - x_2 y_1) P_+$ and $\tilde{\ell}([X, Y]) = 0$ follows from eq. (3.9). Now $\dim A_{\pm} = 3$ and $\dim \text{Lie}(\hat{P}(1, 1)) = 4$; thus $\text{codim } A_{\pm} = 1$; moreover $\dim \Omega = 2$ (proved before). Thus condition (2) is satisfied. That condition (3) is fulfilled will be shown in the next section using a different set of group coordinates better suited for this purpose.

4. Light Cone Coordinates.

As we saw above, the basis in our Lie algebra given by the basis vectors N, P_+, P_- and E , will be more convenient for the discussion to follow. Here $P_{\pm} = H \pm P$ and we may note

$$[N, P_{\pm}] = \pm P_{\pm}, \quad [P_-, P_+] = 2E \quad (4.1)$$

Note the existence of a sequence of ideals generated by $E, (E, P_+), (E, P_+, P_-)$ as stated in the penultimate paragraph of section 2. Corresponding to the present choice of the basis in the Lie algebra, there is a coordinate system for the group manifold. Let us write the typical group element (z, x, y, ω) in the fashion

$$(z, x, y, \omega) = e^{\omega E} e^{x P_+} e^{y P_-} e^{z N} \quad (4.2)$$

The new coordinates are related to the old by $x + y = \eta, x - y = u$; the parametrization of ω above is simply related to that in eq. (3.2), via the Baker-Hausdorff formula (see below). The group multiplication law in these coordinates is given by

$$(z, x, y, \omega)(z', x', y', \omega') = (z + z', x + e^{z'} x', y + e^{-z'} y', \omega + \omega' + 2e^z y x') \quad (4.3)$$

Note the expression for the inverse

$$(z, x, y, \omega)^{-1} = (-z, -e^{-z} x, -e^z y, 2xy - \omega) \quad (4.4)$$

The co-adjoint action in the present coordinate system may be computed exactly as before. Let the typical point ℓ on the dual space be

$$\ell = \delta E^* + \tilde{\beta} P_+^* + \tilde{\gamma} P_-^* + \alpha N^* \quad (4.5)$$

Then under the action of (z, x, y, ω) , $(\delta, \tilde{\beta}, \tilde{\gamma}, \alpha)$ go over to $(\delta', \tilde{\beta}', \tilde{\gamma}', \alpha')$, where

$$\delta' = \delta, \quad \tilde{\beta}' = e^{-z} \tilde{\beta} - 2\delta y, \quad \tilde{\gamma}' = e^z \tilde{\gamma} + 2\delta x \quad (4.6)$$

$$\alpha' = \alpha + x e^{-z} \tilde{\beta} - y e^z \tilde{\gamma} - 2\delta xy$$

and $\tilde{\beta}\tilde{\gamma} - 2\delta\alpha$ is an invariant of the orbit. Indeed, a comparison of eq. (4.5) with eq. (3.7) shows $\tilde{\beta} = \beta + \gamma$, $\tilde{\gamma} = \beta - \gamma$. Further, let us call the parameter ω in eq. (3.2) as $\bar{\omega}$ and ω be as in eq. (4.2); then $\bar{\omega} = \omega - xy$. From these observations and the fact that $\bar{\omega}$ does not appear in eq. (3.11), eq. (4.6) now follows easily from eq. (3.11). As before, the point $(\alpha, 0, 0, \delta) \equiv \alpha N^* + \delta E^*$ belongs to the K-orbit, and the subalgebras $A_{\pm}$ with co-dimension one are subordinate to this point. We may now verify Pukanszky's condition. The annihilators are $A_+^{\perp} = \{\xi P_-^*\}$, ξ real; $A_-^{\perp} = \{\rho P_+^*\}$, ρ real. Consider the case A_- . Then $(\alpha, 0, 0, \delta) + A_-^{\perp}$ is the set $\{(\alpha, \rho, 0, \delta)\}$ as ρ ranges over reals. The question is do these points belong to the K-orbit? The answer is yes, since a point $(\alpha, \rho, 0, \delta)$ may be reached from the point $(\alpha, 0, 0, \delta)$ on the orbit, under the action (4.6) of the group element (z, x, y, ω) with $z = x = \omega = 0$ and $y = -\rho/2\delta$. One similarly verifies the condition for the subalgebra A_+ . Thus, Pukansky's condition stands satisfied and the proof of the fact that $A_{\pm}$ are *admissible* subalgebras is now complete.

5. Representations.

In the Kirillov theory, the group representations are constructed as follows. One selects a point $\tilde{\ell}$ on the co-adjoint orbit and an accompanying admissible subalgebra A . Let K denote the subgroup corresponding to A . The point $\tilde{\ell}$ allows one to construct *one-dimensional* representations of K (since A is subordinate to $\tilde{\ell}$, it is *effectively* abelian on $\tilde{\ell}$). The group representations are the induced representations; induced from the one-dimensional representations of subgroup K .

We may note the following general features of the Kirillov theory. First, the precise choice of $\tilde{\ell}$ plays no role; one can select any point on the orbit depending on one's convenience. Secondly, one may work with any one of the admissible subalgebras, in case there are more than one such, corresponding to a chosen $\tilde{\ell}$. Thus there is a one-to-one correspondence between representations and co-adjoint orbits. *The representation constructed as above will be irreducible.* The representations can be made unitary with respect to a suitable inner product in a Hilbert space.

Let us select the admissible subalgebra A_+ for the chosen point $\tilde{\ell} = (\alpha, 0, 0, \delta)$. Let K denote the subgroup corresponding to A_+ . As before, A_+ is spanned by generators N, P_+ and E and is a real subalgebra; the subgroup K is the set $\{(z, x, 0, \omega)\}$ in the notation of eq. (4.2). Notice K is *not* a normal subgroup; A_+ *not* an ideal. Let $k = (z, x, 0, \omega)$, $k \in K$ and we set

$$\chi(k) = e^{i\delta\omega} e^{i\alpha z} \quad (5.1)$$

Then it follows that χ satisfies the character equation $\chi(k)\chi(k') = \chi(kk')$. Thus χ is a one-dimensional representation of K . We have to carry out the inducing construction from this representation χ .

At this stage, let us recall the basic idea behind the construction of induced representations. Let G be a (Lie) group, K a closed subgroup of G and L_k a (finite dimensional) unitary representation of K in a Hilbert space L . Consider measurable L -valued functions f on G (functions of group parameters) that satisfy the condition

$$f(gk) = L_k^{-1} f(g) \quad ; \quad g \in G, k \in K \quad (5.2)$$

and let

$$(U^L(g_0)f)(g) = \rho_{g_0}(g) f(g_0^{-1}g) \quad (5.3)$$

where $\rho_{g_0}(g)$ is a certain positive function (see below). Then the map $g \rightarrow U^L(g)$ is a representation of G induced by the representation L_k of K . The condition (5.2) is the covariance

condition on left cosets and because of it we shall be able to identify the functions f with certain functions (L -valued) defined on the left coset space $X = G/K$. The $\rho_{g_0}(g)$ appearing in (5.3) is precisely the square root of the Radon-Nikodym derivative $d\mu(g_0^{-1}x)/d\mu(x)$ where $x = \pi(g)$ and $\pi:G \rightarrow X$ is the natural projection. Here μ is a measure on X that is *quasi-invariant* under the action of G . Further restrictions on the functions f will be spelled out as the need arises. In our case L_k is given by χ (see (5.1)) and f are ordinary functions.

We make a remark. The above construction actually gives the group representation on the left coset space. We may similarly construct representations on the right coset space $X_R = K \backslash G$. Now eq. (5.2) will modify to $f(kg) = L_k f(g)$ and eq. (5.3) to $(U^L(g_0)f)(g) = \rho_{g_0}(g) f(g g_0)$. Here $\rho_{g_0}(g)$ is the square root of $d\mu(xg_0)/d\mu(x)$ where x belongs to the right coset and xg_0 is the right action of g_0 on x .

We proceed to construct the group representations. The covariance condition (5.2) becomes

$$f(gk) = \chi^{-1}(k) f(g) \quad (5.4)$$

Let $g = (z', x', y', \omega')$ and $k = (z, x, 0, \omega)$; then using the multiplication law eq. (4.3) we have

$$gk = (z' + z, x' + e^{z'}x, y', \omega' + \omega + 2e^{z'}y'x) \quad (5.5)$$

and eq. (5.4) acquires the form

$$\begin{aligned} & f(z' + z, x' + e^{z'}x, y', \omega' + \omega + 2e^{z'}y'x) \\ &= e^{-i\delta\omega} e^{-i\alpha z'} f(z', x', y', \omega') \end{aligned} \quad (5.6)$$

The solution of (5.6) is shown without difficulty to be given by

$$f(z, x, y, \omega) = e^{-i\alpha z} e^{-i\delta(\omega - 2xy)} F(y) \quad (5.7)$$

where $F(y)$ is a function of y alone. One may check that eq. (5.7) solves equation (5.6). Because of eq. (5.7) the group representations defined on functions f will factor into those defined on functions F ; that is, on the left coset space whose point y is.

Next problem is to figure out the form of the Radon-Nikodym derivative. First, we note that every group element g admits the decomposition $g = sk$ with $k \in K$ and $s \in G$. This is given by

$$(z, x, y, \omega) = (0, 0, y, 0)(z, x, 0, \omega - 2xy) \quad (5.8)$$

and thus the left coset space may be parametrized by y (the element $(0, 0, y, 0) \in G$ goes over to y under natural projection). The left action of the group on the coset space is computed as follows. With g as above we have

$$\begin{aligned} (z, x, y, \omega)(0, 0, y_0, 0) &= (z, x, y + e^{-z} y_0, \omega) \\ &= (0, 0, y + e^{-z} y_0, 0)(z, x, 0, \omega - 2x(y + e^{-z} y_0)) \end{aligned} \quad (5.9)$$

and thus $y_0 \rightarrow e^{-z} y_0 + y$ under the left action of (z, x, y, ω) . Exactly similarly, $y_0 \rightarrow e^z (y_0 - y)$ under the left action of the inverse $(z, x, y, \omega)^{-1}$. Thus the Lebesgue measure dy on the left coset space is not invariant under the group action; it is actually an example of the so called quasi-invariant measure that behaves under group action as $dy \rightarrow e^{-z} dy$ and dy and its transform share the same null set. Here we choose the quasi-invariant Lebesgue measure dy and then the Radon-Nikodym derivative $d\mu(g_0^{-1} y) / d\mu(y)$ is just e^z . Thus $\rho_{g_0}(g)$ is $e^{z/2}$. We are now in a position to write down the explicit form of the group representation. From eq. (5.3) we have

$$\begin{aligned} (U(z, x, y, \omega)f)(z', x', y', \omega') &= e^{z'/2} f((z, x, y, \omega)^{-1}(z', x', y', \omega')) \\ &= e^{z'/2} f(z' - z, e^{-z}(x' - x), e^z(y' - y), \omega' - \omega + 2y(x - x')) \end{aligned} \quad (5.10)$$

where eqs. (4.3) and (4.4) have been utilized in the second step. Eq. (5.10) gives the desired representation; here the functions f satisfy the condition given by eq. (5.7). We proceed to analyze the content of the representation obtained. First, we note

$$\begin{aligned} (U(e^{\omega E})f)(z', x', y', \omega') &\equiv (U(0, 0, 0, \omega)f)(z', x', y', \omega') \\ &= f(z', x', y', \omega' - \omega) = e^{i\delta\omega} f(z', x', y', \omega') \end{aligned} \quad (5.11)$$

where eq. (5.7) was used at the last step. It follows from the above that

$$E = i\delta \quad (5.12)$$

Next we have from eq. (5.10)

$$\begin{aligned} (U(e^{xP_+})f)(z', x', y', \omega') &\equiv (U(0, x, 0, 0)f)(z', x', y', \omega') \\ &= f(z', x' - x, y', \omega') \end{aligned} \quad (5.13)$$

Differentiating the above with respect to x and then setting $x = 0$, we obtain

$$\begin{aligned} (U(P_+)f)(z', x', y', \omega') &= -\frac{\partial}{\partial x'} f(z', x', y', \omega') \\ &= -2i\delta y' f(z', x', y', \omega') \end{aligned} \quad (5.14)$$

where eq. (5.7) was used at the last step. Thus acting on $f(z, x, y, \omega)$, P_+ is the operator of multiplication by $-2i\delta y$. The remaining operators of interest are similarly found. We obtain

$$(U(P_-)f)(z, x, y, \omega) = \left(2i\delta x - \frac{\partial}{\partial y}\right) f(z, x, y, \omega) \quad (5.15)$$

$$(U(N)f)(z, x, y, \omega) = \left(\frac{1}{2} - \frac{\partial}{\partial z} + y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x}\right) f(z, x, y, \omega) \quad (5.16)$$

With the aid of eq. (5.7) the above action of the operators on f may be translated into action on the functions F . The result is

$$(U(P_-)F)(y) = -\frac{d}{dy} F(y) \quad (5.17)$$

$$(U(N)F)(y) = \left(\frac{1}{2} + y \frac{d}{dy} + i\alpha\right) F(y) \quad (5.18)$$

The above expressions satisfy the correct commutation relations of operators N , P_+ , P_- and E , as may be verified.

We should finally consider the restrictions on the function F . From (5.17) we see that for *finite* translation by y to be defined, the function F must be infinitely differentiable; that is, F is an analytic function of the real variable y . Next, we shall restrict F to be square-integrable with respect to the Lebesgue measure dy on the left coset space. The set of all such functions, which may be taken to be real-valued, then constitute a separable Hilbert space H with respect to the inner product

$$(F_1, F_2) = \int F_1(y) F_2(y) dy \quad (5.19)$$

Our representation is then unitary in H , and is strongly continuous (continuous in the strong topology of H); it is irreducible because of reasons explained before. This concludes our discussion of the representations of $\widehat{P}(1, 1)$.

6. Projective representations of the $1 + 1$ Poincaré Group.

The non-trivial projective unitary irreducible representations of the Poincaré group $P(1, 1)$ may now be obtained from the representations of the centrally-extended group $\widehat{P}(1, 1)$ constructed in the preceding section 5.

In the light-cone coordinates the typical element of the Poincaré group $P(1, 1)$ is written in the fashion (z, x, y) and the multiplication law is

$$(z, x, y)(z', x', y') = (z + z', x + e^z x', y + e^{-z} y') \quad (6.1)$$

as follows from eq. (2.2). The desired representations of $P(1, 1)$ follow from eqs. (5.10) and (5.7) upon setting $\omega = \omega' = 0$. Defining $f(z, x, y) \equiv f(z, x, y, 0)$ we thus have

$$(U(z, x, y)f)(z', x', y') = e^{z/2} e^{2i\delta(x' - x)y} f(z' - z, e^{-z}(x' - x), e^z(y' - y)) \quad (6.2)$$

The expressions for the generators N, P_+, P_- of the Lie algebra are exactly the same as those in section 5. The above is a projective representation of $P(1, 1)$. Indeed, from (6.2) we compute

$$(U(0, 0, y)U(0, x, 0)f)(z', x', y') = e^{2i\delta x'y} f(z', x' - x, y' - y) \quad (6.3)$$

On the other hand, from (6.1) and (6.2), it follows

$$\begin{aligned} (U((0, 0, y)(0, x, 0))f)(z', x', y') &= (U(0, x, y)f)(z', x', y') \\ &= e^{2i\delta(x' - x)y} f(z', x' - x, y' - y) \end{aligned} \quad (6.4)$$

and thus

$$U(0, 0, y)U(0, x, 0) = e^{2i\delta xy} U((0, 0, y)(0, x, 0)) \quad (6.5)$$

which shows that our representation is projective.

7. Concluding remarks.

The faithful representations of $P(1, 1)$ corresponding to time-like and light-like representations describe the states of a massive or massless particle in one spatial dimension. The projective representations of $P(1, 1)$ clearly do not admit a free-particle interpretation; the intriguing question is if they possess any physical application. Unfortunately, we have not found answer to this question, thus far. On the other hand, the group $\hat{P}(1, 1)$ does seem to possess physical utility. A *gauge* theory built upon $\hat{P}(1, 1)$ was shown by Cangemi and Jackiw¹⁴ to be efficacious in analyzing the structure of 1 + 1 dimensional "black-hole" models.

The group $\hat{P}(1, 1)$ is closely related to two other groups of interest in physics and it may not be out of place to make a few remarks concerning these relationships. The group $P(1, 1)$ is related to $E(2)$ – the Euclidean group on the plane. Indeed, by replacing the $1 + 1$ Lorentz transformation by a planar rotation we pass from $P(1, 1)$ to $E(2)$. The central extension of these two groups are similarly related. The central extension $\hat{E}(2)$ of $E(2)$ by $\mathbb{R}$ is locally isomorphic with (shares an isomorphic Lie algebra with) the quantum harmonic oscillator group; the difference between the two at the global level resides entirely on the range of one of the group parameters. The parameter of rotation in $\hat{E}(2)$ is an angle that ranges from 0 to 2π ; while the corresponding parameter for the oscillator group — the time parameter conjugate to the Hamiltonian — ranges over the real line. The representations of the oscillator group were constructed by Streater¹⁵ a long time ago, using the Kirillov (actually a generalization of the original Kirillov method involving complex Lie algebras) as well as by the Mackey method. The close parallelism between Streater's treatment and that of the present paper should be noted. It also follows that the projective representations of $E(2)$ can be obtained very simply from the results of Streater.¹⁵ Actually, the projective representations of $E(2)$ are useful in the problem of electrons in a uniform magnetic field. An elegant treatment of this problem has been given by Divakaran and Rajagopal.¹⁶

APPENDIX. The 1 + 1 Galilean Group.

The 1 + 1 Galilean group is the group under composition of the set of spacetime transformations

$$x' = x - vt + u \quad (\text{A.1})$$

$$t' = t + \eta$$

The typical group element is (v, η, u) with the multiplication law

$$(v, \eta, u)(v', \eta', u') = (v + v', \eta + \eta', u + u' - \eta'v) \quad (\text{A.2})$$

The group is isomorphic with the Heisenberg-Weyl group, the isomorphism being explicitly

$$(v, \eta, u) \leftrightarrow \begin{pmatrix} 1 & -v & u \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{A.3})$$

The corresponding Lie algebra is a three-dimensional real vector space in which one can choose basis vectors N, P, H that satisfy the commutation relations

$$[N, H] = P, \quad [N, P] = 0, \quad [H, P] = 0 \quad (\text{A.4})$$

We consider the problem of central extension of the Galilean group. It has two equivalence classes of 2-cocycles. We can choose the representative 2-cocycles (one from each equivalence class) to be

$$m_1(r, r') = \frac{1}{2} v^2 \eta' - v u', \quad m_2(r, r') = \frac{1}{2} v \eta'^2 - \eta u' + v \eta \eta' \quad (\text{A.5})$$

where $r \equiv (v, \eta, u)$ and $r' \equiv (v', \eta', u')$. That the above expressions satisfy the cocycle identities and are non-trivial (not coboundaries) may be checked along the same lines as in ref. 1. We may thus construct the universal centrally extended group as follows. The typical group element is $(r, \theta_1, \theta_2) \equiv (v, \eta, u; \theta_1, \theta_2)$ with the multiplication law

$$(r, \theta_1, \theta_2) (r', \theta'_1, \theta'_2) = (r + r', \theta_1 + \theta'_1 + m_1(r, r'), \theta_2 + \theta'_2 + m_2(r, r')) \quad (\text{A.6})$$

where θ_1 and θ_2 are a pair of real variables. It is useful to note the corresponding Lie algebra central extension, which is given by

$$[N, H] = P, \quad [N, P] = A, \quad [H, P] = B \quad (\text{A.7})$$

where A and B are central charges (commute with every generator). Note P is no longer central.

The centrally-extended group, as defined above, is connected and simply-connected. Furthermore, it is a *nilpotent* Lie group (of dimension five), as is easy to check from eq. (A.6) or from eq. (A.7). At this stage we recall the fact that all nilpotent Lie groups of dimension less than six were classified by Dixmier¹⁷ in 1959. Thus our centrally-extended group must be one in his classification. In fact, *our group is the group $\Gamma_{5,4}$ in the Dixmier classification.* The complete system of unitary, irreducible representation of $\Gamma_{5,4}$ has been given in ref. 17. Lifting the latter to projective representations of the Galilean group is trivial (be setting $\theta_1 = \theta_2 = 0$) and there remains nothing more to discuss.

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