DEFORMATIONS OF COMPACT QUANTUM GROUPS
VIA RIEFFEL'S QUANTIZATION

Shuzhou WANG

Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Juillet 1995

IHES/M/95/70
Deformations of Compact Quantum Groups via Rieffel’s Quantization

Shuzhou Wang

IHES, 35, route de Chartres, 91440–Bures-sur-Yvette, France
E-mail address: SZWANG@ihes.fr

Abstract

It is shown that compact quantum groups containing torus subgroups can be deformed into new compact quantum groups under Rieffel’s quantization. This is applied to showing that the two classes of compact quantum groups $K_q^n$ and $K_q$ studied by Levendorskii and Soibelman are strict deformation quantization of each other, and that the quantum groups $A_u(m)$ has many deformations.

1 Introduction

This paper provides affirmative answers to the following two questions of Rieffel’s: (1) Are the Drinfeld (algebraic) twistings $K_q^n$ of the quantum groups $K_q$ studied in [18, 10, 11] strict deformation quantizations of $K_q^n$? (2) Can the quantum groups $A_u(m)$ constructed in [23, 24] be deformed? The key to answering these questions is a result of independent interest, in the spirit of [15], on deformations of arbitrary compact quantum groups (instead of just compact groups as treated there).

We describe the results of this paper in some more detail. Let $A$ be a Woronowicz Hopf $C^*$-algebra in the sense of [28, 1, 23, 24], whose coproduct being denoted by $\Phi$. We will also call it a compact quantum group, referring to its dual object (cf. [24]). Suppose that some abelian Lie group $T$ is a subgroup of the quantum group $A$. This means that there is a surjective homomorphism $\pi$ of $C^*$-algebras from $A$ to $C(T)$ preserving the coproducts (see [23, 24]). For any element $h$ in $T$, denote by $E_h$ the corresponding evaluation functional on $C(T)$. Assume that $\eta$ is a continuous homomorphism from a vector space Lie group $\mathbb{R}^n$ to $T$, where $n$ is allowed to be different from the dimension of $T$. Define an action $\alpha$ of $\mathbb{R}^n := \mathbb{R}^n \times \mathbb{R}^n$ on the $C^*$-algebra $A$ as follows:

$$\alpha_{(s,v)} = \lambda_{\eta(s)}\rho_{\eta(v)},$$

where

$$\lambda_{\eta(s)} = (E_{\eta(-s)}\pi \otimes id)\Phi, \quad \rho_{\eta(v)} = (id \otimes E_{\eta(v)}\pi)\Phi,$$

$id$ being the identity map on $A$. For any skew-symmetric operator $S$ on $\mathbb{R}^n$, one may apply Rieffel’s quantization procedure [14] for the action $\alpha$ above to obtain a deformed $C^*$-algebra
$A_J$, where $J = S \oplus (-S)$. The family $A_{hJ}$ ($h \in \mathbb{R}$) is a strict deformation quantization of $A$ (see Chapter 9 of [14]). Our main result for answering Rieffel's questions is the following theorem.

1.1. Theorem. (See 3.9) The deformation $A_J$ is also a compact quantum group (namely Woronowicz Hopf $C^*$-algebra) containing $T$ as a (quantum) subgroup; $A_J$ is a compact matrix quantum group if and only if $A$ is.

The construction of the action $\alpha$ above is a rather straightforward reformulation in the more general setting of the construction in [15], and the theorem above is a generalization of the main theorem there. However, the proof of the above theorem is quite different from that of the corresponding theorem in [15]. In [15], Rieffel works with the algebra of smooth functions on the undeformed compact Lie group to obtain the coproduct on the deformed algebra, which is remarkable in that it gives the first example of a deformation quantization of the entire algebra of smooth functions on a smooth manifold. In the present setting, since it is not clear what it should mean by the algebra of smooth functions on a compact (matrix) quantum group, we work with the Krein algebra $A$ of $A$ (in the sense of [23, 26]) to obtain the quantum group structure on $A_J$. We refer the reader to [16] for a generalization of the construction in [15] to the case of non-compact Lie groups, and to [8] for deformations of non-compact Lie groups by using techniques of [7], which are different from those used by Rieffel.

For a simple compact Lie group $K$, two families of compact quantum groups $K_q$ and $K_q^w$ are introduced in [18, 10, 11] based on the construction of the Drinfeld-Jimbo quantum groups [3, 6]. The maximal torus $T$ of $K$ is still a subgroup of both the quantum groups $K_q$ and $K_q^w$. Applying the construction above (with an appropriate skew-symmetric operator $S$), we have the following result, which answers Rieffel's first question.

1.2. Theorem. (See 4.2) The compact quantum groups $K_q$ and $K_q^w$ are strict deformation quantization of each other in the sense of Rieffel [14].

Applying the above construction to the quantum groups $A_{h}(m)$ constructed in [24] (since they contain many abelian Lie subgroups) yields an answer to Rieffel's second question.

1.3. Theorem. (See 5.1) The quantum groups $A_{h}(m)$ can be deformed under Rieffel's quantization. The deformed quantum groups $A_{h}(m)$ are quantum subgroups of $A_{h}(m)$. The quantum groups constructed in [15] are quantum subgroups of $A_{h}(m)$ for suitable $m$.

This version of the paper supersedes the version released earlier.

The author would like to thank Professor Rieffel for drawing his attention to the problems in this paper, and for helpful discussions.

2 Preliminaries

For convenience of the reader, we recall in this section some basic definitions. For use in later sections of this paper, we also collect a few elementary results concerning actions of
groups on Hopf C*-algebras. For a positive integer \( d \), \( M_d = B(\mathbb{C}^d) \) denotes the C*-algebra of \( d \times d \) matrices over the complex numbers \( \mathbb{C} \).

2.1. Definition. (cf. [28, 1, 23, 24]) A Woronowicz Hopf C*-algebra (or a compact quantum group) is a unital C*-algebra \( A \) together with a dense *-subalgebra \( \mathcal{A} \) generated by \( u_{ij}^p \) (where \( p \in \mathbb{N} \) and \( i, j \in \{1, \ldots, d_p\} \), and \( \mathbb{N} \) is an index set), a C*-homomorphism \( \Phi : A \to A \otimes A \), and a linear algebra-antihomomorphism \( \kappa : A \to A \), such that,

1. The matrix \( u^p = (u_{ij}^p) \) is a unitary (or equivalently an invertible) element of \( M_{d_p} \otimes A \), for all \( p \in \mathbb{N} \);
2. For \( p \in \mathbb{N} \), and \( i, j \in \{1, \ldots, d_p\} \), \( \Phi(u_{ij}^p) = \sum_{k=1}^{d_p} u_{ik}^p \otimes u_{kj}^p \);
3. For \( a \in \mathcal{A} \), and \( p \in \mathbb{N} \), \( \kappa(\kappa(a^*)^*) = a \), and \( (id \otimes \kappa)(u^p) = (u^p)^{-1} \).

We denote the above Woronowicz Hopf C*-algebra simply by \( A \). For a Woronowicz Hopf C*-algebra \( A \), we will also call it a compact quantum group, referring to the dual object \( G \) of the Woronowicz Hopf C*-algebra \( A = C(G) \) (cf. [23, 24]). The algebra \( \mathcal{A} \) is the generic example of Krein algebras [26, 23].

2.2. Let \( A \) and \( B \) be compact quantum groups. A homomorphism from the quantum group \( B \) to the quantum group \( A \) is defined to be a unital C*-algebra homomorphism from \( A \) to \( B \) preserving the coproducts. The quantum group \( B \) is called an embedded quantum subgroup (or simply quantum subgroup) of \( A \) if there exists a surjective C*-algebra homomorphism from \( A \) to \( B \) preserving the coproducts (cf [23, 24]).

2.3. We define a Hopf C*-algebra (see e.g. [1]) to be a C*-algebra \( A \) (unital or not) such that there is a nondegenerate (see [1]) C*-algebra homomorphism \( \Phi \) from \( A \) to the multiplier algebra \( M(A \otimes A) \) satisfying the coassociativity condition: \( (id \otimes \Phi)\Phi = (\Phi \otimes id)\Phi \).

A left invariant mean (resp. right invariant mean) on a Hopf C*-algebra \( A \) is defined to be a state \( \phi \) on \( A \) satisfying

\[ (\beta \otimes \phi)\Phi = \phi \quad \text{resp.} \quad (\phi \otimes \gamma)\Phi = \phi \]

for any state \( \psi \) on \( A \). If \( \phi \) is both a left and right invariant mean, we simply call it an invariant mean. On every compact quantum group, there exists a unique invariant mean, which is called the Haar measure of the quantum group ([28, 23, 24, 21]).

2.4. Notation. In the following, let \( A \) be a compact quantum group such that the space \( X(A) \) of non-zero *-homomorphisms from \( A \) to the algebra \( \mathbb{C} \) of complex numbers is non-empty. Then \( X(A) \) is a compact group and it is a compact Lie group if \( A \) is a compact matrix quantum group in the sense of [28]. Consequently the counit of \( A \) is continuous. The group \( X(A) \) is called the maximal compact subgroup of \( A \) (see [23, 24]).

Let \( x, y \in X(A) \). Define endomorphisms \( \lambda_x, \rho_x \) of the C*-algebra \( A \) by

\[ \lambda_x = (E_{x^{-1}} \otimes id)\Phi, \quad \rho_y = (id \otimes E_y)\Phi, \]

where \( E_x \) is identified the homomorphism \( x \) from \( A \) to \( \mathbb{C} \) and \( x^{-1} \) is the inverse of \( x \) in the group \( X(A) \). In the notation of [28, 23],

\[ \lambda_x(a) = a \ast E_{x^{-1}}, \quad \rho_y(a) = E_y \ast a \]
for $a \in A$. Since $A$ is a bimodule over the algebra $A^*$ of continuous functionals of $A$ (in the sense of [23]) and $X(A)$ is a group, we see that both $\lambda_x$ and $\rho_y$ are automorphisms of the C*-algebra $A$ commuting with each other, and that $x \rightarrow \lambda_x$ and $y \rightarrow \rho_y$ define strongly continuous actions of the compact group $X(A)$ on $A$. In other words, we have C*-dynamical systems $(A, X(A), \lambda)$ and $(A, X(A), \rho)$. The actions $\lambda$ and $\rho$ clearly commutes with each other.

2.5. Proposition. Let $x \in X(A)$, and let $(a_{kl})$ be a finite dimensional representation of the quantum group $A$ (not to be confused with the representation of the algebra $A$). Then $\lambda_x(a_{ij})$ (resp. $\rho_x(a_{ij})$) is a linear combination of the coefficients $a_{kj}$ (resp. $a_{il}$), with $k$ (resp. $l$) varying.

Proof. The proof is standard, using $\Phi(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$.

Q.E.D.

2.6. Proposition. We have $(id \otimes \lambda_x)\Phi = (\rho_x \otimes id)\Phi$ for any $x \in X(A)$.

Proof. This follows immediately from the coassociativity of $\Phi$.

Q.E.D.

2.7. Proposition. We have the equalities

$$\Phi \lambda_x = (\lambda_x \otimes id)\Phi, \quad \Phi \rho_x = (id \otimes \rho_x)\Phi,$$

for any $x \in X(A)$

Proof. Let $(a_{ij})$ be any finite dimensional representation of the quantum group $A$, so $\Phi(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$. Then

$$\Phi \lambda_x(a_{ij}) = \sum_{k,l} (E_{x-1} \otimes id \otimes id)(a_{ik} \otimes a_{kl} \otimes a_{lj})$$
$$= (E_{x-1} \otimes id \otimes id)(\Phi \otimes id)(a_{ij})$$
$$= (E_{x-1} \otimes id)(\Phi \otimes id)(\Phi(a_{ij}))$$

From this we see that $\Phi \lambda_x = (\lambda_x \otimes id)\Phi$.

Similarly we have $\Phi \rho_x = (id \otimes \rho_x)\Phi$.

Q.E.D.

2.8. Proposition. The equalities

$$\lambda_x \kappa = \kappa \rho_x, \quad \rho_x \kappa = \kappa \lambda_x$$

are valid as maps on $A$ for any $x \in X(A)$.

Proof. Since $E_{x-1} \kappa = E_x$, using $\Phi \kappa = \sigma(\kappa \otimes \kappa)\Phi$ (see [28, 23]), we have

$$\lambda_x \kappa = (E_{x-1} \otimes id)\Phi \kappa = (E_{x-1} \otimes id)\sigma(\kappa \otimes \kappa)\Phi$$
$$= (id \otimes E_{x-1})(\kappa \otimes \kappa)\Phi = (\kappa \otimes E_{x-1})\Phi$$
$$= \kappa (id \otimes E_x)\Phi = \kappa \rho_x,$$

where $\sigma$ is the flip map on $A \otimes A$ sending $a \otimes b$ to $b \otimes a$. Similarly, we have $\rho_x \kappa = \kappa \lambda_x$.

Q.E.D.
2.9. Remark. We note that the actions $\lambda$ and $\rho$ preserve the Haar measure of the quantum group. This is an immediate consequence of the definition of these actions and invariance properties of the Haar measure.

2.10. If $A$ is a (not necessarily unital) Hopf $C^*$-algebra such that the space $X(A)$ of non-zero $*$-homomorphisms from $A$ to $C$ is non-empty, then $X(A)$ is a locally compact semigroup, and the observations in 2.4, 2.6, 2.9 are still valid if we replace $X(A)$ by a subgroup of $X(A)$, and 2.7 and 2.8 are also valid for $x$ in subgroups of $X(A)$ if $A$ has a continuous antipodal map.

2.11. We now briefly recall Rieffel’s deformation quantization [14] from actions of the abelian Lie group $V = \mathbb{R}^d$ on $C^*$-algebras (not necessarily commutative!). Let $(A, V, \alpha)$ be a $C^*$-dynamical system. Denote by $A^\infty$ the algebra of smooth vectors of the action $\alpha$ endowed with the Frechet topology coming from the action of the Lie algebra of $V$ on $A^\infty$. Let $J$ be any skew-symmetric operator on $V$. For $a, b \in A^\infty$, define $a \times_J b$ by

$$a \times_J b = \int_V \int_V \alpha_J^*(a)\alpha_J(b)e(u \cdot v),$$

where $u \cdot v$ is the inner product of $u$ and $v$ on $V$ and $e$ is the function $e(t) = \exp(2\pi it)$ for a real number $t$. Then $A^\infty$ is an associative $*$-algebra under the product $\times_J$ and the involution of $A$ restricted to $A^\infty$. Let $S^A$ be the space of $A$-valued smooth functions on $V$ such that the products of their derivatives with any complex valued polynomials on $V$ are bounded under the evident super norm of $S^A$. Then $S^A$ is a pre-Hilbert (right) $A$-module with $A$-valued inner product defined by

$$<f, g>_A = \int_V f(v)^*g(v)dv$$

for $f, g \in S^A$. For each $a \in A$, define an operator $L_a$ on $S^A$ by

$$L_a(f)(x) = \int_V \int_V \dot{a}(x + Ju)f(x + v)e(u \cdot v)$$

for $f \in S^A$, where $\dot{a}(x) = \alpha_J^*(a)$ for $x \in V$. Then $L_a$ is a bounded operator having an adjoint on the pre-Hilbert $A$-module $S^A$, and $a \mapsto L_a$ is a $*$-representation of $(A^\infty, \times_J)$ into the $C^*$-algebra of bounded operators on $S^A$. Now define

$$\|a\|_J = \|L_a\|.$$  

Then $\| \cdot \|_J$ is a pre-$C^*$-norm on the algebra $A^\infty$ endowed with the new product $\times_J$. The completion of this pre-$C^*$-algebra is denoted by $A_J$. Therefore for every quadruple $(A, V, \alpha, J)$, Rieffel associates a deformed $C^*$-algebra $A_J$ [14]. The family $A_{hJ}$ ($h \in \mathbb{R}$) is a strict deformation quantization of $A$.

3 Deformations of Compact Quantum Groups

The goal of this section is to prove theorem 1.1. We first fix the set-up of the section.

3.1. Assumptions and Notation. Our standing assumptions throughout this section
are as follows: $A$ is any compact quantum (with the notation as in 2.1), $T$ is a subgroup of the quantum group $A$, where $T$ is a compact abelian Lie group. So there is a surjective homomorphism $\pi$ of $C^*$-algebras from $A$ to $C(T)$ preserving the coproducts. For any element $h$ in $T$, denote by $E_h$ the corresponding evaluation functional on $C(T)$. Let $\eta$ be a continuous homomorphism from a vector space Lie group $\mathbb{R}^n$ to $T$, and $S$ a skew-symmetric operator on $\mathbb{R}^n$. The letters $m, I, id, \Phi, \varepsilon, \kappa$ will denote, respectively, the product map from $A \otimes A$ to $A$, the unit of $A$, the identity map on $A$, the coproduct of $A$, the counit of $A$, the coincidence of $A$.

We remark that the assumptions above on $A$ are equivalent to the assumptions that the space $X(A)$ of non-zero $*$-homomorphism from $A$ to the algebra of complex numbers is a compact group and that there is a continuous injective group homomorphism from $T$ to $X(A)$. In particular the counit of $A$ is continuous. These assumptions are fulfilled by all the nontrivial compact quantum groups constructed so far (see e.g. sections 4 and 5 in the following).

Put

$$\lambda_{\eta(u)} = (E_{\eta(-u)} \pi \otimes id) \Phi, \quad \rho_{\eta(u)} = (id \otimes E_{\eta(u)} \pi) \Phi.$$ 

Note that the meanings of $\lambda$ and $\rho$ here are slightly different from those in section 2, but this should not cause confusion. The action $\alpha$ of $\mathbb{R}^d := \mathbb{R}^n \times \mathbb{R}^n$ (so $d = 2n$) on the $C^*$-algebra $A$ is defined by

$$\alpha_{(s,u)} = \lambda_{\eta(s)} \rho_{\eta(u)},$$

where $s, u \in \mathbb{R}^n$ (the $u$ here is not to be confused with the $u^a$'s in 2.1). Let $J = S \oplus (-S)$. Applying Rieffel's strict dequantization [14] to the quadruple $(A, \mathbb{R}^d, \alpha, J)$, one obtains a deformed $C^*$-algebra $A_J$.

Our goal in this section is to show that $A_J$ is also a compact quantum group. First we show that the space $A$ is an algebra under the product $\times_J$ and is dense in the $C^*$-algebra $A_J$.

3.2. Proposition. The space $A$ is contained in the space $A^\infty$ of smooth vectors of the action $\alpha$. Under the product and involution of $A_J$, the space $A$ is an involutive algebra. The space $A$ is dense in $A^\infty$ under the Frechet topology of $A^\infty$ and is therefore dense in the $C^*$-algebra $A_J$ under the $C^*$-norm of $A_J$.

Proof. Let $a_{ij} \in A$ be a coefficient of a finite dimensional representation $(a_{kl})$ of the quantum group $A$. Then by proposition 2.5, $a_{ij}(a_{kl})$ is a linear combination of the $a_{kl}$'s. The coefficients in this linear combination are smooth complex valued functions on $\mathbb{R}^d$, because the map $\pi$ sends the Krein algebra $A$ of $A$ to that of $C(T)$ (see [23, 24]) and the Krein algebra of $C(T)$ is contained in the algebra of smooth functions on $C(T)$ (note that $\eta$ is automatically smooth). This proves the first statement of the proposition.

Let $(b_{ij})$ be another finite dimensional representation of the quantum group $A$. By definition, the formula for $a_{ij} \times_J b_{kl}$ is

$$a_{ij} \times_J b_{kl} = \int a_{j(s,u)}(a_{ij})a_{(t,v)}(b_{kl})e(st + uv)$$

$$= \int \lambda_{\eta(s)}(a_{ij})\rho_{\eta(v)}(b_{kl})e(st + uv),$$
which by proposition 2.5, is easily seen to be a linear combination of the products $a_{i_1 j_1} b_{k_1 l_1}$'s, so $a_{i j} \times_J b_{k l}$ is in $\mathcal{A}$. We note however that the above integral is defined in the Frechet space $A^\infty$ and requires the completeness of $A^\infty$ (see [14] for the precise definition of the integral). Since elements of $\mathcal{A}$ are linear combinations of the coefficients of finite dimensional representations of the quantum group $A$ in virtue of the Peter-Weyl theorem for compact quantum groups (cf. [28, 23, 24]), we see that $\mathcal{A}$ is indeed an algebra under the product $\times_J$. Since $\mathcal{A}$ is an involutive algebra under the product of $A$ and that the action $\alpha$ preserves its involution, we also see from the above that $\mathcal{A}$ is an involutive algebra.

Let $\iota : A^\infty \rightarrow A$ be the inclusion map from the Frechet space $A^\infty$ to the Frechet space $A$, where the Frechet topology on $A$ is given by the $C^*$-norm thereon. Then $\iota$ is equivariant and continuous, where $A^\infty$ and $A$ are endowed with the actions $\beta := \alpha|_{A^\infty}$ and $\alpha$, respectively. Hence we can invoke proposition 1.1 of [15], with $(A^\infty, \beta)$ and $(A, \alpha)$ here being the $(B, \beta)$ and $(A, \alpha)$ there — the conclusion of that proposition is still true if we replace $B^\infty = (A^\infty)^\infty$ there by $\mathcal{A}$, because $(A^\infty)^\infty = A^\infty$ and for $b \in \mathcal{A}$ the element $\beta_\phi(b)$ defined in the proof there by

$$\beta_\phi(b) = \int_V \int_{\mathbb{T}} \phi(s, u) \alpha_t(b) \rho_{\eta(u)}(b) = \int_V \int_{\mathbb{T}} \phi(s, u) \lambda_{\eta(u)}(b) \rho_{\eta(u)}(b)$$

is in $\mathcal{A}$ because of proposition 2.5. This proves the last statement of the proposition. Q.E.D.

We will denote the algebra $(\mathcal{A}, \times_J)$ simply by $\mathcal{A}_J$. Now we define the coproduct and the coinverse on $\mathcal{A}_J$ and verify the axioms of definition 2.1.

To proceed, as in [15], let

$$C = \{ F \in A \otimes A \mid (\rho_h \otimes \text{id}) F = (\text{id} \otimes \lambda_{h^{-1}}) F, \text{ for all } h \in T \}.$$ 

Then by 2.6, $\Phi(A)$ is contained in $C$. Let $\Psi$ denote $\Phi$ viewed as a homomorphism from $A$ to $C$. Define an action $\bar{\beta}$ of $\mathbb{R}^d$ on $C$ by

$$\bar{\beta}_{\lambda, \rho}(b) = \lambda_b \otimes \rho(b).$$

Note that $\bar{\beta}$ is initially defined on $A \otimes A$, but since $\lambda$ and $\rho$ commute with each other, we see that it can be restricted to an action of $V$ on $C$. By 2.7, we see that the map $\Psi$ is equivariant for the actions $\alpha$ on $A$ and $\beta$ on $C$.

Similarly, define an action $\gamma$ of $\mathbb{R}^d \times \mathbb{R}^d$ on $A \otimes A$ by $\gamma = \alpha \otimes \alpha$. Then $\gamma$ restricts to an action of $\mathbb{R}^{2d}$ on the subalgebra $C$ of $A \otimes A$ because $T$ is abelian and $\lambda$ and $\rho$ commute with each other. Let

$$\delta = J = S \otimes (-S) = S \otimes (-S).$$

As shown in [15], we have $C^*_J = C^d_J$. Let

$$J = \rho \Psi_J,$$

where $\rho$ is the inclusion from $C^*_J$ to $(A \otimes A)^*_J$. Since $(A \otimes A)^*_J = A^*_J \otimes A^*_J$ by 2.2 of [15] and both $\Psi_J$ and $\rho$ are unitary $C^*$-algebra homomorphisms, thus we obtain a unital $C^*$-algebra homomorphism $\Phi_J$ from $A_J$ to $A_J \otimes A_J$.

3.3. Proposition. On the dense subalgebra $A_J$ of $A_J$, we have for each $p \in \mathbb{N}$,

$$\Phi_J(u_{i,j}^p) = \sum_{k=1}^{d_J} u_{ik}^p \otimes u_{kj}^p,$$
Therefore axiom (2) of definition 2.1 is satisfied.

Proof. Recall that on $A^\infty$, we have $\Psi_J = \Psi$ (see [14]), and that $\Psi$ is simply $\Phi$ on $A$. From proposition 3.2, $\mathcal{A}$ is contained in $A^\infty$. The rest is now clear. Q.E.D.

We remark that the coassociativity $(id_J \otimes \Phi_J)\Phi_J = (\Phi_J \otimes id_J)\Phi_J$ of $\Phi_J$ is a consequence of the axiom (2) of definition 2.1, where $id_J$ is the identity map on $A_J$. So the above proposition generalizes theorem 2.3 of [15].

As in [15], we have a surjective homomorphism $\pi_J$ from $A_J$ to $C(T)_J = C(T)$.

3.4. Proposition. The map $\pi_J$ preserves the coproducts. Namely, we have

$$(\pi_J \otimes \pi_J)\Phi_J = \Phi_T \pi_J,$$

where $\Phi_T$ is the coproduct on $C(T)$.

Proof. Since by definition [14], $\pi_J|_{A^\infty} = \pi|_{A^\infty}$, and on the subspace $A_J$ of $A^\infty$,

$$(\pi_J \otimes \pi_J)\Phi_J = (\pi \otimes \pi)\Phi = \Phi_T \pi = \Phi_T \pi_J,$$

by the density of $A_J$ in $A_J$ and the continuity of $\pi_J$, we see that $\pi_J$ preserves the coproducts. Q.E.D.

Let $\epsilon_T$ be the counit of $C(T)$. Define $\epsilon_J := \epsilon_T \pi_J$.

3.5. Proposition. The map $\epsilon_J$ has the property

$$\epsilon_J(u^p_{ij}) = \delta_{ij}$$

for any $u^p_{ij}$.

Proof. This follows immediately from the definition of $\epsilon_J$, because $\epsilon_T \pi = \epsilon$ and $\pi_J|_{\mathcal{A}} = \pi|_{\mathcal{A}}$. Q.E.D.

We remark that as in [15] $\epsilon_J$ satisfies the counit property

$$(id_J \otimes \epsilon_J)\Phi_J = id_J = (\epsilon_J \otimes id_J)\Phi_J$$

on $A_J$ (not just on $A_J$): this follows from the above proposition and proposition 3.2.

Define the map $\kappa_J$ on $A_J$ to be the same as $\kappa$. We can do this because as a vector space $A_J$ is the same as $\mathcal{A}$.

3.6. Proposition. The map $\kappa_J$ is an anti-algebra homomorphism for the product $\times_J$ of $A_J$, and for $a \in A_J$, $\kappa_J(\kappa_J(a^*)^*) = a$.

Proof. Let $a, b \in A$. First we note that

$$\kappa_J(a \times_J b) = \int \int \kappa(\alpha_J(s,u)(a)\alpha_J(t,v)(b))\epsilon(s \cdot t + u \cdot v)$$
even though $\kappa$ is not assumed to be continuous because by proposition 2.5, for fixed $a$ and $b$, there are finite number of finite dimensional representations of the quantum group $A$ such that the expression

$$\alpha_J(s,a) \alpha_J(t,b) e(s \cdot t + u \cdot v)$$

is a linear combination of the products of the matrix coefficients of these representations. Then by proposition 2.8 we have

$$\kappa_J(a \times_J b) = \int \int \kappa(\alpha_J(s,a) \alpha_J(t,b)) e(s \cdot t + u \cdot v)$$

$$= \int \int \kappa(\lambda_{q(s)} \rho_{q(-s)}(a) \lambda_{q(t)} \rho_{q(v)}(b)) e(s \cdot t + u \cdot v)$$

$$= \int \int \kappa(\lambda_{q(t)} \rho_{q(v)}(b)) \kappa(\lambda_{q(s)} \rho_{q(-s)}(a)) e(s \cdot t + u \cdot v)$$

$$= \int \int \lambda_{q(v)} \lambda_{q(t)} \kappa(b) \rho_{q(s)} \lambda_{q(-s)} e(s \cdot t + u \cdot v),$$

which, by proposition 1.13 of [11],

$$= \int \int \rho_{q(-s)} \lambda_{q(s)} \kappa(b) \rho_{q(u)} \lambda_{q(u)} e(s \cdot t + u \cdot v)$$

$$= \kappa_J(b) \cdot_J \kappa_J(a).$$

Now the identity $\kappa_J((a^*J)^*) = a$ is immediate because the involution is not deformed. Q.E.D.

To show that $A_J$ is a compact quantum group, it remains to show that the $u^p$'s are unitary and that $(id_J \otimes \kappa_J) u^p = (u^p)^{-1}$. For this we first show that $\kappa_J$ satisfies the antipodal property on $A_J$. Let $m_J$ be the product map from $A_J \otimes A_J$ to $A_J$.  

3.7. Proposition. On the algebra $A_J$, we have

$$m_J(id_J \otimes \kappa_J) \Phi_J = I_J \epsilon_J = m_J(\kappa_J \otimes id_J) \Phi_J,$$

where $I_J$ is the unit of the algebra $A_J$.

Proof. Let $(a_{ij})$ be a finite dimensional unitary representation of the quantum group $A$, so $\kappa(a_{ij}) = a_{ji}^*$. Then by 2.8 and 2.6 we have

$$m_J(id_J \otimes \kappa_J) \Phi_J(a_{ij}) = m_J(id_J \otimes \kappa_J) \Phi(a_{ij}) = \sum_k a_{ik} \times_J \kappa(a_{kj})$$

$$= \sum_k \int \int m(\alpha_{J(s_a)} \otimes \alpha_{J(u)} \kappa)(a_{ik} \otimes a_{kj}) e(s \cdot t + u \cdot v)$$

$$= \sum_k \int \int m(id \otimes \kappa)(\lambda_{q(s_a)} \rho_{q(-s_u)} \otimes \rho_{q(v)} \rho_{q(t)})(a_{ik} \otimes a_{kj}) e(s \cdot t + u \cdot v)$$

$$= \sum_k \int \int m(id \otimes \kappa)(\lambda_{q(s_a)} \rho_{q(-s_u)} \otimes \rho_{q(t)})(a_{ik} \otimes a_{kj}) e(s \cdot t + u \cdot v),$$

9
which, by 3.1 of [15] and 1.11 of [14], and then 2.8,

\[
\sum_k \int \int m(id \otimes \kappa)(\lambda_{n(S)} \rho_{n(-v)} \otimes \rho_{n(t)})(a_{ik} \otimes a_{kj})\epsilon(s \cdot t + u \cdot v)
\]

\[
= \sum_k \int \int m(id \otimes \kappa)(\lambda_{n(S)} \otimes \rho_{n(t)})(a_{ik} \otimes a_{kj})\epsilon(s \cdot t)
\]

\[
= \sum_{ikr} \int \int E_{n(-S)}(a_{ir})a_{rk}a_{rk}^*E_{n(t)}(a_{rj})\epsilon(s \cdot t)
\]

\[
= \sum_{r} \int E_{n(-S)}(a_{ir})E_{n(t)}(a_{rj})\epsilon(s \cdot t)
\]

\[
= \int \int (E_{n(-S)} : E_{n(t)})\Phi_J(a_{ij})\epsilon(s \cdot t)
\]

\[
= \int \int E_{n(-S)}(a_{ij})\epsilon(s \cdot t) = \int \int E_{n(t)}(a_{ij})\epsilon(s \cdot t)
\]

\[
= E_{n(0)}(a_{ij}) = \epsilon(\pi(a_{ij})) = \epsilon(a_{ij}) = \epsilon_J(a_{ij}).
\]

That is on \(A_J, m_J(id_J \otimes \kappa_J)\Phi_J = I_{fJ}.\)

Similarly \(m_J(\kappa_J \otimes id_J)\Phi_J = I_{fJ} \otimes \Phi_J\) on \(A_J.\) Q.E.D.

3.8. Proposition. For every \(p,\) the matrix \(u^p\) is an unitary element of the \(*\)-algebra \(M_{d_p} \otimes A_J,\) and

\[(id_p \otimes \kappa_J) = (u^p)^{-1},\]

where the inverse \((u^p)^{-1}\) takes place in \(M_{d_p} \otimes A_J.\)

Proof. The argument is contained in the proof of 3.2 of [28]. For reader's convenience we present the detailed proof anyway. The equalities

\[\Phi_J(u^p_{ij}) = \sum_k u^p_{ik} \otimes u^p_{kj}\]

can be written as the equality

\[\sum_{ij} e^p_{ij} \otimes \Phi_J(u^p_{ij}) = \sum_{ijk} e^p_{ij} \otimes u^p_{ik} \otimes u^p_{kj},\]

where \(e^p_{ij}\) is the standard matrix units of \(M_{d_p}.\) Applying the map

\[id_p \otimes m_J(\kappa_J \otimes id_J)\]

to both sides of this equality, and using 3.7 and 3.5 and 2.1(3), where \(id_p\) is the identity map on \(M_{d_p},\) we obtain

\[I_p \otimes I_J = \sum_{ij} e^p_{ij} \otimes \kappa(u^p_{ik}) \times_J u^p_{kj} = \sum_{ij} e^p_{ij} \otimes u^p_{ik} \otimes u^p_{kj} \times_J u^p_{kj}.\]

Similarly, applying the map \(id_p \otimes m_J(id_J \otimes \kappa_J)\) to both sides of the same equality, we obtain,

\[I_p \otimes I_J = \sum_{ij} e^p_{ij} \otimes u^p_{ik} \times_J \kappa(u^p_{kj}) = \sum_{ij} e^p_{ij} \otimes u^p_{ik} \times_J u^p_{kj}.\]
This proves the proposition. Q.E.D.

Summarizing 3.2, 3.3, 3.4, 3.6 and 3.8, we have proved part (1) of the following theorem.

3.9. Theorem. (1) With the dense subalgebra $A_J$, its generating elements $u^*_i$, the coproduct $\Phi_J$, and the coinverse $\kappa_J$, as defined above, $A_J$ is a compact quantum group. The compact abelian Lie group $T$ is still a subgroup of the quantum group $A_J$.

(2) $A_J$ is a compact matrix quantum group if and only if $A$ is.

Proof. The “if” part of (2) is clear from the proof of (1). Because $T$ is again a subgroup of $A_J$, $A_J$ satisfies the assumptions set forth at the beginning of this section. Hence we can apply the deformation process of this section to $A_J$. Note that $(A_J)_{-J} = A$ (see [14]), so the “only if” part follows from the “if” part.

Notation. Let $A = C(G)$ be a Woronowicz Hopf $C^*$-algebra so that the construction $A_J$ in the theorem above can be applied. We will also denote $A_J$ by $C(G_J)$.

3.10. Remarks. (1) We remark that for $A = C(G)$ with $G$ a compact Lie group, the above theorem becomes precisely the main theorem in [15].

(2) As in [15], the Haar measure $h_J$ of the quantum group $A_J$ is still the same as the Haar measure $h$ of $A$, namely on the common subspace $A$ of both $A$ and $A_J$, we have

$$h_J(a) = h(a), \quad \text{for } a \in A.$$ 

From this we see that if $A$ is a compact matrix quantum group of Kac type (one on which the Haar measure is a trace [1]), then $A_J$ is also one such. (See also 5.2 in the following.) Therefore the compact quantum groups constructed in [15] are all of Kac type.

(3) Examining the proof of the above theorem one can check that if we use either of the following two actions:

$$\alpha_{(s,u)} = \lambda_{(s,-s)} \rho_0(u),$$

$$\alpha_{(s,u)} = \lambda_{(s)} \rho_0(-u),$$

the theorem is still valid.

We conclude this section with a generalization of example 4.3 of [25] (see also [23]). Recall [23, 25] that a Woronowicz Hopf $C^*$-dynamical system is defined to be a $C^*$-dynamical system with the additional assumption that the automorphism group preserves the coproduct of the Woronowicz Hopf $C^*$-algebra.

3.11. Proposition. Let $D$ be the subgroup of $\mathbb{R}^d$ consisting of vectors of the form $(s, s)$ with $s$ in $\mathbb{R}^n$. Put $\beta_s = \alpha_{(s,s)}$. Then $(A, D, \beta_s)$ is a Woronowicz $C^*$-dynamical system equivariant for the action $\alpha$, its deformation $(A_J, D, \beta_J)$ is still a Woronowicz $C^*$-dynamical system.

Proof. Same as the proof of 4.3 of [25]. Q.E.D.
4 Deformations of Quantum Groups $K_q$ and $K^u_q$

Based on the earlier work [19], Soibelman [18] studied the representation theory of the function algebra of a general compact quantum simple group $K_q$, which are "compact real forms" of the Drinfeld-Jimbo quantum groups [3, 6]. In [10], Levendorskii introduced a deformation $K_q^u$ of the quantum group $K_q$ by a purely algebraic method (the so-called twisting construction of Drinfeld), and studied the representation theory of its function algebra. See also [11] for a summary of [18, 11], and for related work, see e.g. [29, 20, 17, 2, 9] for the analytical case, and [4, 5] for the algebraic case, as well as the literature cited in these papers. In [15], Rieffel raised the question whether the general quantum groups $K_q$ can be deformed under strict deformation quantization into quantum groups $K^u_q$. This section is devoted to giving an affirmative answer to this question using the deformation developed in section 3.

To establish the notation for this section, we first recall the notation of [18, 10, 11]. Let $G$ be a simple complex Lie group with Lie algebra $\mathfrak{g}$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, together with the corresponding decomposition $\Delta = \Delta_+ \cup \Delta_-$ of the root system and a fixed basis $\{\alpha_i\}_{i=1}^N$ for $\Delta_+$. For each linear functional $\lambda$ on $\mathfrak{h}$, $H_\lambda$ denotes the element in $\mathfrak{h}$ corresponding to $\lambda$ under the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ determined by the Killing form $(\ , \ )$ on $\mathfrak{g}$. Note that if the reader keeps the context in mind, the symbols $\alpha$ and $\lambda$ used in this context should not cause confusion with the ones used in the definition of the action $\alpha$ in the previous sections. Let $\{X^\alpha\}_{\alpha \in \Delta} \cup \{H_i\}_{i=1}^N$ be a Weyl basis of $\mathfrak{g}$, where $H_i = H_{\alpha_i}$. This determines a Cartan involution $\omega_0$ on $\mathfrak{g}$ with $\omega_0(X^\alpha) = -X^{-\alpha}$, $\omega_0(H_i) = -H_i$. Let $t$ be the compact real form of $\mathfrak{g}$ defined as the fixed points of $\omega_0$ and $K$ the associated compact real form of $G$. Put $\mathfrak{h}_t = \oplus_{i=1}^N \mathfrak{h}_i$, and $T = \exp(it\mathfrak{h}_t)$, the later being the associated maximal torus of $K$.

Let $q > 1$. For $n, k \in \mathbb{N}$, $n \geq k$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$[n]_q! = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q[k-1]_q \cdots [1]_q}.$$

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ [3, 6] is the complex associative algebra with generators $X_i^{\pm}$, $K_i^{\pm 1}$ ($i = 1, \cdots, n$) and defining relations:

$$K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i,$$

$$K_i X_j^{\pm} K_i^{-1} = q^{\pm\langle \alpha_i, \alpha_j \rangle} X_j^{\pm},$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0, \quad i \neq j,$$

where $q_i = q^{\langle \alpha_i, \alpha_i \rangle}$.

On $U_q(\mathfrak{g})$ there is a Hopf algebra structure with coproduct

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes K_i + K_i^{-1} \otimes X_i^{\pm},$$

12
and counit and antipode respectively

$$\varepsilon(X^\pm_i) = 0, \quad \varepsilon(K^\pm_i) = 1, \quad S(X^\pm_i) = -q_i^{\pm 1} X^\pm_i, \quad S(K^\pm_i) = K^\mp_i.$$  

Under the *-structure defined by

$$(X^\pm_i)^* = X^\mp_i, \quad K^*_i = K_i,$$

$U_\hbar(g)$ is a Hopf *-algebra.

The algebra $U_\hbar(g)$ is the $\mathbb{C}[[\hbar]]$ algebra generated by $X^\pm_i$ and $H_i$ with the defining relations

$$[H_i, H_j] = 0, \quad [H_i, X^\pm_j] = \pm (\alpha_i, \alpha_j) X^\pm_j,$$

$$[X^+_i, X^-_j] = \delta^{ij} \frac{sh(h_i)}{sh(h^-_i)},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{pmatrix} 1 - a_{ij} \\ k \end{pmatrix}_q (X^\pm_i)^k (X^\pm_j)^{1-a_{ij}-k} = 0, \quad i \neq j.$$

The Hopf *-algebra structure on $U_\hbar(g)$ is defined as above,

$$(X^\pm_i)^* = X^\mp_i, \quad H^*_i = H_i,$$

where the involution of $\mathbb{C}[[\hbar]]$ is given by $(ch)^* = ch$ for $c \in \mathbb{C}$.

Let $u = \sum_{k,l} c_{kl} H_k \otimes H_l \in \Lambda^2 h_2$. Then one can define a new coproduct on $U_\hbar(g)$ by

$$\Delta_u(\xi) = \exp(-i\hbar u/2) \Delta(\xi) \exp(i\hbar u/2),$$

where $\xi \in U_\hbar(g)$ and $\Delta$ is the original coproduct on $U_\hbar(g)$. The new Hopf *-algebra so obtained is denoted by $U_{\hbar,u}(g)$.

The function algebra $\mathbb{C}[K_\hbar]$ of the compact quantum group $K_\hbar$ is defined to be the dual of the Hopf *-algebra $U_\hbar(g)$ consisting of matrix elements of finite dimensional representations $\rho$ of $U_\hbar(g)$ such that eigenvalues of the endomorphisms $\rho(H_i)$ are positive. The function algebra $\mathbb{C}[K_\hbar^u]$ of the compact quantum group $K_\hbar^u$ is defined to have the same elements as $\mathbb{C}[K_\hbar]$ and the same *-structure as $\mathbb{C}[K_\hbar]$, while the product of its elements is defined using $\Delta_u$ instead of $\Delta$. The completions of the algebras $\mathbb{C}[K_\hbar]$ and $\mathbb{C}[K_\hbar^u]$ under their universal $C^*$-norms are denoted by $C(K_\hbar)$ and $C(K_\hbar^u)$ respectively. The algebras $\mathbb{C}[K_\hbar]$ and $\mathbb{C}[K_\hbar^u]$ are examples of Krein algebras $\mathcal{A}$ in the sense of [26, 23].

For each dominant weight $\Lambda$ of $g$, Soibelman introduces [18, 11] the matrix elements $C^\Lambda_{\mu,\nu,\Omega}$ of the highest weight $U_\hbar(g)$ module $(L(\Lambda), \rho_\Lambda)$ as follows. Let $\{v^{(i)}_\nu\}$ be an orthonormal weight basis for the unitary $U_\hbar(g)$ module $L(\Lambda)$, and let $\{j^{(i)}_\nu\}$ be the corresponding dual weight basis in $L^*(\Lambda)$. Put $\Omega = (i, j)$. Then $C^\Lambda_{\mu,\nu,\Omega}$ is defined by

$$C^\Lambda_{\mu,\nu,\Omega}(\xi) = l^{(i)}(\rho_\Lambda(\xi)) v^{(j)}_\mu,$$

where $\xi \in U_\hbar(g)$.

To avoid confusion with the Killing form, we now use $s \oplus v$, instead of $(s, v)$ (as in the previous section), to denote an element of $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^n$. In the present setting, the space
\( \mathbb{R}^n \) is \( \mathfrak{h}_\mathbb{R} \), with inner product \( \langle \cdot , \cdot \rangle = - ( , , ) \), where \( ( , , ) \) is the Killing form of \( \mathfrak{g} \) restricted to \( \mathfrak{h}_\mathbb{R} \). We will also use \( ( , ) \) to denote the inner product on \( \mathfrak{h}_\mathbb{R} \oplus \mathfrak{h}_\mathbb{R} \). Noting that the compact abelian group \( T \) is also a subgroup of both \( K_q^\mu \) and \( K_q^\nu \) (see [18, 10]), we can define as in section 3 an action of \( \mathbb{R}^d \) on \( C(\mathbb{R}) \) by

\[
\alpha_{s\exp(-2\pi is)} = \lambda_{\exp(-2\pi is)}\rho_{\exp(2\pi is)}.
\]

(Thus the map \( \eta \) there is defined by \( \eta(s) = \exp(2\pi is) \)). This action may be viewed as an action of \( H = T \times T \) in the sense of [14]. For each \( \nu \) in the weight lattice \( P \) of \( \mathfrak{g} \), the element \( H_\nu \) is in \( \mathfrak{h}_\mathbb{R} \). Keeping the notation of [14] for the spectral subspaces of the action \( \alpha \), we have the following lemma.

**4.1. Lemma.** The matrix elements \( C_{\nu,\mu,\Omega}^\Lambda \) are in the spectral subspaces \( A_{-\langle H_\nu \oplus H_\mu \rangle} \).

**Proof.** In the notation of [14], we must show that

\[
\alpha_{s\exp(-2\pi i(s + v))} C_{\nu,\mu,\Omega}^\Lambda = e^{-2\pi i\langle H_\nu \oplus H_\mu, s \rangle} C_{\nu,\mu,\Omega}^\Lambda.
\]

Let \( \xi \) be an element in \( U_q(\mathfrak{g}) \). We compute:

\[
\alpha_{s\exp(-2\pi i s)}(C_{\nu,\mu,\Omega}^\Lambda)(\xi) = \langle \xi, (\rho_{\lambda}(\exp(\exp(2\pi is)))\rho_{\Lambda}(\exp(2\pi is)))\rho_{\Lambda}(\exp(2\pi is))\nu_{\mu,\nu,\Omega}^\Lambda \rangle
\]

\[
= \langle \rho_{\Lambda}(\exp(2\pi is))\rho_{\Lambda}(\exp(2\pi is))\nu_{\mu,\nu,\Omega}^\Lambda, \rho_{\Lambda}(\exp(2\pi is))\nu_{\mu,\nu,\Omega}^\Lambda \rangle
\]

\[
= \langle \rho_{\Lambda}(\exp(2\pi i s))\nu_{\mu,\nu,\Omega}^\Lambda, \rho_{\Lambda}(\exp(2\pi is))\nu_{\mu,\nu,\Omega}^\Lambda \rangle
\]

\[
= \rho_{\Lambda}(\exp(2\pi i s))\rho_{\Lambda}(\exp(2\pi is))\langle \nu_{\mu,\nu,\Omega}^\Lambda, \nu_{\mu,\nu,\Omega}^\Lambda \rangle
\]

\[
= e^{2\pi i\langle H_\nu \oplus H_\mu, s \rangle} C_{\nu,\mu,\Omega}^\Lambda(\xi).
\]

This proves the lemma. \( \text{Q.E.D.} \)

In [10, 11], the following identity is obtained:

\[
C_{\nu_1,\mu_1}^\Lambda(\cdot) \circ C_{\nu_2,\mu_2}^\Lambda = \exp\left( \frac{i\hbar}{2} (\langle \nu_1, \cdot \rangle - \langle \mu_1, \cdot \rangle) \right) C_{\nu_1,\mu_1}^\Lambda \circ C_{\nu_2,\mu_2}^\Lambda,
\]

where the left-hand side is the multiplication in \( C(\mathbb{R}) \) and the right-hand side is the multiplication in \( C(\mathbb{R}) \), and \( \hat{u} \) is the map on \( \mathfrak{h}_\mathbb{R}^* \) determined by \( u \) via the Killing form \( ( , , ) \) on \( \mathfrak{g} \). Letting

\[
p = - (H_{\nu_1} \oplus H_{\mu_1}), \quad q = - (H_{\nu_2} \oplus H_{\mu_2}),
\]

\[
J = \frac{\hbar}{4\pi} (S_u \oplus (-S_u)),
\]

where \( S_u \) is the skew-symmetric operator on \( \mathfrak{h}_\mathbb{R} \) defined by

\[
S_u(H_\nu) = \sum_{k, \lambda} c_{k\lambda} \nu(H_k) H_{\lambda},
\]

14
the previous identity becomes
\[ C_{\alpha_1}^{\Lambda_1} \circ C_{\alpha_2}^{\Lambda_2} = \exp(-2\pi i < p, J q >) C_{\alpha_1}^{\Lambda_1} C_{\alpha_2}^{\Lambda_2}. \]

On the other hand from 2.2.2 of [14], we have in virtue of lemma 4.1 that
\[ C_{\alpha_1}^{\Lambda_1} \times_J C_{\alpha_2}^{\Lambda_2} = \exp(-2\pi i < p, J q >) C_{\alpha_1}^{\Lambda_1} C_{\alpha_2}^{\Lambda_2}. \]

Note that the elements \( C_{\alpha, \Omega}^{\Lambda} \) generate \( \mathbb{C}[K_q^w] \) and \((\mathbb{C}[K_q^w], \times_J)\), the later being the algebra with the product \( \times_J \) on \( \mathbb{C}[K_q^w] \). As a matter of fact, the elements \( C_{\alpha, \Omega}^{\Lambda} \) with \( \Lambda \) running over the fundamental weights already generate these algebras. Summarizing these we have reached a proof of the following result.

4.2. Theorem. The Hopf \( \ast \)-algebras \( \mathbb{C}[K_q^w] \) and \((\mathbb{C}[K_q^w], \times_J)\) are isomorphic.

From the viewpoints in [26], we can regard the compact quantum groups \( K_q^w \) and \((K_q^w, J)\) as the same quantum group because the above theorem says that they have the same Krein algebras.

From the universality of the \( C^* \)-algebra \( C(K_q^w) \), there is a map of Woronowicz Hopf \( C^* \)-algebras from \( C(K_q^w) \) to \( C(K_q^w, J) \) sending each of the elements \( C_{\alpha, \Omega}^{\Lambda} \) to itself (viewed as elements in different algebras). We believe this is an isomorphism. One possible way to prove this is to try to adapt the method of 10.2 in [14]: Show that there are no \( \alpha \)-invariant ideals in \( C(K_q^w) \).

5 Deformations of Quantum Groups \( A_u(m) \)

We now apply the construction in section 3 to the quantum groups \( A_u(m) \) [23, 24] to answer Rieffel's second question mentioned at the beginning of this paper. Recall that for any positive integer \( m \geq 2 \), \( A_u(m) \) is the universal \( C^* \)-algebra generated by \( m^2 \) elements \( a_{ij} \) such that both \( (a_{ij}) \) and \( (a_{ij}^*) \) are unitary elements of \( M_m \otimes A_u(m) \), where \( (a_{ij}^*) \) is the matrix obtained from \( (a_{ij}) \) by applying the involution \( * \) to each of its entries. More explicitly, \( A_u(m) \) is the universal \( C^* \)-algebra generated by \( a_{ij} \) subject to the relations

\[
\sum_{k=1}^{m} a_{ik} a_{kj}^* = \delta_{ij}, \quad \sum_{k=1}^{m} a_{ik}^* a_{kj} = \delta_{ij}, \\
\sum_{i=1}^{m} a_{ii} a_{kj}^* = \delta_{ij}, \quad \sum_{k=1}^{m} a_{ik}^* a_{jk} = \delta_{ij}
\]

for \( i, j = 1, \ldots, m \).

The unitary groups \( U(k) \) are subgroups of the quantum groups \( A_u(m) \) for all \( k \leq m \).

This can be seen as follows. Let \( u_{ij} \) be the coordinate functions on the group \( U(k) \). Define

\[ a_{ij}' = u_{ij} \] if \( i, j \leq k \), and \[ a_{ij}' = \delta_{ij} \] if either \( k < i \leq m \) or \( k < j \leq m \).

Then the map \( \pi_0 \) defined by \( \pi_0(a_{ij}) = a_{ij}' \) defines an embedding of the unitary group \( U(k) \) in the compact quantum group \( A_u(m) \).
Let $T$ be a torus subgroup of $U(k)$, say, of dimension $n$. Then $T$ can be viewed as a subgroup of the compact quantum group $A_u(m)$ with the surjective morphism $\pi$ from $A_u(m)$ to $C(T)$ given by the composition of $\pi_0$ defined above and the restriction map from $C(U(k))$ to $C(T)$. The Lie algebra of $T$ can be identified with the vector space $\mathbb{R}^n$ with the zero bracket. Let $\eta$ be the exponential map from $\mathbb{R}^n$ to $T$, and let $S$ be any skew-symmetric operator on $\mathbb{R}^n$. With the ingredients $A_u(m)$, $T$, $\pi$, $\eta$ and $S$, we are in the setting to apply the construction in section 3 to obtain quantum groups $A_u(m)_J$.

5.1. Theorem. (1) The quantum groups $A_u(m)_J$ are quantum subgroups of the quantum groups $A_u(m)$.

(2) For any each compact quantum groups $C(G)_J$ constructed in [15], there is an $m$ such that one may define $A_u(m)_J$ and $C(G)_J$ are quantum subgroups of both $A_u(m)_J$ and $A_u(m)$.

Proof. Since $u = (a_{ij})$ and $\bar{u} = (a_{ij}^*)$ are both unitary representation of the quantum group $A_u(m)$, we see from the proof of 3.8 that $u$ and $\bar{u}$ are still unitary representations of the quantum group $A_u(m)_J$. A moment's reflection shows that $A_u(m)_J$ is also generated by the $a_{ij}$ as a $C^*$-algebra. Thus by the universal property of the $C^*$-algebra $A_u(m)$, there is a unital $C^*$-algebra homomorphism $\pi$ from $A_u(m)$ to $A_u(m)_J$ sending the generators $a_{ij}$ of $A_u(m)$ to the generators $a_{ij}$ of $A_u(m)_J$ for each $(i,j)$. It is clear that $\pi$ preserves the coproducts. This proves (1).

Since $G$ is a compact Lie group, it has faithful finite dimensional unitary representations. Let $(u_{ij})$ be one such, say, of dimension $m$. Hence $G$ is a subgroup of the quantum group $A_u(m)$. Let $\pi$ be the surjective Hopf $C^*$-algebra morphism from $A_u(m)$ to $C(G)$ sending $a_{ij}$ to $u_{ij}$ for each pair $(i,j)$. Then under the obvious action of $\mathbb{R}^d$ on $A_u(m)$ coming from the action of $\mathbb{R}^d$ on $C(G)$, $\pi$ is equivariant, so it can be deformed into a map $\pi_J$, which is easily seen to be still a surjective map of Woronowicz Hopf $C^*$-algebras. Hence $C(G)_J$ is a quantum subgroup of $A_u(m)_J$. Since $A_u(m)_J$ is a quantum subgroup of the quantum group $A_u(m)$, we see that $C(G)_J$ is a quantum subgroup of $A_u(m)$. This completes the proof of (2).

Q.E.D.

5.2. Remarks. (1) From section 5 of [28], one can see that the quantum groups $A_u(m)$ are universal in the following sense (see [22]): Any compact matrix quantum group of Kac type (see remark 3.10.(2)) is a quantum subgroup of $A_u(m)$ for some $m$. It is easy to see that 5.1.(2) (and its proof) can be generalized to the following setting. Let $A$ be a compact quantum group of Kac type with a torus subgroup $T$. Let $\pi$ be a surjection of Woronowicz Hopf $C^*$-algebras from $A_u(m)$ to $A$. We can apply the construction in section 3 to obtain $A_J$ and $A_u(m)_J$. Then the deformed map $\pi_J$ is again a surjection of Woronowicz $C^*$-algebras.

(2) For most invertible matrices $Q$, the quantum groups $A_u(Q)$ and $A_u(Q)$ constructed in [22] have many torus subgroups. Hence, as described in this paper, they are also subject to the Rieffel quantization.

References


