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**ON THE HUGE LIE SUPERALGEBRA
OF PSEUDO SUPERDIFFERENTIAL OPERATORS
AND SUPER KP-HIERARCHIES**

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ABSTRACT

Lie superalgebraic methods are used to establish a connection between the huge Lie superalgebra Ξ of super (pseudo) differential operators and various super KP-hierarchies. We show in particular that Ξ splits into $5 = 2 \times 2 + 1$ graded algebras expected to correspond to five classes of super KP-hierarchies generalizing the well-known Manin-Radul and Figueroa O'Farrill-Ramos supersymmetric KP-hierarchies.

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1 Introduction

Recently, there has been much interest in studying two dimensional integrable models [1, 2]. Known as the generalized KdV-hierarchies, these models are incorporated into a much larger integrable system, namely the KP-hierarchy [3]. This is defined as a set of multitime evolution equations which read in the Lax form as

$$\frac{\partial L}{\partial t_r} = [L_+^r, L], \quad r = 1, 2, 2, \dots, \quad (1)$$

where L is a pseudo-differential operator given by

$$L = \partial + \sum_{i \geq 1} U_i(z) \partial^{1-i}, \quad \partial = \partial/\partial z. \quad (2)$$

There are various remarkable properties of the KP-hierarchy. The most essential ones are its bi-hamiltonian structures [1, 4] and the fact that it admits a Lax formulation [1, 5]. Several extensions of the standard KP-hierarchy Eqs.(1),(2) are possible. The more known ones, given by the supersymmetric extensions of the hierarchy Eqs.(1),(2), are based on the Manin-Radul odd pseudo-superdifferential operator [6]

$$L = D + \sum_{i \geq 0} U_{i+\frac{1}{2}}(\hat{z}) D^{-i} \quad (3)$$

and the Figueroa O'Farrill-Ramos even pseudo superdifferential operator [7]

$$L = D^2 + \sum_{i \geq 0} U_{i+\frac{3}{2}}(z) D^{-i} \quad (4)$$

with $D = \partial_\theta + \theta \partial$.

In this letter we propose a consistently algebraic formulation of the extended KP-integrable systems. We will present a systematic description of the Lie superalgebra Ξ of supersymmetric pseudo-differential operators and we will show, in particular, that it splits into $6 = 3 \times 2$ Lie superalgebras. To establish a connection with the supersymmetric extensions of the KP-hierarchy Eqs.(1),(2) we will define $5 = 2 \times 2 + 1$ graded spaces of pseudo-superdifferential operators containing as a particular element the Figueroa O'Farrill-Ramos Eq.(4) and the Manin-Radul Eq.(3) superdifferential Lax operators and we will conclude by expecting that these five graded spaces correspond to five classes of supersymmetric KP-hierarchies.

2 The algebra Ξ of super-pseudo-differential operators

2.1 The ring $\mathcal{R}$ of analytic superfields

Let us first consider the ring of analytic superfields $U_{k/2}(\hat{z})$, $k \in \mathbf{Z}$, which depend on $(1|1)$ superspace coordinates $\hat{z} = (z, \theta)$. In this supercommutative $\mathbf{Z}_2$ -graded ring $\mathcal{R}$, one can define an odd superderivative $D = \partial_\theta + \theta \partial$, the $N = 1$ supercovariant derivative which obeys the $N = 1$ supersymmetric algebra $D^2 = \partial$ with $\theta^2 = 0$ and $\partial_\theta = \int d\theta$. Following the analysis developed in [8], the ring $\mathcal{R}$ can be decomposed as:

$$\mathcal{R} = \bigoplus_{k \in \mathbf{Z}} \mathcal{R}_{k/2}^{(0,0)} . \quad (5)$$

where $\mathcal{R}_{k/2}^{(0,0)}$ is the set of superfields $U_{k/2}(\hat{z})$ labelled by half integer conformal spin $\frac{k}{2}$, $k \in \mathbf{Z}$. The upper indices $(0,0)$, carried by $\mathcal{R}$ and which we shall drop whenever no confusion can arise, are special values of general indices (p, q) to be introduced later on. One can also define the following product

$$\langle U_{k/2}, U_{\ell/2} \rangle = \delta_{k+\ell,1} \int d\hat{z} \frac{U_k}{2}(\hat{z}) U_{\frac{1-k}{2}}(\hat{z}) , \quad (6)$$

showing that the one dimensional subspaces $\mathcal{R}_{k/2}$ and $\mathcal{R}_{\frac{1-k}{2}}$ are dual to each other. Using dimensional arguments, it is not difficult to see that the product $\langle \cdot, \cdot \rangle$ Eq.(6) carries a conformal spin $\Delta = -\frac{1}{2}$. Later on, we shall introduce a combined scalar product $\ll \cdot, \cdot \gg$ built out of Eq.(6) and the pairing product $(\cdot, \cdot)$ of conformal spin $\Delta = 1/2$, so that we get $\Delta[\ll, \gg] = 0$.

Since the conformal spin product Eq.(6) get induces here, we have that $\mathcal{R} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ and $\mathcal{H}_-$ are two dual semi-infinite tensor subspaces characterized respectively by positive and negative conformal spin as shown here below

$$\mathcal{H}_+ = \bigoplus_{k>0} \mathcal{R}_{k/2} \quad (7)$$

$$\mathcal{H}_- = \bigoplus_{k>0} \mathcal{R}_{\frac{1-k}{2}} .$$

We learn, in particular, that the space $\mathcal{R}_0$ of vanishing conformal spin superfields $U_0(\hat{z})$, is the dual of the space $\mathcal{R}_{1/2}$ generated by the superfield $U_{1/2}(\hat{z})$ of conformal spin $1/2$.

We remark also, that the conformal spin product Eq.(6) agrees with the $\mathbf{Z}_2$ -grading

$$\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1 \quad (8a)$$

with

$$\begin{aligned}\mathcal{R}_{\bar{0}} &= \mathcal{H}_{+\bar{0}} \oplus \mathcal{H}_{-\bar{0}} \\ \mathcal{R}_{\bar{1}} &= \mathcal{H}_{+\bar{1}} \oplus \mathcal{H}_{-\bar{1}}\end{aligned}\tag{8b}$$

where $\mathcal{H}_{\pm, \bar{0}}$ and $\mathcal{H}_{\pm, \bar{1}}$ are subspaces related to each other by the duality conjugation

$$[\mathcal{H}_{\pm}]_{\bar{0}}^* = [\mathcal{H}_{\mp}]_{\bar{1}}\tag{8c}$$

2.2 The superspace $\Xi_{m/2}^{(p,q)}$ and super Lax operators

Consider the superspace $\Xi_{m/2}^{(p,q)}$ labelled by three quantum numbers $\frac{m}{2}$, p and q defining respectively the conformal spin, the lowest and the highest degrees. Typical elements of this superspace are given by

$$L_{m/2}^{(p,q)}[u] = \sum_{i=p}^q U_{\frac{m-i}{2}}(\hat{z}) D^i, \quad p, q, m \in \mathbf{Z},\tag{9}$$

where $U_{\frac{m-i}{2}}(\hat{z})$ are analytic superfield of conformal spin $\frac{m-i}{2}$. $\Xi_{m/2}^{(p,q)}$ behaves then as a $(1+q-p)$ dimensional superspace generated by $L_{m/2}^{(p,q)}$ and whose superspace decomposition is given by the linear sum

$$\Xi_{m/2}^{(p,q)} = \bigoplus_{i=p}^q \Xi_{m/2}^{(i,i)},\tag{10a}$$

with

$$\Xi_{m/2}^{(i,i)} = \mathcal{R}_{\frac{m-i}{2}} \oplus D^i,\tag{10b}$$

where $\mathcal{R}_{k/2} = \Xi_{k/2}^{(0,0)}$ is the ring of analytic superfields $U_{k/2}(\hat{z})$ introduced previously. For any element $L_{m/2}^{(p,q)}$ of the superspace $\Xi_{m/2}^{(p,q)}$, one can introduce the conformal spin Δ , the degrees (deg) and the grading $|\cdot|$, properties which are summarized in the following table

Δ	deg	$ \cdot $	
$\frac{U_i}{2} D^j$	$\frac{i+j}{2}$	(j, j)	$(i+j) \bmod 2$
$\Xi_{m/2}^{(p,q)}$	$\frac{m}{2}$	(p, q)	$m \bmod 2$

An element $L_{m/2}^{(p,q)}$ is called a super Lax operator, if it is homogeneous under the $\mathbf{Z}_2$ -grading

$$|x| = \begin{cases} 0 & \text{for } x \text{ even} \\ 1 & \text{for } x \text{ odd} \end{cases}\tag{12}$$

and have the following form at order $m, m \in \mathbb{N}$,

$$L_{m/2}^{(0,m)} = D^m + \sum_{i=1}^m U_{i/2}(\hat{z}) D^{m-i}.\tag{13}$$

The homogeneity condition simply states that the $\mathbf{Z}_2$ -grading of the analytic superfield $U_{i/2}(\hat{z})$ is defined as

$$|U_{i/2}(\hat{z})| = i \pmod{2} . \quad (14)$$

Setting $m = 2n, p = 1$ and $q = 2n$ into Eq.(9), one recover the Inami–Kanno super Lax operator of $2n$ -th order, [9] associated to the affine Lie superalgebra $A(n-1|n-1)^{(1)}$, namely

$$L_n^{(1,2n)}[u] = D^{2n} + \sum_{i=1}^{n-1} \left[U_i(\hat{z}) D^{2(n-i)} + U_{\frac{2i+1}{2}}(\hat{z}) D^{2(n-i)-1} \right] . \quad (15)$$

The space of supersymmetric Lax operators, Eq.(13) is given by the coset space

$$\mathcal{M}^{(m-1)} = \Xi_{m/2}^{(0,m)} / \Xi_{m/2}^{(m-1/m-1)} , \quad (16)$$

which exhibits a dimension $m-1$.

To define a Lie algebraic structure on the superspace $\Xi_{m/2}^{(p,q)}$ one needs to introduce a Lie bracket defined for two arbitrary operators X and Y as

$$[X, Y] = XY - (-)^{|X||Y|} YX . \quad (17)$$

It follows then that $\Xi_{m/2}^{(p,q)}$ defines a Lie superalgebra provided that

$$m = 0 \quad \text{and} \quad p \leq q \leq 1 . \quad (18)$$

The multiplication of operators in $\Xi_{m/2}^{(p,q)}$ is given by the generalized Leibnitz rule [6]

$$D^i \phi(\hat{z}) = \sum_{k=0}^{\infty} \begin{bmatrix} i \\ 1-k \end{bmatrix} (-)^{|\phi|(1-k)} \phi^{(k)}(\hat{z}) D^{i-k} , \quad (19)$$

where $\phi^{(k)} = D^k \phi$ and $\begin{bmatrix} i \\ 1-k \end{bmatrix}, i \in \mathbf{Z}$, is the superbinomial coefficient given by

$$\begin{bmatrix} i \\ k \end{bmatrix} = \begin{cases} 0, & \text{for } k > i \text{ or } (i, k) \equiv (0, 1) \pmod{2} \\ \binom{[i/2]}{[k/2]}, & \text{otherwise} \end{cases} . \quad (20)$$

The symbol $[x]$ stands for the integer part of $x \in \frac{1}{2} \mathbf{Z}$ and $\binom{i}{j}$ is the usual binomial coefficient.

2.3 The huge Lie superalgebra Ξ

A larger set of supersymmetric pseudo-differential operators than $\Xi_{m/2}^{(p,q)}$ is given by the infinite dimensional vector space $\Xi^{(p,q)}$ of superdifferential operators with given degrees $(p, q), p, q \in \mathbf{Z}$, but indefinite conformal spins,

$$\Xi^{(p,q)} = \bigoplus_{m \in \mathbf{Z}} \Xi_{m/2}^{(p,q)} . \quad (21)$$

The set $\Xi^{(p,q)}$ exhibits a Lie superalgebra structure with respect to the bracket Eq.(17) provided that

$$p \leq q \leq 1 . \quad (22)$$

Using the conformal spin product Eq.(6), it follows that $\Xi^{(p,q)}$ decomposes as

$$\Xi^{(p,q)} = \Lambda_+^{(p,q)} \oplus \Lambda_-^{(p,q)} , \quad (23a)$$

with

$$\Lambda_+^{(p,q)} = \bigoplus_{k>0} \Xi_{k/2}^{(p,q)} \quad (23b)$$

$$\Lambda_-^{(p,q)} = \bigoplus_{k>0} \Xi_{\frac{1-k}{2}}^{(p,q)} .$$

Setting $(p, q) = (0, 0)$, Eqs.(21) and (23) reduce respectively to Eqs.(5) and (7) with $\Xi^{(0,0)} = \mathcal{R}$ and $\Lambda_{\pm}^{(0,0)} = \mathcal{H}_{\pm}$.

A huge Lie superalgebra is obtained by summing from Eq.(21) over all the allowed values of the degrees (p, q) . It is defined by

$$\Xi = \bigoplus_{p \leq q} \Xi^{(p,q)} , \quad (24a)$$

or equivalently

$$\Xi = \bigoplus_{p \in \mathbf{Z}} \left[\bigoplus_{n \in \mathbf{N}} \Xi^{(p, p+n)} \right] . \quad (24b)$$

Remark that the infinite dimensional superspace Ξ is closed under the Lie bracket Eq.(17) without any constraint. Remark also that combining the conformal spin and the degrees quantum numbers, one can write

$$\Xi = \bigoplus_{p \in \mathbf{Z}} \bigoplus_{n \in \mathbf{N}} \bigoplus_{m \in \mathbf{Z}} \Xi_{m/2}^{(p,q)} . \quad (25)$$

Let us now consider $L^{(p,q)}$ and $P^{(r,s)}$, two super (pseudo) differential operators with fixed degrees but indefinite conformal spin. The degree pairing product $(\cdot, \cdot)$ associated to $L^{(p,q)}$ and $P^{(r,s)}$ is defined as

$$(L^{(p,q)}, P^{(r,s)}) = \delta_{p+s+1,0} \delta_{q+r+1,0} \delta res \left(L^{(p,q)} \cdot P^{(r,s)} \right) , \quad (26)$$

where (δres) is the superresidue operation given by

$$\delta res D^i = \delta_{i+1,0} . \quad (27)$$

Using Eqs.(25) and (26), one can easily check that $\Delta[(\cdot, \cdot)] = \frac{1}{2}$. Combining the conformal spin and the degree pairing products Eqs.(6) and (26), one defines the following combined scalar product:

$$\langle\langle L_{m/2}^{(p,q)}, P_{m/2}^{(r,s)} \rangle\rangle = \delta_{p+s+1,0} \delta_{q+r+1,0} \delta_{n+m,0} \int d\hat{z} \delta res \left[L_{m/2}^{(p,q)} \circ P_{n/2}^{(r,s)} \right] \quad (28)$$

so that $\Delta[\ll \cdot, \gg] = 0$.

The combined scalar product we have defined in Eq.(28), plays an important role in the construction of the supersymmetric Gelfand–Dickey (SGD) Poisson bracket, see [10].

As shown in the bosonic case [8], we note that the superspaces $\Xi^{(p,q)}$ and $\Xi^{(-q-1,-p-1)}$ are dual to each other, with respect to the degree pairing product Eq.(26). It is then straightforward to see that the Lie superalgebra Ξ decomposes as

$$\Xi = \Xi^+ \oplus \Xi^- \quad (29a)$$

with

$$\Xi^+ = \bigoplus_{p \geq 0} \bigoplus_{n \in \mathbb{N}} \Xi^{(p,p+n)} \quad (29b)$$

$$\Xi^- = \bigoplus_{p \geq 0} \bigoplus_{n \in \mathbb{N}} \Xi^{(-p-n-1,-p-1)} ,$$

or equivalently, by using the combined scalar product Eq.(28)

$$\Xi^+ = \bigoplus_{p \geq 0} \bigoplus_{n \in \mathbb{N}} \bigoplus_{m \in \mathbb{Z}} \Xi_{m/2}^{(p,p+n)} \quad (29c)$$

$$\Xi^- = \bigoplus_{p \geq 0} \bigoplus_{n \in \mathbb{N}} \bigoplus_{m \in \mathbb{Z}} \Xi_{m/2}^{(-p-n-1,-p-1)} .$$

Next, we introduce the graded algebras Σ_{ij}^k and their dual $(\Sigma_{ij}^k)^* = \Sigma_{-i,j}^{-k}$, with respect to Eq.(28), where the indices $k = \pm, i = 0, \pm$ and $j = \bar{0}, \bar{1}(\text{mod}2)$ refer respectively to the degrees, the conformal spin and the grading quantum numbers. As an example $\Sigma_{+\bar{0}}^+$ is the bosonic Lie algebra of superdifferential operators of positive definite spins and degrees and $\Sigma_{-\bar{0}}^-$, its dual, the Lie algebra of super pseudo differential operators of negative definite spins and degrees. $\Sigma_{+\bar{0}}^+$ is just the Lie algebra of Lorentz scalar differential operators containing the Lie algebra of vector fields on the circle, namely, $\text{diff}(S^1)$ and its dual is $\Sigma_{-\bar{0}}^-$. The graded Lie algebras Σ_{ij}^k are related to the superalgebras $\Xi^\pm$ Eqs.(29) as

$$\begin{aligned} \Xi^+ &= \bigoplus_{i=0,\pm} \bigoplus_{j=\bar{0},\bar{1}} \Sigma_{i,j}^+ \\ \Xi^- &= \bigoplus_{i=0,\pm} \bigoplus_{j=\bar{0},\bar{1}} \Sigma_{-i,j}^- \end{aligned} \quad (30)$$

A remarkable property of the infinite dimensional Lie superalgebra Ξ is that it splits into a linear sum of $6 = 3 \times 2$ graded subalgebras given by

$$\Xi = \bigoplus_{i=0,\pm} (S_i \oplus S_i^*), \quad (31)$$

where the graded Lie subalgebra S_i and its dual S_i^* split for a fixed conformal spin index i into

$$S_i = \Sigma_{i,0}^+ \oplus \Sigma_{i,1}^+ \quad (32a)$$

$$S_i^* = \Sigma_{-i,0}^- \oplus \Sigma_{-i,1}^-$$

or equivalently

$$\begin{aligned} S_- &= \Sigma_{-,0}^+ \oplus \Sigma_{-,1}^+ \\ S_0 &= \Sigma_{0,0}^+ \\ S_+ &= \Sigma_{+,0}^+ \oplus \Sigma_{+,1}^+ \end{aligned} \quad (32b)$$

and

$$\begin{aligned} S_+^* &= \Sigma_{-,0}^- \oplus \Sigma_{-,1}^- \\ S_0^* &= \Sigma_{0,0}^- \\ S_-^* &= \Sigma_{+,0}^- \oplus \Sigma_{+,1}^- \end{aligned} \quad (32c)$$

3 The huge Lie superalgebra Ξ and super-KP-hierarchies

Motivated by the well-known idea that the Lie (super) algebra methods allow for a unifying treatment of non linear integrable systems, we try in this section to find a relation between the Lie superalgebra Ξ of supersymmetric (pseudo) differential operators discussed previously and the various (super) KP integrable hierarchies. We first briefly review what is known about KP-hierarchy. The latter can be thought of as a dynamical system defined on a space whose functions $U_j(z)$ are elements of the ring $\mathcal{R}_j$ of analytic fields of conformal spin $j \in \mathbf{Z}$. It is also defined as the universal family of isospectral deformations of the pseudo-differential operator [3]

$$L = \partial + \sum_{i \geq 1} U_i(z) \partial^{i-1}. \quad (33)$$

The evolution of L is given by a commuting family of flows $\partial_t = \frac{\partial}{\partial t}$, in terms of which we have

$$\partial_t L_1 = [(L^1)_+, L] = -[(L^1)_-, L] . \quad (34)$$

The subscript $+$ (resp. $-$) means taking the purely differential (resp. pseudo-differential) part of L^1 and $t = \{t_i\}$ an infinite system of time variables. The flows Eq.(34) are bi-hamiltonians in the sense that there exist two Poisson bracket structures $\{ \cdot, \cdot \}_{1,2}$ such that we can rewrite Eq.(34) as

$$\partial_t L = \{H_t, L\}_2 = \{H_{t+1}, L\}_1 . \quad (35)$$

Here the hamiltonians for the KP-hierarchy are $H_r = \frac{1}{r} \int \text{res } L^r$, with $\text{res } \partial^{-1} = 1$.

There are two usual supersymmetric extensions of the standard KP-hierarchy Eqs.(33),(34). The first one is given by the Manin Radul supersymmetric KP-hierarchy associated to the odd super Lax operator [6]

$$L = D + \sum_{i \geq 0} U_{\frac{i+1}{2}}(\hat{z}) D^{-i} \quad (36)$$

The second one is given by the supersymmetric KP-hierarchy associated to the even super Lax operator [7]

$$L = D^2 + \sum_{i \geq 0} U_{\frac{i+2}{2}}(\hat{z}) D^{-i} . \quad (37)$$

The interesting consequence of the choice of L Eq.(37), is that under a suitable choice of reduction, it reduces to the Inami-Kanno super Lax operator describing the generalized $N = 2$ super KdV-hierarchy [9].

Using the $6 = 3 \times 2$ decomposition of the Lie superalgebra Ξ Eqs.(31),(32), one suspects that there exist $5 = 2 \times 2 + 1$ classes of supersymmetric KP-hierarchies. The origin of these hierarchies can be traced to the fundamental fact that there exist precisely five graded algebras

$$\begin{aligned} g_1 &= (S_+ \oplus S_-^*)_{\bar{0}} = \Sigma_{+\bar{0}}^+ \oplus \Sigma_{+\bar{0}}^- & (a) \\ g_2 &= (S_+ \oplus S_-^*)_{\bar{1}} = \Sigma_{+\bar{1}}^+ \oplus \Sigma_{+\bar{1}}^- & (b) \\ g &= (S_0 \oplus S_0^*)_{\bar{0}} = \Sigma_{0\bar{0}}^+ \oplus \Sigma_{0\bar{0}}^- & (c) \\ g_2^* &= (S_- \oplus S_+^*)_{\bar{1}} = \Sigma_{-\bar{1}}^- \oplus \Sigma_{-\bar{1}}^+ & (d) \\ g_1^* &= (S_- \oplus S_+^*)_{\bar{0}} = \Sigma_{-\bar{0}}^- \oplus \Sigma_{-\bar{0}}^+ & (e) , \end{aligned} \quad (38)$$

where $g_1 = \Sigma_{+\bar{0}}^+ \oplus \Sigma_{+\bar{0}}^-$ Eq.(38a) is the Lie algebra containing as a particular element, the even super Lax operator Eq.(37) of the Figueroa O'Farrill-Ramos super KP-hierarchy.

$g_2 = \Sigma_{+1}^+ \oplus \Sigma_{+1}^-$ is the graded space generated by elements like the odd super Lax operator Eq.(36) corresponding to the Manin–Radul super KP–hierarchy. The graded spaces g_2^* and g_1^* are just the dual of g_2 and g_1 with respect to the combined scalar product Eq.(28). Note also that the self–dual Lie algebra $g = \Sigma_{00}^+ \oplus \Sigma_{00}^-$ Eq.(38c) containing the super–differential operators of vanishing conformal spin type

$$L = U_{-\frac{1}{2}}(\hat{z}) D + \sum_{i \geq 0} U_{\frac{1}{2}}(\hat{z}) D^{-i} , \quad (39)$$

necessitate in itself a particular interest as it can lead to a new class of super KP–hierarchy. A more explicit description of these five super KP–hierarchies will be considered in a future paper.

4 Conclusion

Using the huge Lie superalgebra Ξ of pseudo superdifferential operators which splits into $6 = 3 \times 2$ Lie superalgebras Eqs.(31),(32), we have defined $5 = 2 \times 2 + 1$ graded algebras Eqs.(38). These graded algebras are expected to correspond to five classes of super KP–hierarchies generalizing the usual Manin–Radul and Figueroa O’Farrill–Ramos supersymmetric KP–hierarchies. We expect moreover that these $5 = 2 \times 2 + 1$ super KP–hierarchies will provide a unified framework that exhibits the underlying structure of $2d$ quantum supergravity.

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