Newman - Penrose Formalism for Gravitational Shock Waves

Koichi HAYASHI¹ and Toshiharu SAMURA² †

¹Department of Mathematics and Physics, Faculty of Science and Technology, Kinki University, Higashi-Osaka, Osaka 577, Japan

²The Graduate School of Science and Technology Kobe University, Nada, Kobe 657, Japan

Abstract
First order perturbations for the fields with spin on the background metric of the gravitational shock waves are discussed. Applying the Newman – Penrose formalism, exact solutions of the perturbation equations are obtained. For particle physics, this would be one approach to the problem of scattering particle at Planck energy.

†Electric mail address: samura@jet.eart.h.s.kobe-u.ac.jp
1 Introduction

The gravitational shock wave (GSW) of a black hole is a solution that is obtained by the black hole moving at the limit of light velocity. Recently, we have calculated the metrics both for Schwarzschild and Kerr black holes [1]. Comparing with the GSW metric derived by Aichelburg and Sexl [2] (below AS), our metrics are interesting for the point that the mass of the black hole is finite.

As physical applications of these metrics, two cases have been considered: the gravitational waves emission when two black holes collide [3], and the scattering of particles at Planck scale as a model of quantum gravity [4]. In these papers, these applications had been calculated using only the AS metric. We have also investigated these using our metric [5].

In this paper, we apply the Newman–Penrose (NP) formalism for GSW metrics and calculate various physical quantities in perturbations. The usual approach to obtain perturbed solution is perturbing the metric directly and solving for the resulting perturbed field equation. However, it is in general very difficult to obtain the direct solution for metric perturbation equation except for simple cases.

On the other hand, the approach of the NP formalism provides soluble perturbation equations for much wider cases. The perturbations of the complicated metrics like the Kerr black hole or Reissner–Nordström one are successful by this approach. In the NP formalism, the perturbation equations for Weyl tensors decouple to independent equations, for which the partial wave analysis are possible using suitable radial functions and spin weighted spherical harmonics.

When perturbed solutions are obtained, we can deal with astrophysical applications: e.g. stability of a black hole, tidal friction effects, superradiant scatterings, and gravitational wave processes. The NP formalism of the GSW metrics derived in this paper is not similarly another mathematical method to obtain the results by ordinary procedures, but the motivations which would shed new lights on physics exist. Especially, this approach is suitable to treat the scattering problem at the Plankian energies. Moreover, we can challenge to several unsolved problems by this method: the scattering of particles with spin, the collision of two GSW’s, the scattering of a particle moving near a black hole, the spontaneous emission of radiation, and so on.

In this paper, we show the first step to these approaches. In section 2, we choose a simple NP tetrad of GSW metric, then, the spin coefficients, the
Wely tensors, the Ricci tensors and the scalar tensor are calculated. The field equation of a scalar field in flat space–time is given by the Klein–Gordon equation, \((\Box + m^2)\phi = 0\). In the curved space–time, this is modified to \((\nabla_\alpha \nabla^\alpha + m^2)\phi = 0\), using covariant derivatives. For the metric of GSW, the solution of this equation is given by ’t Hooft [4], and we can see the effects of GSW metric on the behavior of the scalar field. How about the other fields with spin? This is the main subject of this paper. We calculate the effects of GSW metric on various fields with spin by perturbations. In section 3, we derive the differential equations for a test neutrino field (spin = 1/2), a test electromagnetic field (spin = 1) and a gravitational perturbation (spin = 2) in the background metric of GSW, Exact solutions of these equations are given. Section 4 is devoted to conclusions and discussions.

## 2 Newman–Penrose Formalism

In the Newman-Penrose (NP) formalism, a set of null tetrad \(\ell, n, m, \overline{m}\) is to be introduced, where \(\ell, n\) are real, and \(m, \overline{m}\) are complex conjugates of each other. These must satisfy the orthogonality conditions:

\[
\ell \cdot m = \ell \cdot \overline{m} = n \cdot m = n \cdot \overline{m} = 0 ,
\]

null conditions:

\[
\ell \cdot \ell = n \cdot n = m \cdot m = \overline{m} \cdot \overline{m} = 0 ,
\]

and the normalization conditions:

\[
\ell \cdot n = 1 \quad \text{and} \quad m \cdot \overline{m} = -1 .
\]

Now, we will investigate the NP formalism of the gravitational shock wave (GSW) metrics which are derived by us [1]. For the construction of a null–tetrad frame for the NP formalism, we must find the null tangent vectors for geodesics. The GSW metric is generally written by:

\[
ds^2 = du \, dv - d\rho^2 - \rho^2 d\varphi^2 - \Delta(\rho) \delta(u) \, du^2 ,
\]

where \(u = t - z, v = t + z\), and the concrete forms of \(\Delta(\rho)\) are given in the previous paper. The null geodesic of (4) satisfy:

\[
\frac{1}{2} \dot{u} = \text{constant} \left(= \frac{1}{2}\right)
\]
\[ \frac{1}{2} \dot{v} + A(\rho) \delta(u) \dot{\rho} = \text{constant } (= 0) \]

\[ \rho^2 \dot{\varphi} = \text{constant } (= 0) \]

\[ \ddot{\rho} = \rho \dot{\varphi}^2 - \frac{1}{2} A'(\rho) \delta(u) \dot{\rho}^2, \quad (5) \]

where dot and dash denotes the derivative with respect to the affine parameter \( \lambda \), and \( \rho \), respectively. With no loss of generality, we take the constants of the RHS of (5) to the numbers in the parenthesis. With this choice, \( \dot{\rho} = 1 \), so that \( u \) can be identified as the affine parameter \( \lambda \) itself. The equations in (5) are rewritten as

\[ \dot{\rho} = 1 \]

\[ \dot{\varphi} = 0 \]

\[ \ddot{\rho} = -\frac{1}{2} A' \theta, \quad (6) \]

where \( \theta \) is the step function. For simplicity we put \( A(\rho), \delta(u), \theta(u) \), and \( A'(\rho)|_{\rho=\rho_0} \) as \( A, \delta, \theta \) and \( A' \), respectively.

Then since the null tangent vector

\[ v^i = (\dot{u}, \dot{v}, \dot{\rho}, \dot{\varphi}) = \left( 1, -2A\delta, -\frac{1}{2} A' \theta, 0 \right), \quad (7) \]

the vector \( \ell \) of the NP formalism is taken as \( v^i \) itself:

\[ \ell^i = (\ell^u, \ell^v, \ell^\rho, \ell^\varphi) = \left( 1, -2A\delta, -\frac{1}{2} A' \theta, 0 \right), \quad (7) \]

\[ \ell_i = \left( 0, \frac{1}{2}, \frac{1}{2} A' \theta, 0 \right). \quad (8) \]

It is to be noted that \( \ell^i \ell_i = -A\delta - (\dot{\rho})^2 = 0 \), since the derivative of the equation \( \ddot{\rho} = -A\delta \) with respect to \( \lambda \) gives exactly the last equations in (5).

For the null vector \( n \) and \( m \) we will take

\[ n^i = (a, b, c, d) \]

\[ m^i = (0, \alpha, \beta, i\gamma), \quad (9) \]

From conditions of (1)(2)(3), \( m^i \) is determined uniquely:

\[ \alpha = -\frac{1}{\sqrt{2}} A' \theta, \quad \beta = \frac{1}{\sqrt{2}}, \quad \gamma = \frac{1}{\sqrt{2} \rho}, \quad (10) \]
while for \( n^i \), the following equations are obtained:
\[
\begin{align*}
  b + A'\theta c &= 2 \\
  A'\theta a + 2c &= 0 \\
  c^2 &= ab + A\delta a^2 .
\end{align*}
\]
(11)
The simplest solution is
\[
a = 0, \quad b = 2, \quad c = 0 .
\]
(12)
Let us summarize the null vectors of NP formalism which we have chosen:
\[
\begin{align*}
  \ell^i &= \left( 1, -2A\delta, -\frac{1}{2}A'\theta, 0 \right) , \\
n^i &= \left( 0, 2, 0, 0 \right) , \\
m^i &= \left( 0, -\frac{1}{\sqrt{2}}A'\theta, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}\rho} \right) .
\end{align*}
\]
(13)
and corresponding covariant vectors are
\[
\begin{align*}
  \ell_i &= \left( 0, \frac{1}{2}, \frac{1}{2}A'\theta, 0 \right) , \\
n_i &= \left( 1, 0, 0, 0 \right) , \\
m_i &= \left( -\frac{1}{\sqrt{2}}A'\theta, 0, -\frac{1}{\sqrt{2}}, -\frac{i\rho}{\sqrt{2}} \right) .
\end{align*}
\]
(14)
The nonvanishing spin coefficients (for example, see (286) in Chapter 1 of Chandrasekhar [6]) are
\[
\begin{align*}
  \kappa &= \frac{1}{2\sqrt{2}}A'\delta, \\
  \rho &= -\sigma = \frac{1}{4\rho}A'\theta, \\
  \alpha &= -\beta = -\frac{1}{2\sqrt{2}\rho} .
\end{align*}
\]
(15)
Here, note that \( A' \) is a constant.
Finally, we can calculate the components of the Weyl tensor, \( \Psi_i \), Ricci tensor, \( \Phi_{ij} \), and scalar tensor, \( \Lambda \) (see (294) and (300) in Chap.1 of [6]). Then we see all components of these tensors vanish:
\[
\Psi_i = \Phi_{ij} = \Lambda = 0 \quad (i, j = 0 \sim 4) \quad (\text{For } u \neq 0) .
\]
(16)
Since the space-time of GSW is empty except for \( u = 0 \), this is rather a natural result. The whole space-time is the patchwork of two flat Minkowski spaces pasted at \( u = 0 \), where the continuity is destroyed. Because of this discontinuity, (15) is obtained and physics are not trivial even with (16). In the next section we consider the perturbation over this background space-time. Although this is almost flat, it reflects this discontinuity through (15).
3 Perturbations on the Gravitational Shock Waves Background

The NP equations are system of first – order differential equations linking the tetrad, the spin coefficients, Wely tensors, Ricci tensors and the scalar curvature. The perturbed geometries in NP formalism are specified by:

\[ \ell = \ell^u + \ell^p, \quad n = n^u + n^p, \quad m = m^u + m^p, \] (17)

where the supersuffix \( u \) and \( p \) means ”unperturbed” and ”perturbed”, respectively. All the NP quantities can be written in this form. The spin coefficients \( \kappa, \rho, \sigma, \alpha \) and \( \beta \) have both the unperturbed quantities and the perturbed ones. The other spin coefficients, all Wely tensors, Ricci tensors and the scalar curvature have only the unperturbed quantities. The complete set of perturbation equations are obtained from the NP equations by keeping perturbed terms only to first order.

In this section, we will derive the source free perturbation equations for two component neutrino fields, electromagnetic fields, and gravitational fields on the GSW background.

3.1 Neutrino equations

In the NP formalism for the GSW space – time, the Dirac equations of the massless particle are written by the following equations (see (108) in Chap.10 of [6]):

\[ (D - \rho) F_1 + (\delta^* - \alpha) F_2 = 0 \] (18)
\[ (\delta - \alpha) F_1 + \Delta F_2 = 0 , \] (19)

where \( F_1 \) and \( F_2 \) are 2-spinors. When we consider the realistic problem of the scattering of a massless neutrino off the GSW background, the \( F \)'s can be treated as the first order test fields. \( D, \delta \) and \( \Delta \) are directional derivatives along the basis null vectors defined by:

\[ DF = F_{\mu} \ell^{\mu} , \quad \Delta F = F_{\mu} n^{\mu} , \quad \delta F = F_{\mu} n^{\mu} \] (20)

Eliminating \( F_2 \) from (18) and (19), we obtain the equation for \( F_1 \):

\[ [\Delta (D - \rho) - (\delta^* - \alpha) (\delta - \alpha)] F_1 = 0 , \] (21)
where we have used the relation, $\Delta (\delta^* - \alpha) - (\delta^* - \alpha) \Delta = 0$.

We want to find the solution for $F_i$ which have the form:

$$F_i = e^{i(k_u u + k_v v + m\varphi)} f_i(\rho). \quad (22)$$

where $k_u$ and $k_v$ are constants and $m$ is an integer, General solutions are to be obtained by superpositions of them. When $u > 0$, the equation of $F_1$ is given by:

$$f_{1, \rho, \rho} + \frac{1}{\rho} f_{1, \rho} + \left[(4k_u k_v - k_v^2 A'^2) - \frac{(m - 1/2)^2}{\rho^2}\right] f_1 = 0, \quad (23)$$

and the solution of it is:

$$f_1(\rho) = J_{m - 1/2}(y) \quad (for \ u > 0), \quad (24)$$

where $J$ is the Bessel function, $y = \Omega \rho$ and $\Omega = \sqrt{4k_u k_v - k_v^2 A'}$.

For $u < 0$, $f_1(\rho)$ is similarly obtained:

$$f_1(\rho) = J_{m - 1/2}(y') \quad (for \ u < 0), \quad (25)$$

where $y' = \Omega' \rho$ and $\Omega' = \sqrt{4k_u k_v}$.

Next, we will give the $F_2$. From (19) and (22), $f_2$, which is the $\rho$-direction field of $F_2$, is represented by $f_1$:

$$2\sqrt{2} i k_v f_2 = -f_{1, \rho} + iA' k_v f_1 + \frac{m - 1/2}{\rho} f_1 \quad (for \ u > 0). \quad (26)$$

Using the formula, $J_{\nu}'(z) = -J_{\nu+1}(z) + \nu / z J_{\nu}(z)$, $f_2$ for $u > 0$ is given by:

$$f_2(\rho) = \frac{\Omega}{2\sqrt{2} i k_v} \left\{ J_{m+1/2}(y) + \frac{i k_v A'}{\Omega} J_{m-1/2}(y) \right\} \quad (for \ u > 0), \quad (27)$$

while for $u < 0$,

$$f_2(\rho) = \frac{\Omega'}{2\sqrt{2} i k_v} J_{m+1/2}(y') \quad (for \ u < 0). \quad (28)$$

It is to be noted that these solutions are the exact solutions for the neutrino fields.
3.2 Electromagnetic and Gravitational Equations

Maxwell equations in the NP formalism of the GSW geometry can be written by (see (330)–(333) in Chap.1 of [6]):

\[
\begin{align*}
(D - \rho) \phi_2 - \delta^* \phi_1 &= 0 \quad (29) \\
(\delta - 2\alpha) \phi_2 - \Delta \phi_1 &= 0 \quad (30) \\
\delta \phi_1 - \Delta \phi_0 + \sigma \phi_2 &= 0 \quad (31) \\
(\delta - 2\rho) \phi_1 - (\delta^* - 2\alpha) \phi_0 &= 0 \quad , \quad (32)
\end{align*}
\]

where \(\phi\)'s are Maxwell field strengths. With \(\Delta \times (29) - \delta^* \times (30)\), the equation for \(\phi_2\) is given by the following equation:

\[
[\Delta (\Delta - \rho) - \delta^* (\delta - 2\alpha)] \phi_2 = 0 \quad , \quad (33)
\]

As in the previous subsection, we put

\[
\phi_i = e^{i(k_u u + k_v v + m \phi)} R_i(\rho) \quad . \quad (34)
\]

Then (33) is written by the following differential equation for \(u > 0\):

\[
R_{2,\rho,\rho} + \frac{1}{\rho} R_{2,\rho} + \left[ (4k_u k_v - k_v^2 A'') - \frac{(m - 1)^2}{\rho^2} \right] R_2 = 0 \quad . \quad (35)
\]

The equation is easily solved:

\[
R_2(\rho) = J_{m-1}(y) \quad \text{(for } u > 0) \quad , \quad (36)
\]

as before. On the other hand, the solution for \(u < 0\) is

\[
R_2(\rho) = J_{m-1}(y') \quad \text{(for } u < 0) \quad , \quad (37)
\]

where \(y\) and \(y'\) are given by (24) and (25). As \(\phi_1\) and \(\phi_2\) are related by (30), \(R_1\) is given by

\[
R_1(\rho) = \left( -\frac{\Omega}{2\sqrt{2ik_v}} \right) \left\{ J_m(y) + \frac{i k_v A'}{\Omega} J_{m-1}(y) \right\} \quad \text{(for } u > 0) \quad , \quad (38)
\]

while for \(u < 0\), it is:

\[
R_1(\rho) = \left( -\frac{\Omega'}{2\sqrt{2ik_v}} \right) J_m(y') \quad \text{(for } u < 0) \quad . \quad (39)
\]
Finally, $R_0$ is given from (31)

$$
R_0 (\rho) = \left(-\frac{\Omega}{2\sqrt{2i k_v}}\right)^2 \left\{ J_{m+1} (y) + \frac{2i k_v A'}{\Omega} J_m (y) \right. \\
\left. + \left(\frac{i k_v A'}{\Omega}\right)^2 J_{m-1} (y) \right\} \quad (\text{for } u > 0)
$$

(40)

$$
R_0 (\rho) = \left(-\frac{\Omega'}{2\sqrt{2i k_v}}\right)^2 J_{m+1} (y') \quad (\text{for } u < 0).
$$

(41)

Now we want to discuss about the first–order perturbations of the gravitational field itself. We find that four of the eight NP Bianchi identities provide the following linear homogeneous equations to first order in the perturbations (see (321) in Chap.1 of [6]):

\begin{align}
(\delta^* + 2\alpha) \Psi_3 - (D - \rho) \Psi_4 &= 0 \quad (42) \\
-\Delta \Psi_0 + (\delta - 2\beta) \Psi_1 + 3\sigma \Psi_2 &= 0 \quad (43) \\
-\Delta \Psi_1 + \delta \Psi_2 + 2\sigma \Psi_3 &= 0 \quad (44) \\
-\Delta \Psi_2 + (\delta + 2\beta) \Psi_3 + \sigma \Psi_4 &= 0 \quad (45) \\
-\Delta \Psi_3 + (\delta + 4\beta) \Psi_4 &= 0 \quad (46)
\end{align}

where $\Psi$’s are the perturbed Weyl tensors. We can calculate $\Psi$’s by the same procedure as neutrino and electromagnetic cases, and the final forms of $\Psi$’s are:

\begin{align}
\Psi_n &= \exp(ik u + ik_v v + im \phi) B^{4-n} J_{m-n+2} (y') \quad (\text{for } u < 0), \\
\Psi_n &= \exp(ik u + ik_v v + im \phi) B^{4-n} \sum_{l=1}^{5-n} 4 C_l D^{l-1} J_{m-n+3-l} (y) \quad (\text{for } u > 0),
\end{align}

(47)

(48)

where

$$
B = -\frac{\Omega'}{2\sqrt{2i k_v}}, \quad D = \frac{ik_v A'}{\Omega},
$$

(49)

and $4 C_l$ represents the binomial coefficient.
4 Conclusions

We have derived the first order perturbations for fields with spin on the background metric of GSW. Exact solutions of fields are obtained. From these, we can find the behavior of the field crossing the discontinuity of GSW at $u = 0$. Especially, we can find, for example, the refraction angles and scattering cross sections of various fields near the black hole which is moving with a relativistic speed. For particle physics, this would be one approach to the physics at Planck energy. These physics are now under investigations.

References


