Continuous Self-Similarity and S-Duality

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Abstract

We study the spherically symmetric collapse of the axion/dilaton system coupled to gravity. We show numerically that the critical solution at the threshold of black hole formation is continuously self-similar. Numerical and analytical arguments both demonstrate that the mass scaling away from criticality has a critical exponent of $\gamma = 0.264$.

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1. Introduction

Consider a spherically symmetric spacetime filled by a massless scalar field $\Psi$. The field $\Psi$ evolves according to

$$g^{ab}\nabla_a \nabla_b \Psi = 0 ,$$

(1.1)

where the geometry is determined by Einstein’s field equations in the form

$$R_{ab} = \kappa^2 \partial_a \Psi \partial_b \Psi ,$$

(1.2)

where $\kappa^2 = 8\pi G$. At first sight this problem would appear to be totally predictable and boring. Appearances are however deceptive.

The system of equations (1.1) and (1.2) has been studied numerically, as a Cauchy problem in [1] and as a characteristic initial value problem in [2]. In both cases the initial field can be specified in an arbitrary manner. Choptuik examined the following problem in [1]. Suppose the initial data belong to a one-parameter set. Suppose further that for small values of the parameter $p$, the field is weak. One expects the evolution to be similar to the solution of (1.1) on Minkowski spacetime; the field evaporates leaving an empty spacetime. Suppose that for large values of the parameter $p$, the initial fields are strong. Assuming some form of cosmic censorship hypothesis, we might expect a black hole to form around a spacetime singularity. By continuity there should be a critical parameter $p_{\text{crit}}$ separating the two types of evolution.

Choptuik discovered numerically that nearly critical solutions appear to exhibit discretely self-similar behavior — structure appears on ever finer scales [1]. (This was not discovered in previous numerical work because an adaptive grid algorithm is needed to resolve it.) If this behavior is real then the field and curvature can grow arbitrarily large close to the axis. Further, Choptuik exhibited in [1] a scaling law. For supercritical evolutions ($p > p_{\text{crit}}$) a black hole forms and he obtained a numerical law estimating the black hole mass

$$M_{\text{bh}} = c_i (p - p_{\text{crit}})^\gamma .$$

(1.3)

Here $c_i$ depends on the nature of the initial data but $\gamma \approx 0.37$ appears to be independent of the initial data. These results were confirmed in [2], and similar results for other forms of matter have been established in [3,4].

Discrete self-similarity is hard to treat analytically (see however [5]), and so theoreticians have looked for other forms of matter for which continuous self-similarity is
possible. Assuming continuous self-similarity for the critical solution reduces the problem to an ordinary differential equation. A perfect radiation fluid coupled to gravity was studied in [4], where it was numerically found that the critical solution is indeed continuously self-similar. Analytical studies of continuously self-similar solutions where the scaling symmetry mixes with field symmetries have been made for the complex scalar field [6] and for the axion/dilaton field [7], but it was not demonstrated that the continuously self-similar solution is indeed the critical solution.

This paper reports on a numerical study of the axion/dilaton problem using the numerical techniques of [2]. In §2 we describe the axion/dilaton problem in detail and §3 describes the numerical procedure. Section 4 reviews the conjectured continuously self-similar critical solution. The main result of the paper is given in §5; there is a continuously self-similar solution which appears to be the attractor at criticality. In §6 we consider slightly supercritical solutions and confirm both analytically and numerically the scaling law (1.3). We estimate

\[
\gamma_{\text{analytic}} = 0.2641066, \quad \gamma_{\text{numeric}} = 0.264, \quad (1.4)
\]

giving excellent agreement between theory and experiment. This critical exponent is however very different from the value \( \gamma \approx 0.37 \) found for a massless scalar field [1], black body radiation [4], and gravitational radiation [3].

2. The axion/dilaton system

We use the notation of [7]. The matter used in this paper occurs in the \( 3 + 1 \)-dimensional low-energy effective action of string theory. It consists of gravity coupled to a dilaton \( \phi \) and an axion \( \rho \). The axion arises as the dual of the string three-form \( H \), via

\[
H_{abc} = \frac{1}{\kappa} e^{4\phi} \epsilon_{abcd} \partial^d \rho. \quad (2.1)
\]

The dilaton and axion can be combined into a single complex field

\[
\tau \equiv 2\rho + i e^{-2\phi}. \quad (2.2)
\]

(We have chosen to rescale \( \rho \) by a factor of 2 from the usual definition.) Omitting the (presumably) irrelevant gauge fields, the effective action is

\[
I = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left( R - \frac{1}{2} \frac{\partial_a \tau \partial^a \tau}{(\text{Im } \tau)^2} \right), \quad (2.3)
\]
where $R$ is the scalar curvature.

At the classical level, the model (2.3) has an extra $SL(2, \mathbb{R})$ symmetry that involves only $\tau$ and leaves the metric invariant. The symmetry acts on $\tau$ as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. This $SL(2, \mathbb{R})$ symmetry is presumably broken by quantum effects down to an $SL(2, \mathbb{Z})$ symmetry, which is the popular $S$-duality conjectured to be a non-perturbative symmetry of string theory.

The equations of motions arising from (2.3) are

$$R_{ab} - \frac{1}{4(\text{Im } \tau)^2}(\partial_a \tau \partial_b \bar{\tau} + \partial_a \bar{\tau} \partial_b \tau) = 0,$$  

(2.5a)

$$\nabla^a \nabla_a \tau + \frac{i \nabla^a \tau \nabla_a \tau}{\text{Im } \tau} = 0.$$  

(2.5b)

We will use two approaches to solve these equations. First, we will integrate them numerically given specified initial conditions. This will give us a range of numerical solutions, some ending with black holes, some ending with flat space. Second, we will follow [7] and assume that the exactly critical solution is continuously self-similar. The spacetime can then be derived analytically. We then compare the answers, and find extremely good agreement.

3. The Numerical Approach

The numerical integration is based on the approach described in [2]. Throughout this paper, we assume spherical symmetry. As in [2], the coordinate system most suitable to the problem at hand is a double-null coordinate system. We take the spacetime metric to be

$$ds^2 = -a^2(u, v) du dv + r^2(u, v) d\Omega^2.$$  

(3.1)

Double-null coordinates are not unique. The remaining gauge freedom consists of redefining $u$ and $v$ by $u \rightarrow f(u)$ and $v \rightarrow g(v)$, where $f$ and $g$ are monotonic functions. We fix one of the functions by requiring $u = v$ on the axis where $r(u, v) = 0$. The remaining gauge freedom will be fixed by the initial conditions. See [2] for further details.
With these coordinates, the equations of motion (2.5) become
\[
rr_{uv} + ru rv + \frac{1}{4} a^2 = 0 , \quad (3.2a)
\]
\[
a_{uv}/a - a_u a_v/a^2 + r_{uv}/r + \phi_u \phi_v + e^{4\phi} \rho_u \rho_v = 0 , \quad (3.2b)
\]
\[
r_{uu} - 2a_u r_u /a + r \left( \phi_u^2 + e^{4\phi} \rho_u^2 \right) = 0 , \quad (3.2c)
\]
\[
r_{vv} - 2a_v r_v /a + r \left( \phi_v^2 + e^{4\phi} \rho_v^2 \right) = 0 , \quad (3.2d)
\]
\[
r \phi_{uv} + r_u \phi_v + r_v \phi_u - 2r e^{4\phi} \rho_u \rho_v = 0 , \quad (3.2e)
\]
\[
r \rho_{uv} + r_u \rho_v + r_v \rho_u + 2r \left( \phi_u \rho_v + \phi_v \rho_u \right) = 0 . \quad (3.2f)
\]

It should be noted that when \( \rho = 0 \), the equations (3.2) reduce to a real scalar field coupled to gravity. This allows us to check much of the code.

Since we shall be comparing results in different coordinate systems, we need to know what invariant quantities can be built from the above fields and metric. One quantity is the proper time \( T \) on the \( u = v \) axis, given by
\[
T = \int a(u, u) \, du . \quad (3.3)
\]

All curvature invariants can be constructed from the following three building blocks. First, there is the Ricci scalar curvature
\[
R = -\frac{8}{a^2} \left( \phi_u \phi_v + e^{4\phi} \rho_u \rho_v \right) , \quad (3.4)
\]
where we have used the equations of motion (3.2). Two other curvature invariants exist, defined by
\[
R_2 \equiv \sqrt{\frac{1}{32} \left( R_{ab} R^{ab} - R^2 \right)} = \frac{e^{2\phi}}{a^2} \left( \phi_u \rho_v - \phi_v \rho_u \right) , \quad (3.5)
\]
\[
R_3 \equiv \sqrt{\frac{1}{12} C_{abcd} C^{abcd} - \frac{1}{6} R^2} = \frac{1}{r^2 a^2} \left( a^2 + 4r_u r_v \right) = 2m(u, v)/r^3 , \quad (3.6)
\]
where \( C_{abcd} \) is the Weyl tensor, and \( m(u, v) \) is the Hawking mass. It is simple to show, e.g., using boundary conditions given later in this section, that on axis, \( u = v \), \( R_2 = 0 \), and \( R_3 = -R/6 \).

We can convert the system of equations (3.2) to a first order system by defining new variables
\[
\psi = \phi_v , \ \bar{\psi} = \phi_u , \ \xi = \rho_v , \ \bar{\xi} = \rho_u , \ h = r_v , \ \bar{h} = r_u , \ e = a_v/a . \quad (3.7)
\]
In these new variables, (3.2) becomes

\[ r h_u + h h + \frac{1}{4} a^2 = 0 , \]  \hspace{1cm} (3.8a)

\[ r h_v + h h + \frac{1}{4} a^2 = 0 , \]  \hspace{1cm} (3.8b)

\[ e_u - (h h + \frac{1}{4} a^2)/r^2 + (\psi \psi + e^{4\phi} \xi \xi) = 0 , \]  \hspace{1cm} (3.8c)

\[ h_v - 2 e h + r (\psi^2 + e^{4\phi} \xi^2) = 0 , \]  \hspace{1cm} (3.8d)

\[ r \psi_u + h \psi + h \psi - 2 r e^{4\phi} \xi \xi = 0 , \]  \hspace{1cm} (3.8e)

\[ r \psi_v + h \psi + h \psi - 2 r e^{4\phi} \xi \xi = 0 , \]  \hspace{1cm} (3.8f)

\[ r \xi_u + h \xi + h \xi + 2 r (\psi \xi + \psi \xi) = 0 , \]  \hspace{1cm} (3.8g)

\[ r \xi_v + h \xi + h \xi + 2 r (\psi \xi + \psi \xi) = 0 , \]  \hspace{1cm} (3.8h)

to which we adjoin

\[ a_v - a e = 0 , \]  \hspace{1cm} (3.8i)

\[ r_v - h = 0 , \]  \hspace{1cm} (3.8j)

\[ \rho_v - \xi = 0 , \]  \hspace{1cm} (3.8k)

\[ \phi_v - \psi = 0 , \]  \hspace{1cm} (3.8l)

which follow from eq. (3.7).

Next we write down the boundary conditions which apply on axis, \( u = v \). Obviously, \( r = 0 \) there, which implies \( r_u + r_v = 0 \), i.e., \( h = -h \). Eq. (3.8c) will be regular on axis if and only if \( a = 2 h \) (assuming \( a > 0 \)). Now (3.8g) implies \( \xi = \xi \) and (3.8f) implies \( \psi = \psi \) on axis. From the definitions (3.7), these imply \( \rho_r = \phi_r = 0 \) on axis. It follows from a power series expansion (in powers of \( r \)) that \( a_r = 0 \) on axis.

Suppose that the fields \( e, \psi, \) and \( \xi \) are known as functions of \( v \) on a surface \( u = u_0 = \) constant. We can integrate (3.8l) as an ordinary differential equation in \( v \) for \( \phi \). The initial data come from the boundary condition \( \phi_r = 0 \) on axis. Eq. (3.8k) for \( \rho \) and (3.8i) for \( a \) are treated in identical fashion. Next, eqs. (3.8d), (3.8j) are integrated as simultaneous equations for \( r \) and \( h \), with \( r = 0, h = a/2 \) on axis. Eq. (3.8b) with data \( h = -h \) on axis gives \( h \) as a function of \( v \). Finally we integrate (3.8f), (3.8h) with data \( \psi = \psi, \xi = \xi \) on axis, to obtain \( \psi \) and \( \xi \) as functions of \( v \). Thus we now have all quantities on the surface \( u = u_0 \). Next we determine \( e, \psi, \) and \( \xi \) on the surface \( u = u_0 + \Delta u = \) constant by integrating (3.8c), (3.8e) and (3.8g). Returning to the start of this paragraph, we continue.
the integration process until either a singularity forms or the fields have evaporated away leaving flat spacetime.

In order to start the integration we need to give initial data \( e \), \( \psi \), and \( \xi \) on the initial surface \( u = 0 \). The remaining coordinate freedom is equivalent to the choice of \( e \) (or \( a \)) on the surface \( u = 0 \). This is arbitrary. The initial fields \( \psi \) and \( \xi \) determine the physics.

The \( u \)-integration is done using an explicit finite difference algorithm. The \( v \)-integration is done using an implicit (midpoint) algorithm. However, if the equations are integrated in the manner described above, they are linear and so the method is effectively explicit. We operate exclusively at a Courant number of 1, and the implicit algorithm provides the necessary stability. There is no need for any artificial viscosity.

During the evolution fine structure appears on ever-decreasing scales. We use the adaptive mesh algorithm of Berger and Oliger [8] as modified in [2]. This enables us to insert extra grid points (again with a Courant number of 1) where and when necessary to resolve the fine detail.

4. The analytic critical solution

It was conjectured in [7] that the precisely critical solution of the axion/dilaton system coupled to gravity is continuously self-similar. The continuously self-similar solution is described in detail there, but we review it here. Continuous self-similarity means that there exists a homothetic Killing vector \( \zeta \) that satisfies

\[
\mathcal{L}_\zeta g_{ab} = 2g_{ab} .
\]  

(4.1)

The homothetic symmetry mixes with the \( SL(2, \mathbb{R}) \) symmetry, changing \( \tau \) by an infinitesimal \( SL(2, \mathbb{R}) \) transformation

\[
\mathcal{L}_\zeta \tau = \alpha_0 + \alpha_1 \tau + \alpha_2 \tau^2 ,
\]

(4.2)

with \( \alpha_i \in \mathbb{R} \).

The standard approach to spherically symmetric self-similar solutions is to assume the metric has the form* 

\[
ds^2 = (1 + u(t, r)) \left( -b^2(t, r) dt^2 + dr^2 \right) + r^2 d\Omega^2 .
\]

(4.3)

* Ideally, one would use double-null coordinates as in the preceding section. For unknown technical reasons, we have been unable to repeat the calculations of [7] in a \( u - v \) coordinate system.
The time coordinate \( t \) is chosen so that the singular point occurs at \( t = 0 \), and the metric is regular for \( t < 0 \). The homothetic vector \( \zeta = t \partial_t + r \partial_r \). One then defines a new scale invariant coordinate \( z \equiv -r/t \). In terms of the \( z - t \) coordinates, the metric is

\[
 ds^2 = (1 + u(z)) \left[ (z^2 - b^2(z)) \, dt^2 + 2zt \, dt \, dz + t^2 \, dz^2 \right] + z^2 t^2 \, d\Omega^2 , \tag{4.4}
\]

and \( \zeta \) becomes simply \( \zeta = t \partial_t \). After fixing unimportant constants, the axion/dilaton field has the form

\[
 \tau(t, z) = \frac{1 - |t|^{\mathfrak{i} \omega} f(z)}{1 + |t|^{\mathfrak{i} \omega} f(z)} , \tag{4.5}
\]

where \( \omega \) is a real constant, and \( f(z) \) is a complex function. We will sometimes write \( f(z) = f_m(z) e^{\mathfrak{i} f_a(z)} \), where \( f_m \) and \( f_a \) are real functions.

Details of the construction of the solution can be found in [7]. The \( r = 0 \) axis corresponds to \( z = 0 \) and is a singular point of the equations of motion. One can choose \( b(z = 0) = 1 \) for \( t < 0 \). The past light cone of the origin corresponds to the line where \( b(z_+) = z_+ \), and is also a singular point of the equations of motion. Requiring regularity along the \( r = 0 \) axis (for \( t < 0 \)) and regularity at \( z_+ \) uniquely fixes the solution in terms of \( \omega, z_+, |f(0)|, \) and \(|f(z_+)|\), with

\[
 \omega = 1.17695272200 \pm 0.00000000270 , \tag{4.6a}
\]

\[
 z_+ = 2.60909347510 \pm 0.00000000216 , \tag{4.6b}
\]

\[
 |f(0)| = 0.892555411872 \pm 0.00000000224 , \tag{4.6c}
\]

\[
 |f(z_+)| = 0.364210875022 \pm 0.00000000760 . \tag{4.6d}
\]

We can now construct the same invariant quantities as in the \( u - v \) coordinates. On the \( r = 0 \) axis, \( u(z = 0) = 0 \) and \( b(0) = 1 \), so the proper time is simply \( T = t \). The curvature invariants are

\[
 R = \frac{2}{t^2 z(1 + u(z))} \left[ \frac{2 \omega |f(z)|^2 f_a'(z)}{(1 - |f(z)|^2)^2} - \frac{u(z)}{z} \right] , \tag{4.7}
\]

\[
 R_2 = \frac{\omega |f(z)| f_m'(z)}{2t^2 b(z)(1 + u(z))(1 - |f(z)|^2)^2} , \tag{4.8}
\]

\[
 R_3 = \frac{u(z)}{t^2 z^2 (1 + u(z))} . \tag{4.9}
\]
We can use regularity on the \( r = 0 \) axis to determine that

\[
R(z = 0) = - \frac{2\omega^2 |f(0)|^2}{T^2(1 - |f(0)|^2)^2} = - \frac{53.38}{T^2},
\]

(4.10)

and \( R_2(z = 0) = 0, R_3(z = 0) = -R/6 \), as expected. On the null cone \( b(z_+) = z_+ \), there is no proper time. Define \( \Lambda = (1 + (\omega^2 - 2)|f(z_+)|^2 + |f(z_+)|^4). \) Using regularity, we find

\[
R(z_+) = - \frac{2\omega^2 |f(z_+)|^2(1 - |f(z_+)|^2)^2}{t^2 z_+^2 \Lambda (1 + (\omega^2 - 2)|f(z_+)|^4 + |f(z_+)|^8)},
\]

(4.11a)

\[
R_2(z_+) = - \frac{\omega|f(z_+)|^2(1 - |f(z_+)|^2)^3(1 + |f(z_+)|^2)}{4t^2 z_+^2 \Lambda (1 + (\omega^2 - 2)|f(z_+)|^4 + |f(z_+)|^8)},
\]

(4.11b)

\[
R_3(z_+) = - \frac{\omega^2 |f(z_+)|^2}{t^2 z_+^2 \Lambda}.
\]

(4.11c)

To compare with the numerical solution, we will find it more convenient to consider just the ratios, or

\[
\frac{R_2(z_+)}{R(z_+)} = \frac{1 - |f(z_+)|^4}{8\omega |f(z_+)|^2} = 0.7866,
\]

(4.12a)

\[
\frac{R_3(z_+)}{R(z_+)} = - \frac{1 + (\omega^2 - 2)|f(z_+)|^4 + |f(z_+)|^8}{2|f(z_+)|^2(1 - |f(z_+)|^2)^2} = -4.958,
\]

(4.12b)

\[
\frac{R_2(z_+)}{R_3(z_+)} = \frac{(1 - |f(z_+)|^2)^3(1 + |f(z_+)|^2)}{4\omega(1 + (\omega^2 - 2)|f(z_+)|^4 + |f(z_+)|^8)} = -0.1587.
\]

(4.12c)

5. Comparison of approaches at the critical solution

As in [2], we can tune the initial conditions to be very close to the threshold of black hole formation. This should, in principle, result in a solution very close to the exactly critical solution of the previous section. This behavior is most clearly displayed on the \( r = 0 \) axis, where we expect the numerical solution to resemble most closely the exactly critical solution.

The scalar curvature on axis as a function of proper time is shown in fig. 1. The initial conditions chosen for the numerical solution shown contain much more dilaton than axion, but the late time behavior is independent of initial conditions. The \( x \)-axis is \(- \log(T_{\text{crit}} - T)\) (all logarithms in this paper have base 10), so the system is evolving from left to right. For the analytical solution, \( T_{\text{crit}} = 0 \). For the numerical solutions, we are free to adjust \( T_{\text{crit}} \) to most closely match the analytical solution. The \( y \)-axis is \( \log(-R) \). Three lines are shown in fig. 1. The solid line is a slightly sub-critical numerical evolution. The curvature
becomes strong on axis, but eventually dissipates (when $T > T_{\text{crit}}$). The dashed line is a slightly super-critical solution. It becomes a black hole, as can be seen by its upturn for late $T$. The dotted line is the continuously self-similar solution (4.10).

As expected, on the left side of fig. 1, the numerical and analytical solutions disagree. This is the region where the numerical solutions depends on the initial values of the fields. As the evolution progresses, the numerical solutions approach the analytical solution, becoming almost indistinguishable for a while. This is strong confirmation that the continuously self-similar solution is indeed the attractor. Eventually, the numerical solutions diverge from each other and from the analytical solution.

We compare the solutions on the past lightcone of the singular point in fig. 2, fig. 3 and fig. 4. From the analytic point of view, this is straightforward. Eq. (4.12) tells us that the ratios of the curvature invariants should approach constants. Numerically, it is much trickier. The past lightcone corresponds to a line of constant $v = v_{\text{crit}}$. One must choose $v_{\text{crit}}$ carefully because any error will grow as it approaches the $r = 0$ axis.
Unfortunately, this is precisely where the numerical solution should be approaching the analytical solution. To further complicate matters, since the lightcone is a null curve, there is no invariant definition of distance along the curve. In the figures, we have plotted the curvature ratios as a function of $-\log r$. The function $r$ becomes multivalued for supercritical evolutions (which is why $r$ is a bad choice as a coordinate), so we show a slightly subcritical numerical evolution in the figures. We see that the numerical solution (shown as a solid line) approaches the analytical line (the dotted line), remains there for a while, and then diverges from the analytical solution. When $r = 0$, we expect $R_2 = 0$ and $R_3 = -R/6$, and these are indeed the values seen for very small $r$. The transition from the $z = z_+$ values (4.12) to the $r = 0$ values is fairly abrupt. The transition is at finite $r$ because of errors in choosing $v_{\text{crit}}$ and because the numerical solution is not precisely critical. Note, for example, that $R_3(\log r = -4) \approx 10^7$ and is growing quickly near $\log r = -4$. Thus the fact that the ratios are roughly constant and close to the analytical prediction (4.12) is an
Fig. 3: The ratio $R_3/R$ as a function of $-\log r$ along a line of constant $v$. The solid line is a slightly subcritical numerical solution. The dotted line in the theoretical prediction (4.12b). As $r \to 0$, $R_3/R \to -R/6$.

excellent indication that the solution is indeed the continuously self-similar solution given in §4.

6. Moving away from criticality

One of the most interesting findings in [1] was the discovery of the scaling law (1.3) for the mass of black holes away from criticality. The striking result was that $\gamma$ was completely independent of initial conditions, and was thus dubbed the “universal” critical exponent. For the case of a real scalar field coupled to gravity, numerical work has given $\gamma = 0.374$ [1,2]. Previous numerical work with other types of matter gives similar values for $\gamma$. A study of axisymmetric gravitational wave collapse [3] indicates $\gamma \approx 0.36$. Spherically symmetric perfect radiation fluid [4] gives $\gamma \approx 0.36$. This coincidence of numbers led people to suspect that $\gamma$ might be a universal exponent, independent of the specific type of matter coupled to gravity.
The universality of $\gamma$ has been drawn into question by analytical arguments. Starting with a known critical solution, one can use renormalization group techniques to calculate $\gamma$. This was first carried out for the perfect radiation fluid in [9], which found $\gamma = 0.3558$, in good agreement with [4], but only fairly close to the value for a real scalar field. Because of the numerical uncertainties in [4] and uncertainties in the accuracy of the analytic arguments, it is unclear whether these results indicate conclusively that $\gamma$ depends on the matter content. A more general class of perfect fluids with $p = k\rho$ was studied analytically in [10] under the assumption that the critical solution is continuously self-similar. It was found that $\gamma$ depends strongly on $k$. However, no numerical work has been done for $k \neq 1/3$, and it is not clear that the continuously self-similar solution really is the attractor at criticality.

The case of a complex scalar field has been studied in [11]. They also perturbed away from a continuously self-similar solution, and found $\gamma = 0.3871$, which seems to be distinctly different from the perfect radiation fluid value. However, they also found that
the continuously self-similar solution is unstable, and conjecture that it is an attractor of codimension three. The discretely self-similar solution (the same critical solution as for the real scalar field case) has codimension one, and therefore dominates. Thus numerically it should be difficult to find the continuously self-similar solution. We have numerically evolved the complex scalar field with a variety of initial conditions and in every case have found critical behavior of the discretely self-similar type. Thus, no conclusive evidence has so far been presented that $\gamma$ is not universal.

Given an exactly critical analytical solution, one can perturb the critical solution to find the critical exponent $\gamma$ as follows [9]. Let $h$ be any function of the analytical solution, such as $b$ or $f$. Perturb away from the critical solution

$$h_t(z, t) = h_{\text{ss}}(z) + \epsilon |t|^{-\kappa} h_{\text{pert}}(z), \quad (6.1)$$

where $h_{\text{ss}}(z)$ is the critical solution, $\epsilon$ is a small number, $\kappa$ is a constant, and $h_{\text{pert}}(z)$ depends only on $z$. Replacing $h_t(z, t)$ in the equations of motion and keeping terms to first order in $\epsilon$ gives an eigenvalue equation for $\kappa$. This eigenvalue equation can in principle have a number of possible solutions for $\kappa$. The solution with the largest value of $\text{Re} \kappa$ will cause the fastest growing perturbation in (6.1), and is called the “most relevant mode.” The critical exponent is given by [9,10,11]

$$\gamma = \frac{1}{\text{Re} \kappa}. \quad (6.2)$$

We have carried out a perturbation analysis of the axion/dilaton critical solution described in §4. In principle, $\kappa$ can have an imaginary component, but this greatly complicates the analysis, and we found it to be unnecessary in this instance. The result for perturbation of the continuously self-similar solution is

$$\gamma_{\text{analytic}} = 0.2641066. \quad (6.3)$$

This value is quite different from the critical exponents found in previous numerical studies, and therefore should be easy to distinguish numerically. Our analysis does not rule out other possible values of $\kappa$, but we have seen no numerical evidence for other unstable modes.
We show in fig. 5 the numerical results for a large number of supercritical evolutions along a one-parameter family of initial conditions parameterized by $p$. The mass is defined as in [2]. Pick a constant $v_0$ and determine the point $(u_0, v_0)$ where the apparent horizon intersects $v_0$. The mass of the black hole is defined as $M_{bh} = m(u_0, v_0)$, where $m(u, v)$ is the Hawking mass defined in (3.6). As can be seen from fig. 5, $\log M_{bh}$ is a linear function of $\log(p - p_{\text{crit}})$ until one is far away from the critical solution. Omitting the final seven points in fig. 5 where the curve is no longer linear, we obtain a slope for fig. 5 of

$$\gamma_{\text{numeric}} = 0.264 ,$$

which agrees nearly perfectly with the theoretical prediction (6.3). As in the case of the real scalar field, $\gamma_{\text{numeric}}$ seems to be independent of the initial data, as long as the axion in not identically zero. This is further strong evidence that the critical solution is indeed continuously self-similar, and that the perturbation expansion gives the right answer.

7. Conclusions and unanswered questions
We depict a two-dimensional subspace of the infinite-dimensional space of initial conditions in fig. 6. The $x$-axis is the (normalized) amplitude of the initial axion field, and the $y$-axis is the (normalized) amplitude of the initial dilaton field. The points on the graph represent critical solutions. Not surprisingly, the critical surface has codimension one. When the axion field is initially identically zero, it remains zero, and the problem reduces to a real scalar field coupled to gravity. The critical value for that solution is the point on the $y$-axis, and we know from [1] that the critical solution there is discretely self-similar. The conjecture of [7] is that the points on the critical line with nonzero axion are continuously self-similar. This raises the question: is the discretely self-similar solution simply an artifact of special initial conditions? In other words, is the discretely self-similar solution an isolated point on the critical line, or is there an open set on the critical line near the $y$-axis where the critical solution is discretely self-similar? This is a difficult question to answer numerically, because near the $y$-axis, the axion is quite small and is thus liable to be swamped by numerical noise. The solution in fig. 1 represents a point quite
near the $y$-axis, and is clearly continuously self-similar. By moving along the critical line even closer to the $y$-axis, we push the region in fig. 1 that depends on initial conditions farther to the right. Eventually, this will meet the point where the sub- and super-critical solutions diverge from the analytical solution. Thus we cannot expect to move arbitrarily close to the $y$-axis in a numerical evolution. Except for points at which we believe the numerical accuracy is breaking down, every point near the $y$-axis that we have checked is continuously self-similar. This implies that the discretely self-similar solution is unstable, but analytical work is necessary to confirm this.

The axion/dilaton matter system described in this paper is physically well motivated. However, if the discretely self-similar solution is simply the result of non-generic initial conditions, might the continuously self-similar solution also be the result of overly restricted matter? For instance, string theory contains numerous gauge fields in the low-energy effective action which we have omitted in writing (2.3). Including these fields might result in yet another type of critical solution. If we hope to use critical solutions to probe the small-scale structure of spacetime, we should in principle use the most general solution allowed in our favorite theory of quantum gravity.

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