ABSTRACT

We derive a manifestly gauge invariant low energy blocked action for Yang-Mills theory using operator cutoff regularization, a prescription which renders the theory finite with a regulating smearing function constructed for the proper-time integration. By embedding the momentum cutoff scales in the smearing function, operator cutoff formalism allows for a direct application of Wilson-Kadanoff renormalization group to Yang-Mills theory in a completely gauge symmetry preserving manner. In particular, we obtain a renormalization group flow equation which takes into consideration the contributions of higher dimensional operators and provides a systematic way of exploring the influence of these operators as the strong coupling, infrared limit is approached.
I. INTRODUCTION

An important technical issue in quantum field theory is regularization, the removal of divergences which arise from incorporating the effects of quantum fluctuations. Although various regularization schemes are available to make the theory finite and well defined, it is crucial to subtract off the infinities with a procedure that preserves all the symmetries of the original theory. For example, when the underlying theory possesses gauge symmetry, prescriptions such as dimensional regularization [1], \( \zeta \) function regularization [2], invariant Pauli-Villars procedure [3] and the proper-time regularization [4] are the ideal candidates since they respect gauge symmetry; a sharp momentum cut-off, on the other hand, is not suitable owing to its noninvariant nature. However, for systems having symmetry properties that are dimensionality dependent (e.g. chiral symmetry or supersymmetry), it becomes problematic to use dimensional regularization.

Operator cutoff regularization was proposed in [5] as an invariant prescription which simulates the feature of momentum cutoff regulator. In this formalism, the one-loop contribution to the effective action is written as [5] [6]

\[
\text{Tr}_{\text{oc}} \left( \ln \mathcal{H} - \ln \mathcal{H}_0 \right) = - \int_0^\infty \frac{ds}{s} \rho^{(d)}_k(\Lambda, s) \text{Tr} \left( e^{-\mathcal{H}s} - e^{-\mathcal{H}_0s} \right),
\]

where \( \mathcal{H} \) is an arbitrary fluctuation operator governing the quadratic fluctuations of the fields and \( \mathcal{H}_0 \) its corresponding limit of zero background field. The subscript “oc” implies that the trace sum is to be operator cutoff regularized using the \( d \) dimensional smearing function \( \rho^{(d)}_k(s, \Lambda) \) which contains both the ultraviolet(UV) cutoff \( \Lambda \) and the infrared (IR) cutoff \( k \). We require \( \rho^{(d)}_k(s, \Lambda) \) to satisfy the following conditions: (1) \( \rho^{(d)}_k(s = 0, \Lambda) = 0 \), i.e., it must vanish sufficiently fast near \( s = 0 \) to eliminate the unwanted UV divergence; (2) \( \rho^{(d)}_k(s \to \infty, \Lambda) = 1 \) since the physics in the IR \( (s \sim \infty) \) regime is to remain unmodified; and (3) \( \rho^{(d)}_k(s, \Lambda) = 0 \) so that the one-loop correction to the effective action vanishes at the UV cutoff. In addition, we have

\[
\rho^{(d)}_{k=0}(s, \Lambda \to \infty) = 1,
\]

which reduces the operator cutoff regularization to the original Schwinger’s proper-time formalism [4]:

\[
\text{Tr} \left( \ln \mathcal{H} - \ln \mathcal{H}_0 \right) = - \int_0^\infty \frac{ds}{s} \text{Tr} \left( e^{-\mathcal{H}s} - e^{-\mathcal{H}_0s} \right).
\]

Thus, operator cutoff may be regarded as a special case of proper-time regularization. In fact, various other prescriptions such as sharp proper-time cut-off, point-splitting method, Pauli-Villars regulator, operator cutoff, dimensional regularization and \( \zeta \) function regularization all belong to the generalized class of proper-time and can be represented by a suitably chosen smearing function [7] [8].

What are the advantages of employing operator cutoff regularization? A remarkable feature of this regulator is that it contains momentum cutoff scales yet preserves gauge symmetry. This makes it possible to examine the renormalization group (RG) flow of gauge theories using the Wilson-Kadanoff approach [9]. In the momentum space formulation
of RG, blocking transformation is applied to give the theory an IR cutoff scale $k$ that
separates the fast-fluctuating modes from the slowly-varying components \[10\]; successive
elimination of the fast modes then leads to a low-energy effective blocked action from
which the RG flow pattern can be obtained. The noninvariant nature of this momentum
RG approach is precisely compensated by invoking operator cutoff formalism where the
procedure of blocking is readily taken over by the smearing function $\rho_k^{(d)}(s, \Lambda)$.

As demonstrated in \[5\] and \[6\], when choosing the smearing function to be of the form

$$
\rho_k^{(d)}(s, \Lambda) = \rho^{(d)}(\Lambda^2 s) - \rho^{(d)}(k^2 s) = \frac{2s^{d/2}}{\Gamma(d/2)} \int_k^\Lambda dz \ z^{d-1} e^{-z^2 s} - \frac{2s^{d/2}}{S_d \Gamma(d/2)} \int_z^{\prime} e^{-z^2 s}
$$

\[1.4\]

with

$$
\int_z = \int \frac{d^dz}{(2\pi)^d}, \quad \int_z^{\prime} = S_d \int_k^\Lambda dz \ z^{d-1}, \quad S_d = \frac{2}{(4\pi)^{d/2} \Gamma(d/2)}, \quad \int \quad \text{Integration limits}
$$

\[1.5\]

the characteristic of sharp momentum cutoff is reproduced in the leading order blocked po-
tential in derivative expansion. For the higher order (covariant) derivative terms, $\rho_k^{(d)}(s, \Lambda)$
corresponds to a smooth regulator.

In the present paper, we follow the preliminary work of \[6\] and \[11\] and further extend
the use of operator cutoff to the non-abelian Yang-Mills theory. The gauge-invariant effec-
tive blocked action for the theory is calculated in covariant background formalism. Since
the physics in the low energy regime is dominated by nonperturbative effects, nonpertur-
bative methods such as lattice calculation must be employed. Our RG equation based on
the Wilson-Kadanoff blocking transformation offers a powerful alternative for probing the
physical phenomena near the confining scale. In the past, various attempts were made to
construct the Wilson-Kadanoff RG formalism for gauge theories \[12\] - \[13\]. However, the
constraint of gauge symmetry has been the major obstacle for the success. In \[12\], the
Slavnov-Taylor identities are perturbatively restored by appropriate choice of boundary
conditions for the RG flow equation. On the other hand, in the average action approach
\[13\], additional cutoff-dependent interactions are introduced in such a way that the overall
average action maintains its gauge invariant form. The spirit of our blocking transforma-
tion formalism is parallel to the latter, however, it is implemented in the operator cutoff
formalism by a smearing function. The advantage of this approach is that gauge symmetry
can be seen to persist in a rather transparent manner.

The organization of the paper is as follows: In Sec. II, we review some essential fea-
tures of the operator cutoff regularization and illustrate how it is used in conjunction with
covariant derivative expansion. Similarity can be found between our formalism and the
method of higher (covariant) derivatives \[14\]. Details of computing the gauge-invariant
Yang-Mills blocked action perturbatively in the covariant background formalism are given
in Sec. III. The RG pattern of the blocked action and the $\beta$ function which governs the
evolution of the coupling constant are investigated in Sec IV. In Sec. V we apply operator
cutoff formalism to examine the the flow of the theory in a constant chromomagnetic field
configuration. Since the theory develops an imaginary part which signals instability of
the vacuum as the momentum scale falls below $\sqrt{gB}$, we choose the IR cutoff to be such that $k^2 > gB$, thereby eliminating the difficulties associated with an unstable vacuum. An improved RG equation is proposed to take into account the higher dimensional operators such as $(B^2/2)^2$. Section VI is reserved for summary and discussions. In Appendix A we provide the details of calculating the blocked potentials for $\lambda\phi^4$ theory and scalar electrodynamics in $d = 4$ using operator cutoff regularization. In Appendix B, we compare and contrast various prescriptions that belong to the generalized class of proper-time regularization. In particular, we show how dimensional regularization can be modified to incorporate cutoff scales. Connection between momentum regulator and dimensional regularization is readily established in our “dimensional cutoff” scheme. Momentum cutoff scales can also be brought into the $\zeta$ function regularization in a similar manner.

II. OPERATOR CUTOFF REGULARIZATION

As mentioned in the Introduction, operator cutoff regularization not only allows us to bypass the complications of dealing with divergences, it also encompasses the features of momentum space blocking transformation in a symmetry-preserving manner. We first review some essential properties of the formalism already developed in [6].

With the smearing function $\rho_k^{(d)}(s, \Lambda)$ written in (1.4), the operator cutoff regularized propagator and the one-loop contribution of the blocked action become, respectively,

$$\frac{1}{\mathcal{H}^n}\bigg|_{oc} = \frac{1}{\Gamma(n)} \int_0^\infty ds \, s^{n-1} \rho_k^{(d)}(s, \Lambda)e^{-\mathcal{H}s}$$

$$= \frac{1}{\mathcal{H}^n} \cdot \frac{2\Gamma(n + d/2)}{\Gamma(n)\Gamma(d/2)} \left\{ \left( \frac{\Lambda^2}{\mathcal{H}} \right)^{d/2} F\left(\frac{d}{2}, \frac{1}{2}, \frac{d}{2} + n, 1 + \frac{d}{2}; -\frac{\Lambda^2}{\mathcal{H}} \right) \right.$$

$$- \left( \frac{k^2}{\mathcal{H}} \right)^{d/2} F\left(\frac{d}{2}, \frac{1}{2}, \frac{d}{2} + n, 1 + \frac{d}{2}; -\frac{k^2}{\mathcal{H}} \right) \right\}, \quad (2.1)$$

and

$$\text{Tr}_{oc} \left( \ln \mathcal{H} - \ln \mathcal{H}_0 \right) = - \int_0^\infty \frac{ds}{s} \frac{\rho_k^{(d)}(s, \Lambda)}{s} \text{Tr} \left( e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s} \right)$$

$$= - \frac{2}{d} \text{Tr} \left\{ \left( \frac{\Lambda^2}{\mathcal{H}} \right)^{d/2} F\left(\frac{d}{2}, \frac{1}{2}, \frac{d}{2} + \frac{1}{2}; -\frac{\Lambda^2}{\mathcal{H}} \right) - \left( \frac{\Lambda^2}{\mathcal{H}_0} \right)^{d/2} F\left(\frac{d}{2}, \frac{1}{2}, \frac{d}{2} + \frac{1}{2}; -\frac{\Lambda^2}{\mathcal{H}_0} \right) \right.$$

$$- \left( \frac{k^2}{\mathcal{H}} \right)^{d/2} F\left(\frac{d}{2}, \frac{1}{2}, \frac{d}{2} + \frac{k^2}{\mathcal{H}} \right) + \left( \frac{k^2}{\mathcal{H}_0} \right)^{d/2} F\left(\frac{d}{2}, \frac{1}{2}, \frac{d}{2} + \frac{k^2}{\mathcal{H}_0} \right) \right\}, \quad (2.2)$$

where

$$F\left(a, b, c; \beta \right) = B^{-1}(b, c-b) \int_0^1 dx \, x^{b-1}(1-x)^{c-b-1}(1-\beta x)^{-a} \quad (2.3)$$
is the hypergeometric function symmetric under the exchange between \(a\) and \(b\), and

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_0^1 dt \, t^{x-1}(1-t)^{y-1},
\]

(2.4)

the Euler \(\beta\) function. For \(n = 1\) and \(2\), eq.(2.1) gives, respectively,

\[
\left.\frac{1}{\mathcal{H}}\right|_{oc} = \int_0^\infty ds \, \rho_k^{(d)}(s, \Lambda)e^{-\mathcal{H}s} = \frac{1}{\mathcal{H}} \left\{ \left( \frac{\Lambda^2}{\mathcal{H} + \Lambda^2} \right)^{d/2} - \left( \frac{k^2}{\mathcal{H} + k^2} \right)^{d/2} \right\},
\]

(2.5)

and

\[
\left.\frac{1}{\mathcal{H}^2}\right|_{oc} = \int_0^\infty ds \, s\rho_k^{(d)}(s, \Lambda)e^{-\mathcal{H}s} = \frac{1}{\mathcal{H}^2} \left\{ \left( \frac{\Lambda^2}{\mathcal{H} + \Lambda^2} \right)^{d/2} \left[ 1 + d\frac{\mathcal{H}}{2(\mathcal{H} + \Lambda^2)} \right] - \left( \frac{k^2}{\mathcal{H} + k^2} \right)^{d/2} \left[ 1 + d\frac{\mathcal{H}}{2(\mathcal{H} + k^2)} \right] \right\}.
\]

(2.6)

Consider \(d = 4\) where

\[
\rho_k^{(4)}(s, \Lambda) = (1 + k^2s)e^{-k^2s} - (1 + \Lambda^2s)e^{-\Lambda^2s}.
\]

(2.7)

We have

\[
\left.\frac{1}{\mathcal{H}}\right|_{oc} = \frac{1}{\mathcal{H} + k^2} - \frac{1}{\mathcal{H} + \Lambda^2} - \frac{\Lambda^2}{(\mathcal{H} + \Lambda^2)^2} + \frac{k^2}{(\mathcal{H} + k^2)^2},
\]

(2.8)

and

\[
\text{Tr}_{oc} \left( \ln \mathcal{H} - \ln \mathcal{H}_0 \right) = \text{Tr} \left\{ \ln \left[ \frac{\mathcal{H} + k^2}{\mathcal{H}_0 + k^2} \times \frac{\mathcal{H}_0 + \Lambda^2}{\mathcal{H} + \Lambda^2} \right] - \frac{\Lambda^2 (\mathcal{H} - \mathcal{H}_0)}{(\mathcal{H} + \Lambda^2)(\mathcal{H}_0 + \Lambda^2)} \right\}
\]

\[
+ \frac{k^2 (\mathcal{H} - \mathcal{H}_0)}{(\mathcal{H} + k^2)(\mathcal{H}_0 + k^2)} \right\},
\]

(2.9)

which shows that \(\Lambda\) may be interpreted as the mass of some unitarity-violating ghost states. The interpretation follows from the relative negative sign in the modified propagator. Equivalently, one may also say that the effect of \(\Lambda\) is to make the theory superrenormalizable by incorporating higher order derivative terms. For example, eq.(2.8) implies that the kinetic term in the scalar theory is to be modified as

\[
-\frac{1}{2} \phi \partial^2 \phi \longrightarrow \frac{1}{2} \phi \left[ -\partial^2 + \frac{2}{\Lambda^2} (-\partial^2)^2 + \frac{1}{\Lambda^4} (-\partial^2)^3 \right] \phi.
\]

(2.10)

On the other hand, the IR scale \(k\) may be thought of as an additional mass which makes the overall effective mass parameter \(\mu_{\text{eff}}^2 = \mu^2 + k^2\). The scale \(k\) is useful not only for the purpose of studying RG, but can also be employed as an IR regulator for the theory containing massless modes.
Before examining Yang-Mills theory, we consider the following covariant fluctuation kernel:

\[ H = -D^2 + \mu^2 + Y(x), \quad (2.11) \]

where \( D_\mu \) is the covariant derivative for the gauge group, \( \mu^2 \) the mass for the scalar field interacting with the gauge field \( A_\mu^a(x) \), and \( Y(x) \) a matrix-valued function of \( x \) describing the interaction between the scalar particles. The index \( a \) runs over the dimension of the gauge (color) group. One may also write \( Y = Y^a T^a \) where the \( T^a \)'s are the generators of the gauge group satisfying

\[ [T^a, T^b] = f^{abc} T^c, \quad \text{tr}_c(T^a T^b) = -\frac{1}{2} \delta^{ab} \quad (2.12) \]

with \( f^{abc} \) being the structure constants and \( \text{tr}_c \) the summation over only the color indices.

In the fundamental \( SU(N) \) representation, we have

\[ T^a = \begin{cases} \sigma^a/2i, & a = 1, \ldots, 3 \quad N = 2 \\ \lambda^a/2i, & a = 1, \ldots, 8 \quad N = 3, \end{cases} \quad (2.13) \]

where \( \sigma^a \) and \( \lambda^a \) are, respectively, the Pauli and the Gell-Mann matrices. When operating on \( Y \) with the covariant derivative, we have

\[ D^a_{\mu} Y = (g^{ab} \partial_\mu - gf^{abc} A_\mu^c) Y, \quad (2.14) \]

or \( D_\mu Y = \partial_\mu Y + [A_\mu, Y] \), where \( A_\mu = g A_\mu^a T^a \) and \( g \) is a coupling constant.

The unregularized one-loop contribution to the effective action is

\[ \tilde{S}^{(1)} = \frac{1}{2} \text{Tr} \left( \ln H - \ln H_0 \right) = -\frac{1}{2} \int_x \int_0^\infty \frac{ds}{s} \text{tr}_c (x) (e^{-Hs} - e^{-H_0s}) |x\rangle, \quad \int_x = \int d^dx, \quad (2.15) \]

where the diagonal part of the “heat kernel” is written as

\[ h(s; x, x) = \langle x | e^{-Hs} | x \rangle = \int_p \langle x | p \rangle e^{-H_0s} \langle p | x \rangle = \int_p e^{-ipx} e^{-Hx} e^{ipx} \]

\[ = \int_p e^{-(p^2 - 2ip \cdot D + Hx)s} \mathbf{1} = \int_p e^{-(p^2 + \mu^2)s} e^{(2ip \cdot D + D^2 - Y)s} \mathbf{1}, \quad \int_p = \int \frac{d^dp}{(2\pi)^d}. \quad (2.16) \]

The above expression is derived by employing the plane wave basis \( |p\rangle \) with \( \langle x | p \rangle = e^{-ipx} \) and the commutation relations \([7], [15]:\)

\[ [D_\mu, e^{ipx}] = ip_\mu, \quad [H_x, e^{ipx}] = p^2 - 2ip \cdot D. \quad (2.17) \]
The factor 1 indicates that the operator $D_\mu$ acts on the identity. Making use of the Baker-Campbell-Hausdorff formulae to expand the operators in the exponential, eq.(2.16) can be approximated as

$$h(s;x,x) = e^{-(\mu^2 + Y)s} \int_p e^{-p^2 s} \left\{ 1 + D^2 s + \frac{D^4}{2} s^2 - \frac{[D^2, Y]}{2} s^2 \right\}$$

\[\begin{align*}
&- \frac{2p^2}{d} \left[ D^2 s^2 + \frac{1}{3} \left( [[D^2, D_\mu], D_\mu] + 3D_\mu [D^2, D_\mu] + 3D^4 \right. \right. \\
&\left. \left. - [D_\mu, Y] D_\mu \right) s^3 \right] + \frac{2(p^2)^2}{3d(d+2)} \left[ D^4 + (D_\mu D_\nu)^2 + D_\mu D^2 D_\mu \right] s^4 + \cdots \right\}
\end{align*}\]

(2.18)

where we have used the $O(d)$ invariant property of the momentum integrals:

$$\int_p p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{2m}} e^{-p^2 s} = \frac{T_{\mu_1 \mu_2 \cdots \mu_{2m}}}{2m \Gamma(m + d/2)} \int_p (p^2)^m e^{-p^2 s} = \frac{T_{\mu_1 \mu_2 \cdots \mu_{2m}}}{(4\pi s)^{d/2}(2s)^m},$$

(2.19)

$$T_{\mu_1 \mu_2 \cdots \mu_{2m}} = \delta_{\mu_1, \mu_2} \cdots \delta_{\mu_{2m-1}, \mu_{2m}} + \text{permutations.}$$

(2.20)

As the singularity arising from taking the spacetime trace is transferred to the proper-time integration, we insert the regulating smearing function $\rho_k^{(d)}(s, \Lambda)$ into (2.18) and obtain the following “blocked” heat kernel:

$$h_k(s;x,x) = \frac{e^{-(\mu^2 + Y)s}}{(4\pi)^{d/2}} \rho_k^{(d)}(s, \Lambda) \left\{ 1 + \frac{1}{12} \left[ F_{\mu\nu} F_{\mu\nu} - 2(D^2 Y) \right] s^2 + O(s^3) \right\},$$

(2.21)

which is in agreement with the result found in [15] and [16] for $\rho_k^{(d)}(s, \Lambda) = 1$. Gauge symmetry is easily seen to be preserved by noting that $h_k(s;x,x)$ consists of gauge invariant quantities only. Had we used momentum cutoff regularization instead, there would be contribution from noninvariant operators $D^2, D_\mu Y D_\mu, Y D^2, D^4$ and $D_\mu D^2 D_\mu$ [6]. Higher order contributions to (2.21) can be included in an invariant manner as well. The details can be found in [7].

III. YANG-MILLS THEORY

In the absence of matter field the pure Yang-Mills lagrangian reads

$$\mathcal{L} = \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a,$$

(3.1)

where the field strength is given by

$$G_{\mu\nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a + g f^{abc} A_{\mu}^b A_{\nu}^c,$$

(3.2)
in the matrix-valued representation. In the usual manner, integrating out the irrelevant short-distance (fast-fluctuating) modes $\xi$ having momenta between $k$ and $\Lambda$ leads to a low-energy effective blocked action which depends only on the slowly-varying background fields $\bar{A}$ with momenta below $k$. Our goal is to derive a gauge invariant Yang-Mills blocked action $\tilde{S}_k$ which has an explicit dependence on the IR scale $k$. This would allow us to generate an improved RG flow equation by evolving the blocked action with $k$. As demonstrated in Sec. II, the scale enters in an invariant manner when operator cutoff regularization is used.

To set up the RG formalism, we first separate the modes as

$$A^a_\mu(p) = \begin{cases} \bar{A}^a_\mu(p), & 0 \leq p \leq k, \\ \xi^a_\mu(p), & k < p < \Lambda, \end{cases}$$

and denote the background field strength by $F^a_{\mu\nu} = G^a_{\mu\nu}(\bar{A})$. We next introduce

$$\mathcal{L}_{GF} = -\frac{1}{2\alpha}(D_\mu A^a_\mu)^2,$$  \hspace{1cm} (3.5)

and

$$\mathcal{L}_{FPG} = \chi^\dagger D^2(\bar{A})\chi,$$  \hspace{1cm} (3.6)

as the desirable gauge-fixing condition and the Faddeev-Popov ghost term, respectively. Notice that in order to obtain a gauge invariant expression for the theory, the calculation is performed in the background field formalism [17]. The lagrangian then takes on the form

$$\mathcal{L}(\bar{A}_\mu + \xi_\mu, \chi^\dagger + \eta^\dagger, \bar{\chi} + \eta) = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} \xi_\mu^a \left[ -D^2 g_{\mu\nu} + (1 - \frac{1}{\alpha})D_\mu D_\nu \right]^{ab} \xi^b_\nu + g f^{abc} \xi^a_\mu \xi^b_\nu + \tilde{\chi}^{\dagger a} D^2(\bar{A})^{ab} \bar{\chi}^b + \eta^{\dagger a} D^2(\bar{A})^{ab} \eta^b + \delta \mathcal{L}(\bar{A}_\mu, \xi_\mu),$$  \hspace{1cm} (3.7)

where $\eta^\dagger$ and $\eta$ denote the fast-fluctuating modes for the ghost fields, and

$$\delta \mathcal{L}(\bar{A}_\mu, \xi_\mu) = g f^{abc}(D_\mu \xi_\nu)^a \xi^b_\mu \xi^c_\nu + \frac{1}{4} g^2 f^{abc} f^{ade} \xi^b_\mu \xi^c_\nu \xi^d_\sigma \xi^e_\tau + \ldots$$  \hspace{1cm} (3.8)

represents the higher-order self-interactions. The partition function can be written as

$$Z = \int \mathcal{D}[\bar{A}_\mu]\mathcal{D}[\chi]\mathcal{D}[\chi^\dagger]e^{-S[\bar{A}_\mu + \xi_\mu, \chi^\dagger + \eta^\dagger, \bar{\chi} + \eta]} = \int \mathcal{D}[\bar{\chi}]\mathcal{D}[\bar{A}]\mathcal{D}[\chi^\dagger]e^{-\tilde{S}_k[\bar{A}_\mu, \chi^\dagger, \bar{\chi}]} = \int \mathcal{D}[\eta^\dagger]\mathcal{D}[\eta]e^{-S[\bar{A}_\mu + \xi_\mu, \chi^\dagger + \eta^\dagger, \bar{\chi} + \eta]} = \int \mathcal{D}[\xi_\mu]\mathcal{D}[\eta^\dagger]\mathcal{D}[\eta]e^{-S[\bar{A}_\mu + \xi_\mu, \chi^\dagger + \eta^\dagger, \bar{\chi} + \eta]},$$  \hspace{1cm} (3.9)

where

$$e^{-\tilde{S}_k[\bar{A}_\mu, \chi^\dagger, \bar{\chi}]} = \int \mathcal{D}[\xi_\mu]\mathcal{D}[\eta^\dagger]\mathcal{D}[\eta]e^{-S[\bar{A}_\mu + \xi_\mu, \chi^\dagger + \eta^\dagger, \bar{\chi} + \eta]}.$$  \hspace{1cm} (3.10)
In the above, the functional integrations are performed within the perspective momentum range for each field configuration. In the case of vanishing ghost background fields, by substituting (3.7) into (3.10) and dropping higher order fluctuating terms, the operator cutoff regularized blocked action up to the one-loop order reads

\[
\bar{S}_k[\bar{A}] = \frac{1}{4} \int_x F_{\mu \nu}^a F_{\mu \nu}^a + \frac{1}{2} \text{Tr}_{oc} \left\{ \ln \mathcal{K}(\bar{A}) - \ln \mathcal{K}(0) \right\} - \text{Tr}_{oc} \left\{ \ln \mathcal{O}(\bar{A}) - \ln \mathcal{O}(0) \right\},
\]

(3.11)

where the gauge and the ghost kernels are, respectively,

\[
\mathcal{K}_{\mu \nu}^{ab} = \frac{\partial^2 S}{\partial A^a_\mu(x) \partial A^b_\nu(y)} \bigg|_{\bar{A}} = \left\{ - \left[ D^2 g_{\mu \nu} - \left(1 - \frac{1}{\alpha} \right) D_\mu D_\nu \right]^{ab} + 2g f^{abc} F_{\mu \nu}^c \right\} \delta^4(x - y), \quad (3.12)
\]

and

\[
\mathcal{O}^{ab} = - D^2 (\bar{A})^{ab} \delta^4(x - y). \quad (3.13)
\]

Eq. (3.12) is derived by the help of

\[
D_\mu D_\nu^{bc} - D_\nu D_\mu^{bc} = g f^{abc} F_{\mu \nu}^b.
\]

(3.14)

Here Tr$_{oc}$ denotes the trace sum over (operator cutoff regularized) space-time, Lorentz indices as well as the color indices and tr is for the latter two only. When no confusion arises, internal indices shall be suppressed for brevity.

Since the Yang-Mills blocked action is generally a complicated object even at the one-loop elvel, an approximate solution exists only in a certain energy regime. We shall follow the perturbative formalism developed by Schwinger in [4]. In the momentum space where \( \partial_\mu \rightarrow ip_\mu \), (3.12) and (3.13) may be rewritten as [18]

\[
\mathcal{K}_{\mu \nu}^{ab} = - \left[ D^2 g_{\mu \nu} - \left(1 - \frac{1}{\alpha} \right) D_\mu D_\nu \right]^{ab} + 2g f^{abc} F_{\mu \nu}^c
\]

\[
= g_{\mu \nu} \left[ p^2 \delta^{ab} - ig f^{acb} (p \cdot \bar{A}^c + \bar{A}^c \cdot p) - g^2 f^{amc} f^{clb} \bar{A}_\lambda^m \bar{A}_\ell^l \right] + 2g f^{abc} F_{\mu \nu}^c
\]

\[
- \left(1 - \frac{1}{\alpha} \right) [ p_\mu p_\nu \delta^{ab} - ig f^{acb} (p_\mu \bar{A}_\nu^c + \bar{A}_\nu^c p_\nu) - g^2 f^{amc} f^{cls} \bar{A}_\lambda^m \bar{A}_\ell^l ] \quad (3.15)
\]

and

\[
\mathcal{O}^{ab} = -(D^2)^{ab} = p^2 \delta^{ab} - ig f^{acb} (p \cdot \bar{A}^c + \bar{A}^c \cdot p) - g^2 f^{amc} f^{cls} \bar{A}_\lambda^m \bar{A}_\ell^l.
\]

(3.16)

With \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \) where \( \mathcal{H}_I \) accounts for the interactions, we and expand the fluctuation kernel in power of \( s \) and obtain

\[
\text{Tr} (e^{-\mathcal{H}s}) = \text{Tr} (e^{-\mathcal{H}_0s}) + \int_0^\infty d\lambda \text{ Tr} \left( \mathcal{H}_I e^{-(\mathcal{H}_0 + \lambda \mathcal{H}_I)s} \right)
\]

\[
= \text{Tr} \left\{ e^{-\mathcal{H}_0s} + (-s) e^{-\mathcal{H}_0s} \mathcal{H}_I + \frac{(-s)^2}{2} \int_0^1 du_1 e^{-(1-u_1)\mathcal{H}_0s} \mathcal{H}_I e^{-u_1\mathcal{H}_0s} \mathcal{H}_I \right. \\
+ \frac{(-s)^3}{3} \int_0^1 du_1 u_1 \int_0^1 du_2 e^{-(1-u_1)\mathcal{H}_0s} \mathcal{H}_I e^{-(1-u_2)\mathcal{H}_0s} \mathcal{H}_I e^{-u_1\mathcal{H}_0s} \mathcal{H}_I + \ldots \right\}.
\]

(3.17)
Applying the above expansion formula to (3.15) and (3.16), we have

\[
\text{Tr}(e^{-Q^{ab}}) = \delta^{ab} \left\{ e^{-p^2s} - se^{-p^2s}(-g^2 f^{abc} f^{d\ell b} \bar{A}_m^c \bar{A}_\ell^d) \right.
\]
\[
+ \frac{s^2}{2} \int_0^1 du e^{(1-u)p^2s} \left[ -ig f^{a\ell c}(p \cdot \bar{A}_c^c + \bar{A}_c^c \cdot p) \right]
\]
\[
\times e^{-up^2s} \left[ -ig f^{\ell mb}(p \cdot \bar{A}_m^c + \bar{A}_m^c \cdot p) + \cdots \right],
\]

\quad (3.18)

and, in Feynman gauge where \( \alpha = 1 \),

\[
\text{Tr}(e^{-K^{ab}}) = \delta^{ab} \left\{ g_{\mu\nu} e^{-p^2s} - se^{-p^2s} \left( -g^2 f^{abc} f^{d\ell b} g_{\mu\nu} \bar{A}_m^c \bar{A}_\ell^d \right) \right.
\]
\[
+ \frac{s^2}{2} \int_0^1 du e^{(1-u)p^2s} \left[ -ig f^{a\ell c} g_{\mu\nu}(p \cdot \bar{A}_c^c + \bar{A}_c^c \cdot p) - 2gf^{a\ell c} F_{\mu\nu}(\bar{A}) \right]
\]
\[
\times e^{-up^2s} \left[ -ig f^{\ell mb} g_{\rho\nu}(p \cdot \bar{A}_m^c + \bar{A}_m^c \cdot p) - 2gf^{\ell mb} F_{\rho\nu}(\bar{A}) + \cdots \right].
\]

\quad (3.19)

By inserting a complete orthonormal set of momentum states \( |p\rangle \) satisfying:

\[
\int_p |p\rangle \langle p| = 1, \quad \langle p|p'\rangle = (2\pi)^4 \delta^4(p - p')
\]

\quad (3.20)

\[
\langle x|p\rangle = e^{ip \cdot x}, \quad \langle p|A_\mu|p'\rangle = A_\mu(p - p'),
\]

we carry out the calculations explicitly in \( d = 4 \) and obtain

\[
\text{Tr} \int_0^1 du e^{-(1-u)p^2s}(p \cdot \bar{A}_c^c + \bar{A}_c^c \cdot p) e^{-up^2s}(p \cdot \bar{A}_c^c + \bar{A}_c^c \cdot p)
\]
\[
= \text{tr} \int_0^1 du \int_{p,q} e^{-[(1-u)p^2 + uq^2]s}(p + q)_\mu(p + q)_\nu \bar{A}_c^c(p - q) \bar{A}_c^c(q - p)
\]
\[
= \frac{1}{16\pi^2 s^2} \text{tr} \int_0^1 du \int_p \bar{A}_\mu^c(p) \bar{A}_\nu^c(-p) e^{-u(1-u)p^2s} \left[ \frac{2g_{\mu\nu}}{s} + (2u - 1)^2 p_\mu p_\nu \right].
\]

\quad (3.21)

The above expression is arrived by shifting the variable \( p \to p + q \) followed by \( q \to q - (1 - u)p \), and the \( q \) integration using \( O(4) \) invariance. Making use of the regulating
smearing function $\rho_k^{(4)}(s, \Lambda)$ then leads to

\[
\text{Tr}_\text{oc} \ln \mathcal{O}(\bar{A}) = \frac{g^2 C_2(G)}{16\pi^2} \int_p \tilde{A}_\mu^c(p) \tilde{A}_\nu^c(-p) \left\{ g_{\mu\nu} \int_0^\infty \frac{ds}{s^2} \rho_k^{(4)}(s, \Lambda) ight\} \\
- \frac{1}{2} \int_0^1 du \int_0^\infty \frac{ds}{s^2} \rho_k^{(4)}(s, \Lambda) e^{-u(1-u)p^2 s} \left[ \frac{2g_{\mu\nu}}{s} + (2u - 1)^2 p_\mu p_\nu \right] \\
= g^2 C_2(G) \frac{1}{32\pi^2} \int_p \tilde{A}_\mu^c(p) \tilde{A}_\nu^c(-p) \int_1^\infty du \left\{ p_\mu p_\nu (2u - 1)^2 \left[ \frac{\tilde{\Lambda}^2}{\Lambda^2 + u(1-u)} \right] \\
- \frac{\kappa^2}{\kappa^2 + u(1-u)} \right] - \left[ 2g_{\mu\nu} p^2 u(1 - u) - (2u - 1)^2 p_\mu p_\nu \right] \ln \left( \frac{\kappa^2 + u(1-u)}{\Lambda^2 + u(1-u)} \right) \right\} \\
= g^2 C_2(G) \frac{1}{192\pi^2} \int_p F_{\mu\nu}^c(p) F_{\mu\nu}^c(-p) \left\{ \ln \left( \frac{\Lambda^2}{k^2} + \frac{5}{3} - 4\kappa^2 - (2\kappa^2 - 1)(4\kappa^2 + 1)f(\kappa) \right) \right\}
\]  
(3.22)

where $\kappa = k/p$ and

\[
f(\kappa) = \frac{1}{\sqrt{4\kappa^2 + 1}} \ln \left( \frac{\sqrt{4\kappa^2 + 1} - 1}{\sqrt{4\kappa^2 + 1} + 1} \right).
\]  
(3.23)

In the above, we have made the substitution

\[
\frac{1}{2} F_{\mu\nu}^c(p) F_{\mu\nu}^c(-p) = (p^2 g_{\mu\nu} - p_\mu p_\nu) \tilde{A}_\mu^c(p) \tilde{A}_\nu^c(-p) + \cdots,
\]  
(3.24)

by neglecting higher order gauge field contributions, as well as the expansion

\[
-\frac{1}{2} \ln \left( \frac{1 - x}{1 + x} \right) = \tanh^{-1} x = x + \frac{x^3}{3} + \cdots,
\]  
(3.25)

by taking the limit $\Lambda \rightarrow \infty$. Notice that $f^{ac} f^{\ell m} = -\delta^{cm} C_2(G)$, where $C_2(G)$ is a Casimir operator with $C_2(G) = N$ for $G = SU(N)$. The quadratic divergence naively expected from dimensional counting also disappears, as required by gauge invariance.

Similarly, the gauge field contribution reads

\[
\text{Tr}_\text{oc} \ln \mathcal{K}(\bar{A}) = \frac{g^2 C_2(G)}{4\pi^2} \int_p \tilde{A}_\mu^c(p) \tilde{A}_\nu^c(-p) g_{\mu\nu} \int_0^\infty \frac{ds}{s^2} \rho_k^{(4)}(s, \Lambda) \\
- \frac{1}{2} \text{Tr} \int_0^\infty ds \rho_k^{(4)}(s, \Lambda) \left( \int_0^1 du e^{-(1-u)p^2 s} \delta^{ab} \left\{ -ig f^{ac} [(p \cdot \tilde{A}^c + \tilde{A}^c \cdot p) g_{\mu\rho} \\
- 2i F_{\mu\rho}^c \right] \right) e^{-up^2 s} \left\{ -ig f^{\ell m} [(p \cdot \tilde{A}^m + \tilde{A}^m \cdot p) g_{\rho\nu} - 2i F_{\rho\nu}^m \right] \right) \right) \\

= 4\text{Tr}_\text{oc} \ln \mathcal{O}(\bar{A}) - \frac{g^2 C_2(G)}{8\pi^2} \int_p F_{\mu\rho}^c(p) F_{\mu\rho}^c(-p) \int_0^1 du \int_0^\infty \frac{ds}{s^2} \rho_k^{(4)}(s, \Lambda) e^{-u(1-u)p^2 s} \\

= 4\text{Tr}_\text{oc} \ln \mathcal{O}(\bar{A}) - \frac{g^2 C_2(G)}{8\pi^2} \int_p F_{\mu\rho}^c(p) F_{\mu\rho}^c(-p) \left\{ \ln \left( \frac{\Lambda^2}{k^2} \right) + 1 + (2\kappa^2 + 1)f(\kappa) \right\}
\]  
(3.26)
Adding up these terms, the perturbative Yang-Mills blocked action becomes

\[
\tilde{S}_k = \frac{1}{4} \int_p F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} Tr_{\alpha\beta} \ln K(\tilde{A}) - Tr_{\alpha\beta} \ln O(\tilde{A}) = \frac{1}{4} \int_p \tilde{Z}^{-1}_k(p) F^a_{\mu\nu}(p) F^a_{\mu\nu}(-p), \tag{3.27}
\]

where

\[
\tilde{Z}^{-1}_k = 1 - g^2 C_2(G) \left\{ 11 \ln \left( \frac{\Lambda^2}{k^2} \right) + \frac{31}{3} + 4\kappa^2 + (8\kappa^4 + 22\kappa^2 + 11) f(\kappa) \right\}
\]

\[
= Z^{-1}_k + \delta \tilde{Z}^{-1}_k,
\]

with

\[
Z^{-1}_k = 1 - \frac{g^2 C_2(G)}{48\pi^2} \left[ 11 \ln \left( \frac{\Lambda^2}{k^2} \right) + \frac{1}{3} \right] = 1 - \frac{11g^2 C_2(G)}{48\pi^2} \ln \left( \frac{\Lambda^2}{k^2} \right), \tag{3.29}
\]

and

\[
\delta \tilde{Z}^{-1}_k = - \frac{g^2 C_2(G)}{48\pi^2} \left[ 10 \frac{1}{3} + 4\kappa^2 + (8\kappa^4 + 22\kappa^2 + 11) f(\kappa) \right]. \tag{3.30}
\]

Here we recover the familiar factor of \(-11g^2 C_2(G)/48\pi^2\) associated with the \(\ln \Lambda^2\) term, with \(-10g^2 C_2(G)/48\pi^2\) coming from the gauge kernel \(Tr_{\alpha\beta} \ln K/2\) and \(-g^2 C_2(G)/48\pi^2\) from the ghost sector \(-Tr_{\alpha\beta} \ln O\). The constant 1/3 inside the bracket of (3.29) is regularization scheme dependent and can be adjusted by changing the cutoff \(\Lambda \rightarrow \tilde{\Lambda}\), i.e. a finite renormalization.

However, eq.(3.27) is not gauge invariant since the complicated non-polynomial \(p\) dependence coming from \(\delta \tilde{Z}^{-1}_k\) implies the presence of nonlocal field strength coupling which manifestly violates gauge symmetry. Although nonlocality may be characteristic of momentum cutoff brought about by the regulating smearing function, this is a general feature for the low energy blocked action irrespective of how the theory is regularized. Nevertheless, the difficulty can be avoided by taking the large \(k\) limit where

\[
\delta \tilde{Z}^{-1}_k \approx \frac{g^2 C_2(G)}{288\pi^2} \left\{ \frac{7}{6} \left( \frac{p^2}{k^2} \right) + \frac{1}{6} \left( \frac{p^2}{k^2} \right)^2 + \cdots \right\}. \tag{3.31}
\]

Notice that as \(k \rightarrow \infty\), \(\delta \tilde{Z}^{-1}_k \rightarrow 0\) and the theory is completely local. The expansion enables us to “upgrade” the momentum \(p\) to the generalized momentum \(\Pi\) (up to terms which are of higher order in \(\tilde{A}\)) as

\[
\int_p F^c_{\mu\nu}(p)p^2 F^c_{\mu\nu}(-p) \rightarrow \int_p F^c_{\mu\nu}(p)\Pi^2 F^c_{\mu\nu}(-p) \rightarrow \int_x D_\mu F^c_{\mu\nu}(x) D_\sigma F^c_{\sigma\nu}(x)
\]

\[
\int_p F^c_{\mu\nu}(p)(p^2)^2 F^c_{\mu\nu}(-p) \rightarrow \int_p F^c_{\mu\nu}(p)(\Pi^2)^2 F^c_{\mu\nu}(-p) \rightarrow \int_x D^2 F^c_{\mu\nu}(x) D^2 F^c_{\mu\nu}(x), \tag{3.32}
\]

which leads to the following gauge invariant blocked potential:

\[
U_k = \frac{Z^{-1}_k}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{g^2 C_2(G)}{1152\pi^2} \left\{ \frac{7}{k^2} \left( D_\mu F^a_{\mu\nu} D_\sigma F^a_{\sigma\nu} \right) + \frac{1}{6k^4} \left( D^2 F^a_{\mu\nu} D^2 F^a_{\mu\nu} \right) + \cdots \right\}. \tag{3.33}
\]
These higher dimensional operators which are completely absent at the cutoff scale \( k = \Lambda \) may be thought of as being generated by blocking transformation as \( k \) is lowered. While the presence of logarithmic divergence in \( Z_{k}^{-1} \) can be readily absorbed by a redefinition of the coupling constant \( g \) associated with the operator \( F_{\mu \nu}^{a} F_{\mu \nu}^{a} \), the contribution from \( \delta Z_{k}^{-1} \) accounts for the finite renormalization of the coupling strengths for the higher dimensional operators. Notice that the coefficients for the higher operators generated from the above heuristic argument are expected to differ from the formal covariant derivative expansion. A thorough treatment on this subject may be found in [7].

**IV. RENORMALIZATION GROUP EQUATION**

We now examine the RG flow pattern of the Yang-Mills blocked action derived in the last Section. Using Slavnov-Taylor identities, the field renormalization constant \( Z_{k} \) written in (3.28) can be related to the coupling constant renormalization by

\[
\frac{1}{g^2(k)} = 1 - \frac{C_2(G)}{48\pi^2} \left[ 11 \ln \left( \frac{\Lambda^2}{k^2} \right) + \frac{1}{3} \right] = \frac{1}{g^2} - \frac{11C_2(G)}{48\pi^2} \ln \left( \frac{\Lambda^2}{k^2} \right). \tag{4.1}
\]

The running coupling constant \( g(k) \) reduces to the usual bare coupling \( g \) at \( k = \tilde{\Lambda} \), as expected from perturbation theory. However, in the low energy regime where \( k \to 0 \), IR singularity appears due to the masslessness of the gluon fields. The renormalized coupling constant is usually defined at an arbitrary off-shell scale \( k_0 \) as

\[
g_R^2(k_0) = g^2(k^2 = k_0^2) = \frac{g^2}{1 - \frac{11C_2(G)g^2}{48\pi^2} \ln \left( \frac{\tilde{\Lambda}^2}{k_0^2} \right)}.	ag{4.2}
\]

Similarly, the \( \beta \) function reads

\[
\beta(g(k)) = k \frac{\partial g(k)}{\partial k} = - \frac{11C_2(G)}{48\pi^2} g^3(k). \tag{4.3}
\]

On the other hand, the evolution of the blocked action is obtained by differentiating (3.27) with respect to \( k \) to give

\[
k \partial_k \tilde{S}_k = \frac{1}{4} k \frac{\partial Z_{k}^{-1}}{\partial k} \int_p F_{\mu \nu}^{a}(p) F_{\mu \nu}^{a}(-p), \tag{4.4}
\]

which along with (3.29) and (4.1), implies that this “independent-mode” approximation [19] simply reduces (4.4) to the \( \beta \) function (4.3) which governs the flow of the coupling constant. The result is to be expected on the ground that \( g \) is the only free parameter in the theory. It also justifies the truncation of the background fields beyond quadratic order in our perturbative evaluation of the Yang-Mills blocked action.

While perturbation works for large \( k \) where the theory exhibits asymptotic freedom, it breaks down in the low energy limit as the coupling constant grows stronger. Thus, for small \( k \), in addition to \( Z_{k}^{-1} \), one must also consider the nonlocal sector \( \delta Z_{k}^{-1} \), for nonlocal
interactions may be a crucial ingredient for explaining confinement. Hence, from (3.28), we have the following nonlocal running coupling constant (denoted with a tilde):

\[
\frac{1}{\bar{g}^2(\kappa)} = \frac{1}{g^2} - \frac{C_2(G)}{4\pi^2} \left[ 11 \ln \left( \frac{\Lambda^2}{k^2} \right) + \frac{31}{3} + 4\kappa^2 + (8\kappa^4 + 22\kappa^2 + 11)f(\kappa) \right],
\]

which gives

\[
\bar{\beta}(\bar{g}(\kappa), \kappa) = \kappa \partial \bar{g}(\kappa) = \frac{C_2(G)}{8\pi^2} \bar{g}^3(\kappa) \frac{\kappa^2}{4\kappa^2 + 1} \left\{ -3 + 4\kappa^2 + 2\kappa^2(5 + 4\kappa^2)f(\kappa) \right\}.
\]

In other words, the behavior of the coupling constant in the IR region is characterized by the complicated expression in (4.6). Notice that as \( k \to \Lambda \), (4.5) and (4.6) reduce to the usual perturbative expressions found in (4.1) and (4.3), respectively.

Besides nonlocal interactions, higher dimensional operators may also play an important role in the physics of confinement in spite of their initial suppression at the high energy scale. The RG equation in (4.6) was obtained by an expansion in \( s \) up to \( O(s^2) \); contributions from higher order operators such as \( (F^2)^2 \) are therefore neglected. To improve the evolution equation (4.4) in the spirit of Wilson-Kadanoff, we first turn to the simplest noninvariant momentum cutoff regularization since its main features may be reproduced in the invariant operator cutoff scheme. From (3.11), we have

\[
k\partial \tilde{S}_k = \frac{1}{2} k \partial_k \left\{ \text{Tr}' \left[ \ln \mathcal{K}_{\mu\nu}^{ab}(\bar{A}) - \ln \mathcal{K}_{\mu\nu}^{ab}(0) \right] - 2\text{Tr}' \left[ \ln \mathcal{O}_{ab}(\bar{A}) - \ln \mathcal{O}_{ab}(0) \right] \right\},
\]

where the prime notation in \( \text{Tr}' \) indicates the presence of cutoff scale in the momentum integration. In going beyond the one-loop approximation to probe the physics near the energy scale \( \sim k \) using the RG improved idea, the first step is to divide the momentum integration volume defined between \( k \) and the UV cutoff \( \Lambda \) into a large number of thin shells each having a width \( \Delta k \). By lowering the cutoff infinitesimally from \( \Lambda \to \Lambda - \Delta k \to \Lambda - 2\Delta k \) until the scale \( k \) is reached, we then arrive at a nonlinear partial differential RG equation which takes into account the continuous feedbacks of the high mode to the lower ones as it gets integrated over[19]. Following this prescription, we obtain the following RG improved equation:

\[
k\partial \tilde{S}_k = \frac{1}{2} k \partial_k \left\{ \text{Tr}' \left[ \ln \left( \frac{\partial^2 \tilde{S}_k}{\partial A^a_m \partial A^b_n} \right)_{\bar{A}} - \ln \left( \frac{\partial^2 \tilde{S}_k}{\partial A^a_m \partial A^b_n} \right)_{0} \right] - 2\text{Tr}' \left[ \ln \mathcal{O}_{ab}(\bar{A}) - \ln \mathcal{O}_{ab}(0) \right] \right\},
\]

which is similar to that derived in [13]. Notice that there is no “dressing” in the ghost sector since the ghost fields enter the action as an “initial” boundary condition. By comparing (4.8) with (4.7), we observe that the “trick” to implement RG seems to be a simple replacement of the bare action \( S \) in the definition of \( \mathcal{K}_{\mu\nu}^{ab} \) by its corresponding \( k \)-dependent blocked action \( \tilde{S}_k \). This dressing is equivalent to summation over the higher order nonoverlapping graphs such as the daisies and the superdaisies when investigating theory at finite temperature [19].
Alternatively, we may first go back to (4.7) and split the fluctuation kernels as
\[ K_{\mu\nu}^{ab}(\bar{A}) = K_{\mu\nu}^{ab}(0) + \delta K_{\mu\nu}^{ab}(\bar{A}), \quad O^{ab}(\bar{A}) = O^{ab}(0) + \delta O^{ab}(\bar{A}), \] (4.9)

where \( K_{\mu\nu}^{ab,0} = \hat{1}_{\mu\nu} p^2 \) and \( O^{ab,0} = \delta^{ab} p^2 \) with \( \hat{1}_{\mu\nu} \) being the unit matrix in Lorentz and color space. In case where \( \delta K \) and \( \delta O \) are constant, instead of the blocked action, it suffices to consider the blocked potential whose evolution equation is given by
\[ k \partial_k U_k = -\frac{k^4}{16\pi^2} \ln \left( \frac{\hat{1}_{\mu\nu,k}^2 + \delta K_{\mu\nu}^{ab}(\bar{A})}{\hat{1}_{\mu\nu,k}^2} \right) + \frac{k^4}{8\pi^2} \ln \left( \frac{\delta^{ab}k^2 + \delta O^{ab}(\bar{A})}{k^2} \right), \] (4.10)

subject to the boundary condition
\[ \lim_{k \to \Lambda} U_k = -\frac{1}{2g^2} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) \bigg|_{k=\Lambda} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \bigg|_{k=\Lambda} = \frac{1}{4} G_{\mu\nu} G_{\mu\nu} = \mathcal{L}, \] (4.11)

i.e., the bare lagrangian is recovered at the UV cutoff scale. Summing over the Lorentz and color indices, (4.10) simplifies to
\[ k \partial_k U_k = -\frac{k^4}{16\pi^2} \ln \left( \frac{k^2 + \delta K(\bar{A})}{k^2} \right) + \frac{k^4}{8\pi^2} \ln \left( \frac{k^2 + \delta O(\bar{A})}{k^2} \right), \] (4.12)

which implies the following improved equation:
\[ k \partial_k U_k = -\frac{k^4}{16\pi^2} \ln \left( \frac{k^2 + U_k''}{k^2} \right) + \frac{k^4}{8\pi^2} \ln \left( \frac{k^2 + \delta O(\bar{A})}{k^2} \right). \] (4.13)

Eq. (4.13) is applicable in the case of homogeneous background such as constant magnetic or chromomagnetic field which we explore in the next Section.

Going to the invariant operator cutoff prescription, (4.8) takes on the form
\[ k \partial_k \tilde{S}_k = -\frac{1}{2} \text{Tr} \int_0^\infty ds \frac{k^{(4)} \partial \rho_k(s, A)}{\partial k} \left\{ e^{-K_{\mu\nu,k}^{ab}(\bar{A})s} - e^{-K_{\mu\nu,k}^{ab}(0)s} \right\} - 2 \left[ e^{-O^{ab}(\bar{A})s} - e^{-O^{ab}(0)s} \right], \] (4.14)

where
\[ K_{\mu\nu,k}^{ab}(\bar{A}) = \left. \frac{\partial^2 \tilde{S}_k}{\partial A^a_{\mu} \partial A^b_{\nu}} \right|_{\bar{A}}. \] (4.15)

Again we see that the improved RG takes on the form of a nonlinear partial differential equation. If (4.14) is approximated by expanding the integrand in power of \( s \) and keeping only up to \( O(A^2) \), the flow would reduce to the usual \( \beta \)-function, as we have seen before. Since treating \( s \) as a small expansion parameter corresponds to exploring the high energy regime of the theory, it is not surprising after all that the short-distance property of asymptotic freedom is easily recovered from such a perturbative approximation of the blocked
action. However, if one is interested in the IR behavior of the theory, the approximation becomes unreliable. Actually a complete $s$ integration without expansion is possible and it gives

$$k \partial_k \tilde{S}_k = k^4 \text{Tr} \left\{ \left[ \hat{1}_{ab} k^2 + \mathcal{K}_{\mu \nu, k}(\hat{A}) \right]^{-2} - \left[ \hat{1}_{\mu \nu} k^2 + \mathcal{K}_{\mu \nu, k}(0) \right]^{-2} - 2 \left[ \delta_{ab} k^2 + \mathcal{O}^{ab}(\hat{A}) \right]^{-2} + 2 \left[ \delta_{ab} k^2 + \mathcal{O}^{ab}(0) \right]^{-2} \right\}. \quad (4.16)$$

The role played by higher dimensional operators at the energy scale $\sim k$ can now be elucidated by solving (4.16). The above equation may be compared with (4.8) which is obtained using the noninvariant momentum cutoff regularization. The resulting effective blocked actions generated from these nonlinear partial differential equations are expected to be nonlocal. Nevertheless, if the relevant local operators, be they of higher dimension or not, can be identified to account for the physical phenomena in the infrared, a suitable expansion may then be possible to make the theory local.

V. CONSTANT CHROMOMAGNETIC FIELD

Enormous efforts have been devoted to the study of the vacuum structure of $SU(2)$ gauge theory in a constant chromomagnetic background since the pioneering work of Matinyan and Savvidy [20]-[25]. By the help of (4.16), we now explore the RG evolution associated with this configuration.

For simplicity, we choose the background to be a constant chromomagnetic field $B$ in the $\hat{z}$-direction produced by

$$\hat{A}_\mu^a = \delta^{a3} \delta_{\mu 2} B x, \quad (5.1)$$

with

$$F^a_{\mu \nu} F^a_{\mu \nu} = 2 B^2. \quad (5.2)$$

An alternative choice $\hat{A}_\mu^a = \frac{1}{2} B \delta^{a3} (x \delta_{\mu 2} - y \delta_{\mu 1})$ has also been used in [21]. Working in the background gauge, the eigenvalues for the kernels $\mathcal{K}$ and $\mathcal{O}$ can be obtained by a diagonalization in the color space, which then reduces the equation of motion into a harmonic oscillator equation and yields the Landau energy levels labeled by $n$, where $n = 0, 1, 2, \cdots$ [22] [23]. Thus, the Yang-Mills blocked potential can be written as [24]

$$U_k \sim \int \frac{dp}{2\pi} \sum_{n=0}^{\infty} \sum_{S_z=\pm 1} \sqrt{p^2 + k^2 + 2gB(n + \frac{1}{2}) - 2gBS_z}, \quad (5.3)$$

where $S_z$ is the $\hat{z}$-component of the gluon spin along the direction of the chromomagnetic field, and the factor 2 in $\vec{B} \cdot \vec{S}$ comes from the gyromagnetic ratio $g_L = 2$ for the gluon fields. Since gluons are massless vector particles, $S_z = 0$ is an unphysical degree of freedom and the associated contribution will be cancelled by the Faddeev-Popov ghost [24]. For $n = 0$ and $S_z = 1$, we notice that $U_k$ becomes complex below certain momentum scale. The unstable mode gives an imaginary contribution to the blocked potential and signals an instability for the vacuum. Thus, to stabilize the theory, we choose the IR scale $k$ to be such that $k^2 > gB$. 

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Using
\[ \int \frac{dp}{2\pi} \sqrt{p^2 + E^2} = \int \frac{d^2p}{(2\pi)^2} \ln(p^2 + E^2), \]
up to an \( E \)-independent constant, the trace sum in the operator cutoff formalism may be represented as
\[ \text{Tr}_{\Omega} = \frac{gB}{2\pi} \rho^{(2)}_k(s, \Lambda) \int \frac{d^2p}{(2\pi)^2} \sum_n, \]
where \( \Omega \) is the space-time volume. Taking into account the multiplicity factors for the eigenvalues, it follows from (4.16) that the RG equation for the theory reads
\[ k \partial_k U_k = \frac{k^2 gB}{2\pi} 2 \int \frac{d^2p}{(2\pi)^2} \left\{ \left[ \frac{1}{p^2 + k^2 - gB} - \frac{1}{p^2 + k^2} \right] + 3 \left[ \frac{1}{p^2 + k^2 + gB} - \frac{1}{p^2 + k^2} \right] + 4 \sum_{n=1}^{\infty} \left[ \frac{1}{p^2 + k^2 + (2n + 1)gB} - \frac{1}{p^2 + k^2} \right] \right\} \]
\[ = -\frac{k^2 gB}{4\pi^2} \left\{ \ln\left( \frac{k^2 - gB}{k^2} \right) + 3\ln\left( \frac{k^2 + gB}{k^2} \right) + 4 \sum_{n=1}^{\infty} \ln\left( \frac{k^2 + (2n + 1)gB}{k^2} \right) - 2 \sum_{n=0}^{\infty} \ln\left( \frac{k^2 + (2n + 1)gB}{k^2} \right) \right\}, \]
where the overall factor of 2 accounts for the color charge degeneracy in the \( SU(2) \) gauge group. While the last term in the curly bracket represents the contribution from the Faddeev-Popov ghost kernel, the first term is due to the mode which becomes unstable for \( k^2 < gB \). Notice that the multiplicity factors for the eigenvalues \( gB \) and \( (2n + 1)gB \) for \( n > 1 \) were miswritten in [22]; the correct factors should be 3 and 4, respectively. The reason is due to the negligence of the unphysical \( S_z = 0 \) sector which yields eigenvalues \( (2n + 1)gB \) for \( n = 0, 1, \cdots \). As explained before, this mode must be considered fully in the presence of Faddeev-Popov ghost.

With \( g(k) = Z^{1/2} \) and
\[ Z_k^{-1} = \frac{\partial U_k}{\partial F} = \frac{1}{B} \frac{\partial U_k}{\partial B}, \]
the \( \beta \) function can be rewritten as
\[ \beta(g(k), \tau) = g_k \frac{\partial g_k}{\partial \tau} = -\frac{1}{2} g \dot{Z}_k^{3/2} k \partial_k Z_k^{-1} = -g \dot{Z}_k^{3/2} k \partial_k \left( \frac{\partial U_k}{\partial B^2} \right) = -\frac{g Z_k^{3/2}}{2B} \frac{\partial}{\partial B} (k \partial_k U_k) \]
\[ = \frac{g^3(k)}{8\pi^2 \tau} \left\{ \ln(1 - \tau) + 3\ln(1 + \tau) + 4 \sum_{n=1}^{\infty} \ln[1 + (2n + 1)\tau] - 2 \sum_{n=0}^{\infty} \ln[1 + (2n + 1)\tau] \right\} \]
\[ - \tau \left[ \frac{1}{1 - \tau} - \frac{3}{1 + \tau} - 4 \sum_{n=1}^{\infty} \frac{(2n + 1)}{1 + (2n + 1)\tau} + 2 \sum_{n=0}^{\infty} \frac{(2n + 1)}{1 + (2n + 1)\tau} \right], \]
where \( \tau = gB/k^2 \). It is rather interesting to note that the \( \beta \) function here not only depends on \( g(k) \), but also the dimensionless parameter \( \tau \). With the help of the Euler formula [23]:

\[
\sum_{n=0}^{\infty} h(n + \frac{1}{2}) = \int_{0}^{\infty} dx \ h(x) - \frac{1}{24} h'(x)\bigg|_{0}^{\infty} + \cdots, \quad h(\infty) = 0, \quad (5.9)
\]

the \( \beta \) function reads

\[
\beta(g(k), \tau) = \frac{g^3(k)}{8\pi^2 \tau} \left\{ \ln(1-\tau) - \ln(1+\tau) + \frac{\tau}{3} - \frac{\tau}{6} + \cdots - \tau \left[ \frac{1}{1-\tau} + \frac{1}{1+\tau} - \frac{1}{3} + \frac{1}{6} + \cdots \right] \right\}, \quad (5.10)
\]

which in the limit of large \( k \) or vanishing \( \tau \), gives

\[
\beta(g(k)) = -\frac{g^3(k)}{4\pi^2} \left\{ \frac{1}{3} + \frac{2}{3} + \frac{1}{6} \right\} = -\frac{11g^3(k)}{24\pi^2}, \quad (5.11)
\]

in complete agreement with the that obtained from (4.3) for \( SU(2) \). Notice that the contributions to the \( \beta \) function from the “unstable mode” (the first term) and the ghost kernel (the third term) are, respectively, \(-g^3(k)/4\pi^2\) and \(-g^3(k)/24\pi^2\), in accord with the analyses of Nielsen and Olsen [24].

We mentioned before that two multiplicity factors used in [22] were incorrect due to the negligence of the unphysical \( S_z = 0 \) sector albeit the same \( \beta \) function was given. The way it was obtained is as follows: The original expression which makes no reference of the unphysical sector \( S_z = 0 \) actually gives

\[
\beta_M(g(k)) = -\frac{g^3(k)}{4\pi^2} \left\{ 1 + \frac{3}{4} + \frac{1}{6} \right\} = -\frac{23g^3(k)}{48\pi^2}, \quad (5.12)
\]

instead of (5.11). Removing the contribution from the would-be unstable mode entirely by the subtraction

\[
\int \frac{d^2p}{(2\pi)^2} \ln(p^2 + gB) - \int_{p^2 > gB} \frac{d^2p}{(2\pi)^2} \ln(p^2 - gB) \quad (5.13)
\]

followed by the substitution

\[
\int \frac{ds}{s} = \ln s \longrightarrow 2 \times \ln \left( \frac{\mu^2}{gB} \right) \quad (5.14)
\]

in the proper-time representation then gives the correct result:

\[
\beta_M(g(k)) \rightarrow -\frac{g^3(k)}{4\pi^2} \left\{ 0 + \frac{3}{4} + \frac{1}{6} \right\} = -\frac{11g^3(k)}{24\pi^2}. \quad (5.15)
\]

In other words, eliminating the contribution from the would-be unstable mode completely followed by multiplying the extra factor of 2 as appeared on the right-hand-side of (5.14) is how the expected \( \beta \) function was arrived in [22].

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Taking into account the imaginary contribution which gets generated for \( p^2 < gB \), the complete complex effective potential reads

\[
U = \frac{B^2}{2} + \frac{11g^2}{48\pi^2}B^2 \left[ \ln \left( \frac{gB}{\mu^2} \right) - \frac{1}{2} \right] - \frac{i g^2 B^2}{8\pi^2},
\]

subject to the renormalization condition [20]

\[
\left. \frac{\partial (\text{Re } U)}{\partial \mathcal{F}} \right|_{\tilde{\mu}^2} = 1,
\]

with \( \mathcal{F} = B^2/2 \) being the gauge invariant quantity of the theory. The condition is readily fulfilled by choosing \( \tilde{\mu}^4 = 2g^2 \mathcal{F} \).

The arbitrary scale \( \mu^2 \) and the IR cutoff \( k \) arising from the operator cutoff regularization can be related to each other by noting that

\[
\mu \frac{\partial}{\partial \mu} (\text{Re } U) = - \frac{11g^2 B^2}{24\pi^2},
\]

and

\[
k \frac{\partial}{\partial k} U_k = - \frac{k^2 gB}{4\pi^2} \left[ \ln \left( \frac{k^2 - gB}{k^2 + gB} \right) + 2 \sum_{n=0}^{\infty} \ln \left( \frac{k^2 + (2n + 1)gB}{k^2} \right) \right] \rightarrow \frac{11g^2 B^2}{24\pi^2} + \frac{1}{6\pi^2} \frac{g^4 B^4}{k^4} + \cdots
\]

in the large \( k \) limit. This implies that we must have \( \mu \frac{\partial}{\partial \mu} \equiv -k \frac{\partial}{\partial k} \), i.e., the two scales run in the opposite manner. This connection can also be seen from

\[
\frac{1}{g^2(k)} = \frac{1}{g^2} - \frac{11}{24\pi^2} \ln \left( \frac{\tilde{\Lambda}^2}{k^2} \right)
\]

given by (4.1) and

\[
\frac{1}{g^2(\mu)} = \frac{1}{g^2} - \frac{11}{24\pi^2} \ln \left( \frac{\mu^2}{gB} \right).
\]

In other words, with \( k^2 \) being the usual IR cutoff, \( \mu^2 \) should be interpreted as an UV cutoff. By replacing then right-hand-side of (5.19) with the corresponding \( k \)-dependent running parameters, the RG evolution equation for \( U_k \) becomes

\[
k \frac{\partial}{\partial k} U_k = - \frac{2\beta(g(k))}{g(k)} U_k \left( 1 + \frac{8g^2(k)}{11} \frac{U_k}{k^4} \right) + \cdots.
\]

Solving this differential equation by retaining only the leading order contribution, we have the following RG improved blocked potential:

\[
\ln U_k = -2 \int \frac{dk}{k} \frac{\beta(g(k))}{g(k)}.
\]
Eq. (5.23) is analogous to the result obtained in [20] by Matinyan and Savvidy. However, it only takes into consideration the effect the $\mathcal{F} = B^2/2$ term. In order to explore the influence of the higher order operators, one must solve (5.22) completely without truncation.

We emphasize that the above treatments are limited to the regime where $k$ is large and the theory is asymptotically free. Continuing to evolve the system to a lower $k$ will result in a complicated blocked action which invalidates perturbation theory. In the IR region where $\tau$ is large, eq. (5.22) is no longer a good approximation. Serious difficulties are already encountered near $\tau = 1$ as the $\beta$ function in (5.8) develops pole and divergence. The source of the singularity is undoubtedly due to the unstable mode which becomes unsuppressed for $k \leq \sqrt{gB}$ ($\tau \geq 1$). In [20], Savvidy considered only the real part of the potential in (5.18) and obtained a nontrivial minimum:

$$gB_{\text{min}} = \mu^2 e^{-24\pi^2/11g^2}.$$  \hfill (5.24)

However, the existence of this vacuum configuration which lies in the deep IR regime has been a subject of intense debates for quite some time. The persistence of the unstable mode in the IR region lead Maiani et al to argue that the problem associated with unstable configurations can only be treated nonperturbatively. On the other hand, lattice calculations seem to support the formation of chromomagnetic condensate [21] [25]. To provide a consistent check to these claims, a successful nonperturbative RG approach would be desirable. We therefore propose to modify (5.6) and write

$$k \partial_k U_k = -\frac{k^2 g(k)}{4\pi^2} \sqrt{2U_k} \left\{ \ln \left( \frac{k^2 - g(k)\sqrt{2U_k}}{k^2 + g(k)\sqrt{2U_k}} \right) + 2\sum_{n=0}^{\infty} \ln \left( \frac{k^2 + (2n+1)g(k)\sqrt{2U_k}}{k^2} \right) \right\},$$  \hfill (5.25)

where the chromomagnetic field $B$ on the right-hand-side of (5.6) has been replaced by $Z_k^{1/2}\sqrt{2U_k}$ as suggested by (5.7). Since no assumption has been made on the value of $k$ in (5.25), it remains valid even in the deep IR regime where $k$ is small. Our nonlinear RG equation for the blocked potential $U_k$ provides an improvement beyond the standard perturbative treatment to the IR problem; in particular, the RG dependence of the unstable mode can be explored.

VI. SUMMARY AND DISCUSSIONS

In this paper we derived the effective Yang-Mills blocked action $\tilde{S}_k$ in a manifestly gauge invariant manner by the help of the operator cutoff regularization. The regulating smearing function $\rho^{(d)}_k(s, \Lambda)$ introduced in the proper-time integration simulates a sharp momentum cutoff for the leading order blocked potential $U_k$ and is reminiscent to the Pauli-Villars regulator. The blocked action $\tilde{S}_k$ provides a smooth interpolation between the bare action defined at $k = \Lambda$ and the effective action at an arbitrary energy scale $k$. The presence of the IR scale $k$ readily allows us to study the RG flow of the theory using the Wilson-Kadanoff blocking approach.

The conventional perturbative one-loop differential flow equation for the Yang-Mills blocked action corresponds to the evolution for the gauge coupling constant $g$. This is due to the fact that $g$ is the only free parameter in the theory and the inclusion of the
one-loop contributions from both gauge and Faddeev-Popov ghost kernels is equivalent to replacing $g$ by the running parameter $g(k)$. However, the situation changes when we consider the RG improved equation (4.16) which invariably incorporates the contributions from other higher dimensional operators that are being generated in the course of blocking transformation. These operators, though initially absent in the bare lagrangian, may play an important role in the effective lagrangian for the low energy IR theory. The advantage of (4.16) is that it allows practical calculations and it remains applicable even when $k$ is small.

In probing the vacuum structure of QCD, nonperturbative method such as lattice gauge theory has often been employed. We find the RG approach outlined in this paper a powerful alternative for investigating the vacuum. Though the complicated nonlinear partial differential equation written in (4.16) seems to make the analytical form for the low energy blocked action rather hopeless, its numerical solutions may nevertheless provide a consistent check for the lattice results. For the simplest $SU(2)$ theory in the presence of a static chromomagnetic field considered in Sec. V, a complete solution for the RG flow equation (5.25) may yield additional insights on the role of the unstable mode as well as the effect of the higher order gauge invariant operators such as $F^2 = B^4/4$. It may even help resolve the longstanding issue of the reliability of the energetically more favored ground state given in (5.24). For realistic theories, the effects of matter fields too must be considered. Works along these directions are now in progress.

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APPENDIX A: SCALAR FIELD THEORY

In this Appendix, we give the details of how blocked potentials for scalar field theory are computed using operator cutoff formalism. To be definite, the calculations will be carried out in $d = 4$ dimension. Consider for simplicity the following bare lagrangian:

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 + V(\phi). \quad (A.1)$$

In the presence of a slowly-varying background field $\Phi(x)$ whose Fourier modes are constrained by an upper cutoff scale $k$, by integrating out the fast-fluctuating modes, the one-loop contribution to the low-energy blocked potential is given by

$$U_{k}^{(1)}(\Phi) = -\frac{1}{2} \int_{0}^{\infty} \frac{ds}{s} \rho_k^{(4)}(s, \Lambda) \int_{p} e^{-p^{2}s} \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right)$$

$$= -\frac{1}{32\pi^2} \int_{0}^{\infty} \frac{ds}{s^3} \rho_k^{(4)}(s, \Lambda) \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right). \quad (A.2)$$
Notice that the scales set by momentum regularization are now taken over by $\rho_k^{(4)}(s, \Lambda)$. As shown in [6], smearing function of the form

$$\rho_k^{(4)}(s, \Lambda) = \left[ 1 - (1 + \Lambda^2 s) e^{-\Lambda^2 s} \right] - \left[ 1 - (1 + k^2 s) e^{-k^2 s} \right] = \rho(\Lambda^2 s) - \rho(k^2 s) \quad (A.3)$$

is equivalent to imposing sharp momentum cutoffs. That is, inserting (A.3) into (A.2) leads to the cutoff expression:

$$U_k^{(1)}(\Phi) = \frac{1}{2} \int_0^\infty \ln \left( \frac{p^2 + V''(\Phi)}{p^2 + V''(0)} \right) = \frac{1}{64\pi^2} \left\{ (\Lambda^2 - k^2)(V''(\Phi) - V''(0)) \right. $$

$$+ \Lambda^4 \ln \left( \frac{\Lambda^2 + V''(\Phi)}{\Lambda^2 + V''(0)} \right) - k^4 \ln \left( \frac{k^2 + V''(\Phi)}{k^2 + V''(0)} \right) - V''(\Phi) \ln \left( \frac{\Lambda^2 + V''(\Phi)}{k^2 + V''(\Phi)} \right) + \cdots \right\} ,$$

up to some $\Phi$-independent constant. Taking the $\lambda \phi^4$ theory as an example, the blocked potential up to the one-loop order becomes

$$U_k(\Phi) = V(\Phi) - \frac{1}{2} \int_0^\infty ds \, \rho_k^{(4)}(s, \Lambda) e^{-s(\rho^2 + \mu_R^2)} \left( e^{-\lambda R s^2/2} - 1 \right)$$

$$= \frac{1}{2} \left[ \mu^2 + \frac{\lambda R}{32\pi^2} \left( \Lambda^2 + \mu_R^2 \ln \frac{\mu_R^2}{\Lambda^2} \right) - \frac{\lambda R}{64\pi^2} k^2 \right] \Phi^2 + \frac{1}{4!} \left[ \lambda + \frac{3\lambda_R^2 R}{32\pi^2} \left( 1 + \ln \frac{\mu_R^2}{\Lambda^2} \right) \right] \Phi^4$$

$$+ \frac{1}{64\pi^2} \left[ \left( \mu_R^2 + \frac{\lambda R}{2} \Phi^2 \right)^2 - k^4 \right] \ln \left( \frac{k^2 + \mu_R^2 + \lambda R \Phi^2/2}{\mu_R^2} \right)$$

$$= \frac{\mu_R^2}{2} \Phi^2 \left[ 1 - \frac{\lambda R}{64\pi^2} \left( 1 + \frac{k^2}{\mu_R^2} \right) \right] + \frac{\lambda R}{4!} \Phi^4 \left( 1 - \frac{9\lambda R}{64\pi^2} \right)$$

$$+ \frac{1}{64\pi^2} \left[ \left( \mu_R^2 + \frac{\lambda R}{2} \Phi^2 \right)^2 - k^4 \right] \ln \left( 1 + \frac{k^2 + \mu_R^2 + \lambda R \Phi^2/2}{\mu_R^2} \right) ,$$

where the renormalized parameters are given by

$$\begin{align*}
\mu_R^2 &= \mu^2 + \frac{\lambda R}{32\pi^2} \left( \Lambda^2 + \mu_R^2 \ln \frac{\mu_R^2}{\Lambda^2} \right) \\
\lambda_R &= \lambda + \frac{3\lambda_R^2 R}{32\pi^2} \left( 1 + \ln \frac{\mu_R^2}{\Lambda^2} \right) .
\end{align*} \quad (A.6)$$

It is easily seen that in the limit $k = 0$, (A.5) reduces to the usual effective potential obtained in [26]. For this theory, the improved RG equation reads

$$k \partial_k U_k(\Phi) = -\frac{k^4}{16\pi^2} \ln \left( \frac{k^2 + U_k''(\Phi)}{k^2 + U_k''(0)} \right) ,$$

which is a nonlinear differential equation that takes into account the coupling between the high and the low momentum modes.
The results obtained above can be readily extended to scalar electrodynamics. The lagrangian is given by
\begin{equation}
\mathcal{L}_{\text{SQED}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 \\
+ |(\partial_\mu + i e_0 A_\mu) \phi(x)|^2 + \frac{\mu^2}{2} \phi(x)^\dagger \phi(x) + \frac{\lambda}{6} (\phi(x)^\dagger \phi(x))^2,
\end{equation}
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\alpha$ is the gauge-fixing parameter. The complex field $\phi(x)$ may be rewritten in terms of real fields $\phi_1$ and $\phi_2$ as $(\phi_1(x) + i \phi_2(x))/\sqrt{2}$. Considering the special case where $A_\mu = 0$ and $\Phi^a = \Phi \delta^{a,1}$ with $\Phi$ being the constant background configuration, the blocked potential in the Landau gauge with $\alpha = 0$ becomes
\begin{equation}
U_1^{(1)}(\Phi) = -\frac{1}{32\pi^2} \int_0^\infty \frac{ds}{s^3} \left\{ e^{-\mu_R^2 s} \left[ (1 + k^2 s) e^{-k^2 s} - (1 + \Lambda^2 s) e^{-\Lambda^2 s} \right] \times \left[ (e^{-\lambda_R \Phi^2 s/2 - 1}) + (e^{-\lambda_R \Phi^2 s/2 - 1}) \right] + 3 \left[ 1 - (1 + \Lambda^2 s) e^{-\Lambda^2 s} \right] \left( e^{-\phi_0^2 \Phi^2 s} - 1 \right) \right\}.
\end{equation}

Even though blocking is performed only for the scalar fields, it can be implemented in a similar fashion for gauge fields as well. Notice that the extra factor of three in the photon loop contribution arises from the trace of the propagator in the Landau gauge. We also comment that the form of $U_k(\Phi)$ is generally gauge dependent although physical quantities must be gauge independent [27].

The theory, however, is plagued by IR singularity due to the presence of massless photons. The problem could be avoided if blocking is also done for the gauge fields, i.e., using $\rho_{k,0}^{(4)}(s, \Lambda)$ instead of $\rho_{k,0}^{(4)}(s, \Lambda)$. The conventional regularization scheme is an off-shell subtraction condition for the coupling constant [26]:
\begin{equation}
\lambda_R = \frac{\partial^4 U_k(\Phi)}{\partial \Phi^4} \bigg|_{\Phi=M,k=0}
\end{equation}
which, in the language of operator cutoff, is equivalent to using the following coupling constant counterterm [28]:
\begin{equation}
\delta \lambda = \frac{1}{32\pi^2} \int_0^\infty \frac{ds}{s} \left[ 1 - (1 + \Lambda^2 s) e^{-\Lambda^2 s} \right] \left\{ \lambda_R^2 R e^{-(\mu_R^2 + \lambda_R M^2/2)s} \left( 3 - 6 \lambda_R M^2 s + \lambda_R^2 M^4 s^2 \right) \right. \\
+ \frac{\lambda_R^2}{81} e^{-(\mu_R^2 + \lambda_R M^2/6)s} \left( 27 - 18 \lambda_R M^2 s + \lambda_R^2 M^4 s^2 \right) \\
+ 12 \epsilon_0^4 e^{-\phi_0^2 \Phi^2 s} \left( 3 - 12 \epsilon_0^2 M^2 s + 4 \epsilon_0^4 M^4 s^2 \right) \left. \right\} \\
= -\frac{1}{64\pi^2} \left\{ \frac{20}{3} \lambda_R^2 + \frac{4 \lambda_R^2 M^2 (\lambda_R M^2 + 9 \mu_R^2)}{81(\mu_R^2 + \lambda_R M^2/6)^2} + \frac{4 \lambda_R^2 M^2 (\lambda_R M^2 + 3 \mu_R^2)}{(\mu_R^2 + \lambda_R M^2/2)^2} \right. \\
+ 24 \epsilon_0^4 \left[ 1 + 3 \ln \left( \frac{M^2}{\Lambda^2} \right) \right] + \frac{2 \lambda_R^2}{3} \ln \left( \frac{\mu_R^2 + \lambda_R M^2/6}{\Lambda^2} \right) + 6 \lambda_R^2 \ln \left( \frac{\mu_R^2 + \lambda_R M^2/2}{\Lambda^2} \right) \left. \right\}. \quad \text{(A.11)}
\end{equation}
After removing the $\Lambda$ dependence, the blocked potential becomes

$$U_k(\Phi) = \frac{\mu^2}{2} \Phi^2 + \frac{\lambda_R}{4!} \Phi^4 + \frac{1}{64\pi^2} \left\{ -\frac{2\lambda_R}{3} (k^2 + \mu^2_R) \Phi^2 - \frac{5\lambda^2_R}{12} \Phi^4 
+ \left[ (\mu^2_R + \frac{\lambda_R}{2} \Phi^2)^2 - k^4 \right] \ln \left( \frac{k^2 + \mu^2_R + \lambda_R \Phi^2/2}{\mu^2_R} \right) 
+ \left[ (\mu^2_R + \frac{\lambda_R}{6} \Phi^2)^2 - k^4 \right] \ln \left( \frac{k^2 + \mu^2_R + \lambda_R \Phi^2/6}{\mu^2_R} \right) 
+ \frac{\lambda^2_R}{4} \Phi^4 \ln \left( \frac{\mu^2_R}{\mu^2_R + \lambda_R M^2/2} \right) 
+ \frac{\lambda^2_R}{36} \Phi^4 \ln \left( \frac{\mu^2_R}{\mu^2_R + \lambda_R M^2/3} \right) 
- \frac{\lambda^3_R M^2}{6} \Phi^4 \left[ \frac{\lambda_R M^2 + 9\mu^2_R}{81(\mu^2_R + \lambda_R M^2/6)^2} + \frac{\lambda_R M^2 + 3\mu^2_R}{(\mu^2_R + \lambda_R M^2/2)^2} \right] 
+ 3\epsilon_0^4 \Phi^4 \left[ \ln \left( \Phi^2/M^2 \right) - \frac{25}{6} \right] \right\},$$

which for $\mu^2_R = k^2 = 0$ reduces to

$$U_{k=0}(\Phi) = \frac{\lambda_R}{4!} \Phi^4 + \frac{\Phi^4}{64\pi^2} \left( \frac{5}{18} \lambda^2_R + 3\epsilon_0^4 \right) \left[ \ln \left( \Phi^2/M^2 \right) - \frac{25}{6} \right].$$

The theory in this limit shows spontaneous symmetry breaking driven by radiative corrections [26]. Once more, the symmetry-preserving nature of the operator cutoff is seen from the absence of cutoff scales in the $p$ integration.

**APPENDIX B: GENERALIZED PROPER-TIME CLASS**

Numerous regularization schemes can all be shown to belong to the generalized class of proper-time since they can be represented by a suitable definition of smearing function. A detailed discussion has already been given by Ball [7] and also in [8]. However, for the sake of completeness and comparative purpose, we recapitulate here various examples and examine how they modify the propagator and the corresponding one-loop kernel. We also show how cutoff scales can be implemented in dimensional regularization as well as $\zeta$ function regularization. The “hybrid” prescriptions of dimensional cutoff and $\zeta$ function cutoff allows us to establish a direct connection with the momentum regularization. We specifically apply each of these techniques to regularize the divergences encountered in the computation of the two- and the four-point vertex functions for $\mathcal{H} = p^2 + \mu^2$ in $d = 4$.

(1) operator cut-off:

For $d = 4$, the smearing function becomes

$$\rho_k^{(4)}(s, \Lambda) = (1 + k^2 s) e^{-k^2 s} - (1 + \Lambda^2 s) e^{-\Lambda^2 s} = (e^{-k^2 s} - e^{-\Lambda^2 s}) + s(k^2 e^{-k^2 s} - \Lambda^2 e^{-\Lambda^2 s}),$$

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which leads to the following regularized propagator and one-loop kernel:

\[
\frac{1}{\mathcal{H}^n} \bigg|_{\text{oc}} = \frac{1}{\Gamma(n)} \int_0^\infty ds \ s^{n-1} \left[ (1 + k^2 s)e^{-k^2 s} - (1 + \Lambda^2 s)e^{-\Lambda^2 s} \right] e^{-\mathcal{H} s}
\]

\[
= \frac{1}{(\mathcal{H} + k^2)^n} - \frac{1}{(\mathcal{H} + \Lambda^2)^n} + \frac{n k^2}{(\mathcal{H} + k^2)^{n+1}} - \frac{n \Lambda^2}{(\mathcal{H} + \Lambda^2)^{n+1}},
\]

and

\[
\text{Tr}_{\text{oc}} \ln \left( \frac{\mathcal{H}}{\mathcal{H}_0} \right) = \text{Tr} \left\{ \ln \left[ \frac{\mathcal{H} + k^2}{\mathcal{H}_0 + k^2} \times H_0 + \Lambda^2 \right] - \frac{\Lambda^2 (\mathcal{H} - \mathcal{H}_0)}{(\mathcal{H} + \Lambda^2)(\mathcal{H}_0 + \Lambda^2)} + \frac{k^2 (\mathcal{H} - \mathcal{H}_0)}{(\mathcal{H} + k^2)(\mathcal{H}_0 + k^2)} \right\}.
\]

As demonstrated in Sec. II, eq.(B.1) simulates a sharp cutoff at the level of blocked potential. Using (A.2), the one-loop correction to the two- and four-point vertex functions for scalar theory can be written as

\[
\delta \Gamma^{(2)}_{\text{oc}} = \frac{\partial^2 U_k^{(1)}}{\partial \Phi^2} \bigg|_{\Phi=0} = \frac{\lambda}{32\pi^2} \int_0^\infty \frac{ds}{s^2} \rho_k^{(4)}(s, \Lambda)e^{-\mu^2 s},
\]

and

\[
\delta \Gamma^{(4)}_{\text{oc}} = \frac{\partial^4 U_k^{(1)}}{\partial \Phi^4} \bigg|_{\Phi=0} = -\frac{3\lambda^2}{32\pi^2} \int_0^\infty \frac{ds}{s} \rho_k^{(4)}(s, \Lambda)e^{-\mu^2 s},
\]

which for \( k = 0 \) become

\[
\delta \Gamma^{(2)}_{\text{oc}} = \int_p \frac{1}{p^2 + \mu^2} \left( \frac{\Lambda^2}{p^2 + \mu^2 + \Lambda^2} \right)^2
\]

\[
= \frac{\lambda}{32\pi^2} \int_0^\infty \frac{ds}{s^2} \rho_{k=0}^{(4)}(s, \Lambda)e^{-\mu^2 s} = \frac{\lambda}{32\pi^2} \left[ \Lambda^2 - \mu^2 \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) \right],
\]

and

\[
\delta \Gamma^{(4)}_{\text{oc}} = \int_p \frac{1}{(p^2 + \mu^2)^2} \left( \frac{\Lambda^2}{p^2 + \mu^2 + \Lambda^2} \right)^2 \left[ 1 + \frac{2(p^2 + \mu^2)}{p^2 + \mu^2 + \Lambda^2} \right]
\]

\[
= -\frac{3\lambda}{32\pi^2} \int_0^\infty \frac{ds}{s} \rho_{k=0}^{(4)}(s, \Lambda)e^{-\mu^2 s} = -\frac{3\lambda^2}{32\pi^2} \left[ \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) - \frac{\Lambda^2}{\mu^2 + \Lambda^2} \right].
\]

On the other hand, using the momentum cutoff procedure, one also has

\[
\delta \Gamma^{(2)}_{\Lambda} = \frac{\lambda}{2} \int_p \frac{1}{p^2 + \mu^2} = \frac{\lambda}{32\pi^2} \left[ \Lambda^2 - \mu^2 \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) \right],
\]

and

\[
\delta \Gamma^{(4)}_{\Lambda} = -\frac{3\lambda^2}{2} \int_p \frac{1}{(p^2 + \mu^2)^2} = -\frac{3\lambda^2}{32\pi^2} \left[ \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) - \frac{\Lambda^2}{\mu^2 + \Lambda^2} \right].
\]
Pauli-Villars:

The conventional Pauli-Villars scheme can be parameterized in the proper-time representation by taking the smearing function to be

$$\rho_k^{pv}(s, \Lambda) = \sum_i (a_i e^{-k_i^2 s} - b_i e^{-\Lambda_i^2 s}), \quad (B.10)$$

where $\Lambda_i$ are the masses of some ghost states, and $k_i$ the extra masses added to the spectra. To render the theory finite, the coefficients $a_i$ and $b_i$ as well as $i$, the number of ghost terms are appropriately chosen. Physical limit, however, corresponds to taking $\Lambda_i \to \infty$ and $k_i \to 0$ since $\Lambda_i$ and $k_i$ control, respectively, the divergent behaviors of the theory in the UV and the IR regimes. Eq. (B.10) implies

$$\left. \frac{1}{\mathcal{H}^n} \right|_{pv} = \sum_i \frac{1}{\Gamma(n)} \int_0^\infty ds \ s^{n-1} (a_i e^{-k_i^2 s} - b_i e^{-\Lambda_i^2 s}) e^{-\mathcal{H}s} = \sum_i \left[ \frac{a_i}{(\mathcal{H} + k_i^2)^n} - \frac{b_i}{(\mathcal{H} + \Lambda_i^2)^n} \right], \quad (B.11)$$

and

$$\text{Tr}_{pv} \ln \left( \frac{\mathcal{H}}{\mathcal{H}_0} \right) = -\sum_i \int_0^\infty ds \ \frac{a_i e^{-k_i^2 s} - b_i e^{-\Lambda_i^2 s}}{s} \text{Tr} \left( e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s} \right)$$

$$= \text{Tr} \sum_i \ln \left[ \left( \frac{\mathcal{H} + k_i^2}{\mathcal{H}_0 + k_i^2} \right)^{a_i} \times \left( \frac{\mathcal{H}_0 + \Lambda_i^2}{\mathcal{H} + \Lambda_i^2} \right)^{b_i} \right]. \quad (B.12)$$

The similarity between the operator cutoff and the Pauli-Villars is now apparent. By choosing $a_i = b_i = i = 1$, we notice that the two smearing functions differ from one another only by a higher order correction.

In computing $\delta \Gamma^{(2)}_{pv}$ using the Pauli-Villars regulator, it is necessary to introduce two ghost terms since the integral in (B.8) is quadratically divergent. Thus, we write [29]

$$\left. \frac{1}{p^2 + \mu^2} \right|_{pv} = \frac{1}{p^2 + \mu^2} - \frac{b_1}{p^2 + \mu^2 + \Lambda_1^2} - \frac{b_2}{p^2 + \mu^2 + \Lambda_2^2}, \quad (B.13)$$

$$\frac{f(p^2, \mu^2, \Lambda_1^2, \Lambda_2^2)}{(p^2 + \mu^2)(p^2 + \mu^2 + \Lambda_1^2)(p^2 + \mu^2 + \Lambda_2^2)},$$

where

$$f(p^2, \mu^2, \Lambda_1^2, \Lambda_2^2) = (1 - b_1 - b_2) p^4 + \left[ 2(1 - b_1 - b_2) \mu^2 + (1 - b_1) \Lambda_2^2 + (1 - b_2) \Lambda_1^2 \right] p^2$$

$$+ \mu^2 \left[ (1 - b_1) \Lambda_2^2 + (1 - b_2) \Lambda_1^2 \right] + \Lambda_1^2 \Lambda_2^2, \quad (B.14)$$

and demand that

$$\left. \frac{1}{p^2 + \mu^2} \right|_{pv} \to \frac{1}{p^6}, \quad \text{as} \quad p^2 \to \infty. \quad (B.15)$$

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The condition is satisfied if
\[ b_1 + b_2 - 1 = 0, \quad (1 - b_2)\Lambda_1^2 + (1 - b_1)\Lambda_2^2 = 0, \tag{B.16} \]
which implies
\[ b_1 = \frac{\Lambda_2^2}{\Lambda_2^2 - \Lambda_1^2}, \quad b_2 = -\frac{\Lambda_1^2}{\Lambda_2^2 - \Lambda_1^2}. \tag{B.17} \]
The correction to the two-point function can now be obtained as
\[
\delta \Gamma^{(2)}_{pv} = \int_p \frac{1}{p^2 + \mu^2} \bigg|_{pv} = \int_p \frac{\Lambda_1^2 \Lambda_2^2}{(p^2 + \mu^2)(p^2 + \mu^2 + \Lambda_1^2)(p^2 + \mu^2 + \Lambda_2^2)} \\
\quad \rightarrow \int_p \frac{1}{p^2 + \mu^2} \left( \frac{\Lambda^2}{p^2 + \mu^2 + \Lambda^2} \right)^2 = \frac{1}{16\pi^2} \left[ \Lambda^2 - \mu^2 \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) \right], \quad (\Lambda_1, \Lambda_2 \to \Lambda), \tag{B.18} \]
which is in agreement with (B.8). As for \( \delta \Gamma^{(4)}_{pv} \), since it is logarithmically divergent, only one ghost term is sufficient and we obtain
\[
\delta \Gamma^{(4)}_{pv} = \int_p \frac{1}{(p^2 + \mu^2)^2} \bigg|_{pv} = \int_p \left[ \frac{1}{(p^2 + \mu^2)^2} - \frac{1}{(p^2 + \mu^2 + \Lambda^2)^2} \right] = \frac{1}{16\pi^2} \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right). \tag{B.19} \]

(3) proper-time cutoff:
Since divergences generated from taking the trace in space-time are transferred into singularities in the proper-time integration, one may regularize the theory by a direct truncation of the integration regime(s) to avoid singularity. For example, we may simply take the smearing function to be a sharp proper-time cutoff:
\[ \rho^pc_k(s, \Lambda) = \Theta(s - \frac{1}{\Lambda^2})\Theta(s), \tag{B.20} \]
In this manner, we have
\[
\frac{1}{\mathcal{H}^n} \bigg|_{pc} = \frac{1}{\Gamma(n)} \int_0^\infty ds \ s^{n-1} \Theta(s - \frac{1}{\Lambda^2})\Theta(s) e^{-\mathcal{H}s} = \frac{1}{\Gamma(n)} \int_{1/\Lambda^2}^{1/k^2} ds \ s^{n-1} e^{-\mathcal{H}s} \\
= \frac{1}{\mathcal{H}^n} \cdot \frac{1}{\Gamma(n)} \left( \Gamma[n, 0, \mathcal{H}/k^2] - \Gamma[n, 0, \mathcal{H}/\Lambda^2] \right), \tag{B.21} \]
and
\[
\text{Tr}_{pc} \ln \left( \frac{\mathcal{H}}{\mathcal{H}_0} \right) = -\int_{1/\Lambda^2}^{1/k^2} \frac{ds}{s} \text{Tr} \left( e^{-\mathcal{H}s} - e^{-\mathcal{H}_0s} \right) \\
= \text{Tr} \left\{ -\text{Ei}(\mathcal{H}/k^2) + \text{Ei}(\mathcal{H}_0/k^2) + \text{Ei}(\mathcal{H}/\Lambda^2) - \text{Ei}(\mathcal{H}_0/\Lambda^2) \right\} \tag{B.22} \]
\[ = \text{Tr} \ln \left( \frac{\mathcal{H}}{\mathcal{H}_0} \right) + \cdots, \]
where we have employed the asymptotic forms of the exponential-integral function
\[
\text{Ei}(s_0) = -\int_{s_0}^{\infty} \frac{ds}{s} e^{-s} = \begin{cases} 
\ln s_0 + \gamma + \sum_{n=1}^{\infty} \frac{(-s_0)^n}{n \ln n} & (s_0 \to 0^+), \\
-\frac{e^{-s_0}}{s_0} & (s_0 \to \infty). 
\end{cases}
\] (B.23)

Correspondingly, we have
\[
\delta \Gamma_{\text{pc}}^{(2)} = \int_p \frac{e^{-(p^2+\mu^2)/\Lambda^2}}{p^2 + \mu^2} = \frac{1}{16\pi^2} \left[ \Lambda^2 e^{-\mu^2/\Lambda^2} + \mu^2 \text{Ei}\left(\frac{\mu^2}{\Lambda^2}\right) \right] = \frac{1}{16\pi^2} \left[ \Lambda^2 - \mu^2 \ln \left(\frac{\Lambda^2}{\mu^2}\right) \right] + \cdots,
\] (B.24)

and
\[
\delta \Gamma_{\text{pc}}^{(4)} = \int_p \frac{e^{-(p^2+\mu^2)/\Lambda^2}}{(p^2 + \mu^2)^2} \left[ 1 + \frac{p^2 + \mu^2}{\Lambda^2} \right] = \frac{1}{16\pi^2} \left[ -(1 - \frac{2\mu^2}{\Lambda^2} - \frac{2\mu^4}{\Lambda^4}) \text{Ei}\left(\frac{\mu^2}{\Lambda^2}\right) + 2(1 + \frac{\mu^2}{\Lambda^2}) e^{-\mu^2/\Lambda^2} \right] = \frac{1}{16\pi^2} \ln \left(\frac{\Lambda^2}{\mu^2}\right) + \cdots.
\] (B.25)

(4) point-splitting:
One may also choose the smearing function to be of the form
\[
\rho_{\text{ps}}^s(p, \Lambda) = e^{-1/\Lambda^2 s} - e^{-1/k^2 s},
\] (B.26)

which corresponds to the so-called point-splitting regularization scheme. This smearing function yields
\[
\frac{1}{H^n} \bigg|_{\text{ps}} = \frac{1}{\Gamma(n)} \int_0^{\infty} ds \ s^{n-1} \left( e^{-1/\Lambda^2 s} - e^{-1/k^2 s} \right) e^{-Hs} = \frac{1}{H^n} \cdot \frac{2}{\Gamma(n)} \left[ \left( \frac{H}{\Lambda^2} \right)^{n/2} K_n\left( \frac{2H^{1/2}}{\Lambda} \right) - \left( \frac{H}{k^2} \right)^{n/2} K_n\left( \frac{2H^{1/2}}{k} \right) \right] = \frac{1}{H^n} + \cdots,
\] (B.27)

and
\[
\text{Tr}_{\text{ps}} \ln \left( \frac{H}{H_0} \right) = -\int_0^{\infty} \frac{ds}{s} \left( e^{-1/\Lambda^2 s} - e^{-1/k^2 s} \right) \text{Tr} \left( e^{-Hs} - e^{-H_0 s} \right) = 2\text{Tr} \left[ K_0\left( \frac{2H^{1/2}}{\Lambda} \right) - K_0\left( \frac{2H^{1/2}}{k} \right) + K_0\left( \frac{2H_0^{1/2}}{k} \right) \right]
\] (B.28)

where we have expanded the modified Bessel function asymptotically as [30]:
\[
K_n(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left[ 1 + \frac{(4n^2 - 1^2)}{1! 8x} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2! (8x)^2} + \cdots \right] \quad (x \to \infty),
\] (B.29)
and

$$K_n(x) \sim \begin{cases} 2^{n-1}(n-1)!x^{-n} + \cdots & n \geq 1, \\ -\ln x^2 - \gamma & n = 0. \end{cases} \quad (x \to 0^+). \quad (B.30)$$

The two- and four-point functions in this scheme are as follows:

$$\delta \Gamma^{(2)}_{ps} = \int_p \frac{1}{p^2 + \mu^2} \mid_{ps} = \int_0^\infty dy \ e^{-y-\mu^2/\Lambda^2 y} \int_p \frac{e^{-p^2/\Lambda^2 y}}{p^2 + \mu^2}$$

$$= \frac{1}{16\pi^2} \int_0^\infty dy \ e^{-y-\mu^2/\Lambda^2 y} \left[ \Lambda^2 y + \mu^2 e^{\mu^2/\Lambda^2 y} \text{Ei}(-\mu^2/\Lambda^2 y) \right]$$

$$\approx \frac{\mu^2}{16\pi^2} \left\{ 2K_2(2\mu/\Lambda) + \int_0^\infty dy e^{-y} \ln\left( \frac{\mu^2/\Lambda^2 y}{\mu^2} \right) \right\} = \frac{1}{16\pi^2} \left[ \Lambda^2 - \mu^2 \ln\left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) \right] + \cdots, \quad (B.31)$$

and

$$\delta \Gamma^{(4)}_{ps} = \int_p \frac{1}{(p^2 + \mu^2)^2} \mid_{ps} = \int_0^\infty dy \ e^{-y-\mu^2/\Lambda^2 y} \int_p \frac{e^{-p^2/\Lambda^2 y}}{(p^2 + \mu^2)^2}$$

$$= -\frac{1}{16\pi^2} \int_0^\infty dy \ e^{-y-\mu^2/\Lambda^2 y} \left\{ 1 + (1 + \frac{\mu^2}{\Lambda^2 y}) e^{\mu^2/\Lambda^2 y} \text{Ei}(-\mu^2/\Lambda^2 y) \right\}$$

$$= \frac{1}{16\pi^2} \ln\left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) + \cdots. \quad (B.32)$$

We remark that the four smearing functions presented so far in a certain sense can all be viewed as a special case of the generalized momentum regularization in which the regularized integral for an arbitrary momentum-dependent function $f(p)$ is written as

$$\int_{a(\Lambda)}^{a(\Lambda)} dp f(p), \quad (B.33)$$

where $a(\Lambda)$ and $b(k)$ are arbitrary functions of the cutoffs $\Lambda$ and $k$, respectively. This is readily seen by noting that the prescriptions presented previously can be related to the generalized momentum regularization via

$$\rho^{\text{reg}}_k(s, \Lambda) \int_p e^{-p^2 s} = \frac{1}{(4\pi s)^{d/2}} \rho^{\text{reg}}_k(s, \Lambda) = \int_p e^{-p^2 s} = S_d \int_{b(k)}^{a(\Lambda)} dp p^{d-1} e^{-p^2 s}, \quad (B.34)$$

or

$$\rho^{\text{reg}}_k(s, \Lambda) = \frac{2s^{d/2}}{\Gamma(d/2)} \int_{b(k)}^{a(\Lambda)} dp p^{d-1} e^{-p^2 s}. \quad (B.35)$$

For example, in the $d = 4$ Pauli-Villars case, we have

$$\rho^{\text{pv}}_k(s, \Lambda) = e^{-k^2 s} - e^{-\Lambda^2 s} = 2s^2 \int_{b(k)}^{a(\Lambda)} dp p^3 e^{-p^2 s}$$

$$= (1 + b^2(k)s) e^{-b^2(k)s} - (1 + a^2(\Lambda)s) e^{-a^2(\Lambda)s}, \quad (B.36)$$

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where \( a(\Lambda) \) obeys the transcendental equation
\[
e^{-\Lambda^2 s} = (1 + a^2(\Lambda)s)e^{-a^2(\Lambda)s}.
\] (B.37)

The IR cutoff function \( b(k) \) can be obtained in a similar manner.

(5.a) dimensional regularization:

One can also show that dimensional regularization falls into the generalized class of proper-time by taking the smearing function to be
\[
\rho_\epsilon(s) = \left(\frac{4\pi}{\epsilon}\right)^{\epsilon/2},
\] (B.38)

which suggests
\[
\frac{1}{\mathcal{H}^n}\bigg|_\epsilon = \frac{(4\pi)^{\epsilon/2}}{\Gamma(n)} \int_0^\infty ds \ s^{\epsilon/2+n-1}e^{-\mathcal{H}s} = \frac{\Gamma(n+\epsilon/2)}{\Gamma(n)} (4\pi)^{\epsilon/2} \mathcal{H}^{-(n+\epsilon/2)},
\] (B.39)

and
\[
\text{Tr}_\epsilon \ln\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right) = -(4\pi)^{\epsilon/2} \int_0^\infty ds s^{-1+\epsilon/2} \text{Tr}\left(e^{-\mathcal{H}s} - e^{-\mathcal{H}_0s}\right) = -(4\pi)^{\epsilon/2} \Gamma(\epsilon/2) \text{Tr}\left(\mathcal{H}^{-\epsilon/2} - \mathcal{H}_0^{-\epsilon/2}\right).
\] (B.40)

We remark here that this proper-time version of dimensional regularization differs from the conventional one in the sense that calculations are done in the original dimension. The manner in which the theory is regularized is to increase the power of the propagator \( \mathcal{H}^{-1} \) by \( \epsilon/2 \), thereby decreasing the degree of divergence. The advantage of using this method is that no difficulty is encountered when dealing with objects such as \( \gamma_5 \) which are defined on a specific dimension. In fact, this corresponds to adding extra degrees of freedom that are not directly coupled to the background fields [7].

The corrections to the two- and four-point functions are, respectively,
\[
\delta\Gamma^{(2)}_\epsilon = \int_p \frac{1}{p^2 + \mu^2} \bigg|_\epsilon = (4\pi)^{\epsilon/2} \int_0^\infty ds \ s^{\epsilon/2} e^{-\mu^2 s} \int_p e^{-p^2 s} = \frac{1}{(4\pi)^{2-\epsilon/2}} (\mu^2)^{1-\epsilon/2} \Gamma(-1 + \epsilon/2) = \frac{1}{16\pi^2} \left[\frac{2\mu^2}{\epsilon} - \mu^2 \ln\left(\frac{4\pi}{\mu^2}\right)\right] + \cdots,
\] (B.41)

and
\[
\delta\Gamma^{(4)}_\epsilon = \int_p \frac{1}{(p^2 + \mu^2)^2} \bigg|_\epsilon = \frac{1}{(4\pi)^{2-\epsilon/2}} (\mu^2)^{-\epsilon/2} \Gamma(\epsilon/2) = \frac{1}{16\pi^2} \left[\frac{2}{\epsilon} + \ln\left(\frac{4\pi}{\mu^2}\right)\right] + \cdots,
\] (B.42)

where we have used [31]
\[
\Gamma(-n+\epsilon/2) = \frac{(-1)^n}{n!} \left[\frac{2}{\epsilon} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\right) + O(\epsilon)\right].
\] (B.43)
The divergences now appear as poles for $\epsilon = 0$ and 2 since
\[
\Gamma(-1 + \epsilon/2) = \frac{1}{(-1 + \epsilon/2)(\epsilon/2)} \Gamma(1 + \epsilon/2).
\] (B.44)

These poles can be mapped onto the divergent expressions obtained using the momentum cutoffs.

(5.b) dimensional cutoff regularization:
The most direct way to establish the connection between dimensional regularization and the momentum cutoff regulator is by means of the “dimensional cutoff regularization” defined by
\[
\rho_\epsilon(s, \Lambda) = \rho_\epsilon(s)\rho_k^{(d)}(s, \Lambda) = \frac{(4\pi)^{\epsilon/2}}{S_d \Gamma(d/2)} s^{(d+\epsilon)/2} \int_z e^{-z^2 s},
\] (B.45)

which is simply the product of the two smearing functions taken from each scheme in $d$ dimension. The modified propagator and kernel in this $\epsilon'$ scheme read
\[
\frac{1}{H'^n} |_{\epsilon'} = \frac{1}{\Gamma(n)} \int_0^\infty ds \ s^{n-1} \rho_\epsilon^{(d)}(s, \Lambda) e^{-H s}
\]
\[
= \frac{1}{H'^n} \cdot \frac{2(4\pi)^{\epsilon/2} \Gamma(n + d/2)}{d \Gamma(n) \Gamma(d/2)} \left\{ \left( \frac{\Lambda^2}{H} \right)^{d/2} F \left( \frac{d}{2}, \frac{d + \epsilon}{2}, n + 1 + \frac{d}{2}; -\frac{\Lambda^2}{H} \right) - \left( \frac{k^2}{H} \right)^{d/2} F \left( \frac{d}{2}, \frac{d + \epsilon}{2}, n + 1 + \frac{d}{2}; -\frac{k^2}{H} \right) \right\},
\] (B.46)

and
\[
\text{Tr}_{\epsilon'} \ln \left( \frac{H}{H_0} \right) = - \int_0^\infty ds \ s \rho_\epsilon^{(d)}(s, \Lambda) \text{Tr} \left( e^{-H s} - e^{-H_0 s} \right)
\]
\[
= - \frac{2(4\pi)^{\epsilon/2}}{d} \text{Tr} \left\{ \left( \frac{\Lambda^2}{H} \right)^{d/2} F \left( \frac{d}{2}, \frac{d + \epsilon}{2}, 1 + \frac{d}{2}; -\frac{\Lambda^2}{H} \right) - \left( \frac{\Lambda^2}{H_0} \right)^{d/2} F \left( \frac{d}{2}, \frac{d + \epsilon}{2}, 1 + \frac{d}{2}; -\frac{\Lambda^2}{H_0} \right) - \left( \frac{k^2}{H} \right)^{d/2} F \left( \frac{d}{2}, \frac{d + \epsilon}{2}, 1 + \frac{d}{2}; -\frac{k^2}{H} \right) + \left( \frac{k^2}{H_0} \right)^{d/2} F \left( \frac{d}{2}, \frac{d + \epsilon}{2}, 1 + \frac{d}{2}; -\frac{k^2}{H_0} \right) \right\}.
\] (B.47)

Eq. (B.41) and (B.42) are now modified as:
\[
\delta \Gamma_{\epsilon'}^{(2)} = \int_p \frac{1}{p^2 + \mu^2} |_{\epsilon'} = (4\pi)^{\epsilon/2} \int_0^\infty ds \ s^{\epsilon/2} \left[ 1 - (1 + \Lambda^2 s) e^{-\Lambda^2 s} \right] e^{-\mu^2 s} \int_p e^{-p^2 s}
\]
\[
= \frac{\Gamma(-1 + \epsilon/2)}{(4\pi)^{2-\epsilon/2}} \left\{ (\mu^2)^{1-\epsilon/2} - \frac{\epsilon \Lambda^2 / 2 + \mu^2}{(\Lambda^2 + \mu^2)^{\epsilon/2}} \right\},
\] (B.48)

and
\[
\delta \Gamma_{\epsilon'}^{(4)} = \int_p \frac{1}{(p^2 + \mu^2)^2} |_{\epsilon'} = \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}} \left\{ (\mu^2)^{-\epsilon/2} - \frac{(1 + \epsilon/2) \Lambda^2 + \mu^2}{(\Lambda^2 + \mu^2)^{1+\epsilon/2}} \right\}.
\] (B.49)
To recover the cutoff results, we take the limit $\epsilon \to 0$ first and obtain

$$
\delta \Gamma^{(2)}_\epsilon = -\frac{\mu^2}{8\pi^2} \left\{ \left[ \frac{1}{\epsilon} - \frac{1}{2} \ln 4\pi \mu^2 \right] - \left[ \frac{1}{\epsilon} + \frac{\Lambda^2}{2\mu^2} - \frac{1}{2} \ln 4\pi (\Lambda^2 + \mu^2) \right] \right\} + O(\epsilon)
$$

(B.50)

and

$$
\delta \Gamma^{(4)}_\epsilon' = \frac{1}{16\pi^2} \left[ \Lambda^2 - \mu^2 \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) \right] + O(\epsilon),
$$

(B.51)

The above equations explicitly demonstrate how with this hybrid dimensional cutoff regulator, the $1/\epsilon$ singular term coming from dimensional regularization and cutoff regularization cancels each other and gives back the $\Lambda$ dependence of the cutoff theory shown in (B.8) and (B.9). On the other hand, taking the limit $\Lambda \to \infty$ before $\epsilon \to 0$ allows us to recover the usual dimensional regularization scheme. In other words, depending on the order in which the limits $\Lambda \to \infty$ and $\epsilon \to 0$ are taken, different regularization schemes are actually achieved.

(6.a) $\zeta$-function regularization:

$\zeta$-function regularization has been discussed extensively by Elizalde et al [32] and in the context of operator regularization by McKeon et al [18].

In the $\zeta$-function regularization, the logarithm of an operator is represented by

$$
\ln \mathcal{H} = -\lim_{t \to 0} \frac{d}{dt} \mathcal{H}^{-t}.
$$

(B.52)

Noting that

$$
\frac{1}{\mathcal{H}^t} = \frac{1}{\Gamma(t)} \int_0^\infty ds \ s^{t-1} e^{-s \mathcal{H}},
$$

(B.53)

one may define the $\zeta$-function as

$$
\zeta(t) = \frac{1}{\Gamma(t)} \int_0^\infty ds \ s^{t-1} \text{Tr} e^{-s \mathcal{H}},
$$

(B.54)

which implies

$$
\det \mathcal{H} = \exp \left[ \text{Tr} \ln \mathcal{H} \right] = \exp \left\{ \text{Tr} \lim_{t \to 0} \left[ -\frac{d}{dt} \mathcal{H}^{-t} \right] \right\} = \exp \left[ -\lim_{t \to 0} \frac{d}{dt} \zeta(t) \right].
$$

(B.55)
The equivalent of $\zeta$-function regularization in the proper-time formulation can be obtained by choosing the following smearing function:

$$\rho_\zeta^t(s) = \lim_{t \to 0} \frac{d}{dt} \frac{1}{\Gamma(t)} s^t,$$

which gives

$$\frac{1}{\mathcal{H}^n} \left|_\zeta \right. = \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{\Gamma(t)\Gamma(n)} \int_0^{\infty} ds \, s^{n+t-1} e^{-\mathcal{H}s} \right\} = \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{\Gamma(n+t)}{\Gamma(n)\Gamma(t)} \mathcal{H}^{-(n+t)} \right\}$$

$$= \lim_{t \to 0} \left\{ \frac{\Gamma(n+t)}{\Gamma(n)\Gamma(t)} \mathcal{H}^{-(n+t)} \left[ \psi(n+t) - \psi(t) - \ln \mathcal{H} \right] \right\} \quad \text{(B.57)}$$

and

$$\text{Tr}_\zeta \ln \left( \frac{\mathcal{H}}{\mathcal{H}_0} \right) = - \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{\Gamma(t)} \int_0^{\infty} ds \, s^{t-1} \text{Tr} \left( e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s} \right) \right\}$$

$$= \text{Tr} \left[ - \lim_{t \to 0} \frac{d}{dt} \left( \mathcal{H}^{-(n+t)} - \mathcal{H}_0^{-(n+t)} \right) \right] = \text{Tr} \ln \left( \frac{\mathcal{H}}{\mathcal{H}_0} \right),$$

where we have used [33]

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad \text{(B.59)}$$

and

$$\psi(n+t) = \psi(t) + \sum_{\ell=0}^{n-1} \frac{1}{t+\ell}.$$

(6.b) $\zeta$-function cutoff regularization:

In an analogous manner to the dimensional cutoff regularization scheme, one can introduce cutoff scales to the $\zeta$-function cutoff regularization as well. This again can be done with the product of two smearing functions:

$$\rho_{\zeta^{(d)}}(s, \Lambda) = \rho_\zeta^t(s) \rho_k^{(d)}(s, \Lambda) = \lim_{t \to 0} \frac{d}{dt} \frac{1}{\Gamma(t)} s^t \rho_k^{(d)}(s, \Lambda).$$

To show that the same $\rho_k^{(d)}(s, \Lambda)$ obtained in (1.4) can be used to reproduce the momentum cutoff structure, we consider again the simple scalar theory. In this $\zeta$-function cutoff formalism, the one-loop correction to $U_k$ is written as

$$U_k^{(1)}(\Phi) = \frac{1}{2} \int_{\mathcal{P}} \ln \left( \frac{p^2 + V''(\Phi)}{p^2 + V''(0)} \right)$$

$$\longrightarrow - \frac{1}{2(4\pi)^{d/2}} \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{\Gamma(t)} \int_0^{\infty} ds \, s^{t-1-d/2} \rho_k^{(d)}(s, \Lambda) \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right) \right\}. \quad \text{(B.62)}$$
By demanding that (B.62) yields the same differential flow equation for $U_k$ as that obtained from momentum cutoff regularization, we are lead to

$$
\lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{\Gamma(t)} \int_0^\infty ds \ s^{t-1-d/2} \left( k \frac{\partial \rho_k^{(d)}(s, \Lambda)}{\partial k} \right) \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right) \right\}
$$

$$
= -\frac{2k^d}{\Gamma(d/2)} \int_0^\infty ds e^{-k^2s} \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right).
$$

One can then verify by direct substitution that the above expression is indeed satisfied by the smearing function given in (1.4). Thus the same $\rho_k^{(d)}(s, \Lambda)$ can be used to bring the momentum cutoffs into $\zeta$-function regularization although this may seem redundant since no divergence is encountered in this prescription. Nevertheless, by retaining the cutoff scales, the flow pattern of the theory may be explored in a lucid manner.

As an explicit demonstration of $\zeta$ function cutoff regularization, we compute the one-loop contribution of the blocked potential in $d = 4$ and obtain

$$
U_k^{(1)}(\Phi) = -\frac{1}{32\pi^2} \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{\Gamma(t)} \int_0^\infty ds \ s^{t-3} \left[ (1 + k^2s)e^{-k^2s} - (1 + \Lambda^2s)e^{-\Lambda^2s} \right] \right\}
$$

$$
\times \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right)
$$

$$
= -\frac{1}{32\pi^2} \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{(t-1)(t-2)} \left[ (k^2 + V''(\Phi))^2 - (1 + \Lambda^2s)e^{-\Lambda^2s} \right] \right\}
$$

$$
= \frac{1}{64\pi^2} \left\{ V''(\Phi)(\Lambda^2 - k^2) + \Lambda^4 \ln \left( 1 + \frac{V''(\Phi)}{\Lambda^2} \right) - k^4 \ln \left( 1 + \frac{V''(\Phi)}{k^2} \right) \right\}
$$

$$
\times \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right)
$$

$$
= -\frac{1}{32\pi^2} \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{k^2 + V''(\Phi)^2}{(t-1)(t-2)} + \frac{k^2(2 + V''(\Phi))^{1-t}}{t-1} + \cdots \right\}
$$

$$
= -\frac{1}{64\pi^2} \left\{ k^2V''(\Phi) + \frac{3}{2} V''(\Phi)^2 + (k^4 - V''(\Phi)^2) \ln \left( \frac{k^2 + V''(\Phi)}{k^2} \right) \right\} + \cdots,
$$

(B.64)
which is precisely the finite one-loop contribution of the blocked potential. It is interesting to note that taking the limit $\Lambda \to \infty$ before and after the $s$ integration actually yields different results. In fact, the two limits correspond to two different regularization procedures. The connection between the $\zeta$-function cutoff formalism and the momentum cutoff regularization actually be established by the following integral transformation:

\[
U_k^{(1)}(\Phi) = -\frac{1}{32\pi^2} \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{\Gamma(t)} \int_0^\infty ds \ s^{t-3} \rho_k^{(d)}(s, \Lambda) \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right) \right\}
\]

\[
\rightarrow -\frac{1}{2} \int_z^\prime \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{\Gamma(t)} \int_0^\infty ds \ s^{t-1} e^{-z^2s} \left( e^{-V''(\Phi)s} - e^{-V''(0)s} \right) \right\}
\]

\[
= -\frac{1}{2} \int_z^\prime \lim_{t \to 0} \frac{d}{dt} \left\{ \frac{1}{(z^2 + V''(\Phi))^t} - \frac{1}{(z^2 + V''(0))^t} \right\} = \frac{1}{2} \int_z^\prime \ln \left( \frac{z^2 + V''(\Phi)}{z^2 + V''(0)} \right).
\]

In other words, equality with cutoff regularization is obtained by keeping the $z$ integration till the end and interpreting $z$ as the momentum scale $p$.

REFERENCES