Pursuing parameters for critical density dark matter models

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ABSTRACT
We present an extensive comparison of models of structure formation with observations, based on linear and quasi-linear theory. We assume a critical matter density, and study both cold dark matter models and cold plus hot dark matter models. We explore a wide range of parameters, by varying the fraction of hot dark matter \( \Omega_\nu \), the Hubble parameter \( h \), the spectral index of density perturbations \( n \), and allowing for the possibility of gravitational waves from inflation influencing large angle microwave background anisotropies. New calculations are made of the transfer functions describing the linear power spectrum, with special emphasis on improving the accuracy on short scales where there are strong constraints. For assessing early object formation, the transfer functions are explicitly evaluated at the appropriate redshift. The observations considered are microwave background anisotropies, peculiar velocity flows, the galaxy correlation function, and the abundances of galaxy clusters, quasars and damped Lyman alpha systems. Each observation is interpreted in terms of the power spectrum filtered by a top-hat window function. We find that there remains a viable region of parameter space for critical density models when all the dark matter is cold, though \( h \) must be less than 0.5 before any fit is found and \( n \) significantly below unity is preferred. Once a hot dark matter component is invoked, a wide parameter space is acceptable, including \( n \approx 1 \). The allowed region is characterized by \( \Omega_\nu 0.30 \) and \( 0.6n1.2 \), at 95 per cent confidence on at least one piece of data. There is no useful lower bound on \( h \), and for curious combinations of the other parameters it is possible to fit the data with \( h \) as high as 0.70.

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1 INTRODUCTION
The concept of cosmological inflation has motivated an enormous amount of research into the formation of structure in the universe. It has long been known that the simplest particle physics models for inflation typically predict that the universe is spatially flat and that the gravitational seeds for structure are adiabatic, Gaussian distributed density fluctuations with a nearly Harrison-Zel'dovich spectrum (power spectrum index of \( n \sim 1 \)). In order to proceed with detailed calculations from this starting point, one needs to pick a value for the Hubble constant. The standard choice has been to assume \( h = 0.5 \), where the present Hubble constant is parametrized as \( H_0 \equiv 100 \ h \ \text{km s}^{-1} \ \text{Mpc}^{-1} \). Finally, the character of the dark matter must be decided. Big bang nucleosynthesis implies that the bulk of dark matter cannot be baryonic.

The choice for this dark matter which involves the fewest assumptions is relic neutrinos, as we know they exist and expect that they fill the universe. Neutrinos are referred to as hot dark matter (HDM) because they remain relativistic until the horizon size is comparable to large scale struc-
tures. Unfortunately HDM does not give a satisfactory picture of structure formation, because galaxies form too late and phase space of the halos of small galaxies is not large enough to accommodate the required number of neutrinos. A more popular alternative for the dark matter was to assume the existence of a cold relic particle, known as cold dark matter (CDM). Typical candidate particles for CDM are axions and lightest supersymmetric particles.

Initial studies of galaxy formation found that the galaxies in CDM models were too clustered when used with inflationary type fluctuations (see e.g., Davis et al. 1985). This problem was surmounted by introducing the concept of biasing, in which the fluctuations in the galactic distribution are much larger or 'biased' compared to the underlying density field. This model (with the ingredients $\Omega_{\text{CDM}} = 0.95$, $f_{\text{b}} = 0.05$, $h = 0.5$ and strongly biased scale-invariant adiabatic Gaussian fluctuations), proved to be quite successful at explaining many properties of galaxies and clusters, mostly on smaller scales. However, because the amplitude of density fluctuations needed to be reduced to properly account for galaxies, this also meant that there would be insufficient power for making much larger scale structures.

Standard CDM's problems with large scale structure had already been noticed in the 1980s. In particular, the observed spatial correlation of galactic clusters was much stronger than predicted by the model. In the meantime, it was noted (Holman, Lazarides & Shafi 1983; Shafi 1983; Mohapatra & Senjanovic 1983) that certain realistic particle physics grand unification models predict the simultaneous presence of cold and hot dark matter. Such a mixture was recognized (Shafi & Stecker 1984) to have the desirable properties of reduced small scale power to make galaxies properly, while still having significant amounts of power on larger scales. [Models which mixed hot and warm dark matter were also studied (Bonometto & Vakilmini 1984; Fang, Li & Xiang 1984; Vakilmini & Bonometto 1985).] However, it was pointed out (Acchilli, Occhionero & Scaramella 1985; Bardeen, Bond & Efstathiou 1987; van Dalen & Schaefer 1992) that if the mixture contained more HDM than CDM, the model would have difficulty forming galaxies early enough. Thus this model became known as the cold plus hot dark matter model (CHDM), to indicate that there should be more CDM than HDM.

As evidence for more large scale power than expected in CDM models continued to accumulate during the 1980s the outlook for the CHDM model brightened. The number densities of cosmic structures and large cluster correlation lengths in CHDM models were shown to be in better agreement with observations (Occhionero & Scaramella 1989; van Dalen & Schaefer 1992; Holtzman & Primack 1993). In addition to showing that CHDM predictions of observed large scale bulk flows agreed better with observations, two papers predicted the amplitude of the temperature fluctuations expected on large angular scales in the cosmic microwave background radiation (Schaefer, Shafi & Stecker 1989; Holtzman 1989) well before the launch of the Cosmic Background Explorer (COBE) satellite. The verification by COBE of these anisotropy predictions brought about a wave of intense interest in the CHDM model.

The recent rapid increase in the quality of the observations of large scale structure and microwave background temperature fluctuations have led to a new precision in investigations of theoretical models. Until recently, it was standard practice to derive conclusions about models of structure formation within a fairly rigid subset of assumptions about the relevant parameters. Two of these parameters are the spectral index $n$, and a parameter $\tau$ specifying the relative contribution of gravitational waves to the cosmic microwave background anisotropy (Liddle & Lyth 1992), and they have usually been fixed at the canonical values $n = 1$ and $\tau = 0$. However, within a given model of inflation their values are determined or at least constrained, and while many models do accurately give these canonical values there are other models which do not. It is therefore realistic to allow $n$ and $\tau$ to vary when confronting a model with the data, and we shall adopt that viewpoint in this paper. An observational determination of these parameters in the future will be a powerful constraint on models of inflation, and hence on the nature of the fundamental interactions at very high energy scales. Another parameter which is often fixed is the Hubble constant, usually to the value 0.5. Its variation can be important; the amount of small scale power is extremely sensitive to the value of the Hubble parameter as the redshift of matter domination scales quadratically with $h$. The choice of baryon density also can have a modest impact, as we discuss shortly. Our intent here is to realistically study the CHDM model, by varying the parameters of inflation and $h$ to see which are the most favorable values by testing it against data.

The above set of parameters is by no means complete, even within the limited context of inflation. For instance, inflation says nothing about whether or not there might be a relic cosmological constant $\Lambda$ contributing to the present-day spatial flatness, although it may be difficult to understand the magnitude of the residual $\Lambda$ within the philosophical context of inflation. In keeping with the original spirit of inflation, we set $\Lambda = 0$ here. Recently it has also been emphasised that one can obtain genuinely open universes from inflation, albeit at present only with considerable tuning of parameters. Structure formation is apparently viable in these models (Ratra & Peebles 1994; Gorski et al. 1995; Liddle et al. 1995), but we shall not consider them here. Our assumption, therefore, is that the universe possesses a critical density of matter.

Further impetus has been delivered to the CHDM model by the observations of neutrino oscillations from the sun, atmospheric cosmic ray cascades, and possibly by the LSND experiment (Caldwell 1994; Athanassopoulos et al. 1995). These observations suggest that some of the neutrino masses are non-zero, and that one or more neutrino species may provide a significant HDM density. [It has even been suggested that a multiple (2 or 3) neutrino CHDM scenario may provide an even better fit to observational data (Primack et al. 1995; Pogosyan & Starobinsky 1995b; Babu, Schaefer & Shafi 1995).] While promising, this remains speculative physics. Here we will only consider the situation of a single few eV mass neutrino. Indeed such a scenario can be made reasonably compatible with all of the oscillation experimental results (see Babu et al. 1995 and references therein). We are also, of course, assuming the standard cosmology for the neutrinos (for some alternative proposals, see [Kaiser, Malaney & Starkman 1993; Bonometto, Caldara & Masiero 1994; Pierpaoli & Bonometto 1994]).

While detailed $N$-body simulations, necessitating the
selection of particular parameter values, are required to provide a detailed comparison of models against observations, it is vital to carry out an investigation of the wider parameter space using the less intensive strategy of linear perturbation theory in order to find those regions of parameter space best suited to matching the data. Linear and quasi-linear theory offer powerful tools for investigating the shape of the density perturbation power spectrum across a very wide range of scales, as there are now copious data addressing scales large enough to still be linear today. Further, even the shorter scales which are non-linear today can be investigated by examining phenomena such as the abundance of quasars and damped Lyman alpha systems at moderate redshift, corresponding to times when those scales were still in the linear regime.

A choice is required for the baryon density, which is taken to agree with standard nucleosynthesis. The theory of nucleosynthesis has seen some developments recently, and the range advocated by Walker et al. (1991) is now seen as too stringent, especially their upper limit. We choose to take a value compatible with more recent analyses by Copi, Schramm & Turner (1995a,b) and Hata et al. (1995) which is \( \Omega_b h^2 = 0.016 \). Copi et al. (1995a,b) claim that a plausible range of \( \Omega_b \), clearly intended to be thought of as 95 per cent confidence, extends to 50 per cent in either direction around that value. The slightly higher value is helpful, especially in models without a hot component, as it helps remove short scale power from the spectrum. In CDM models, there may be further motivation to raise it further towards the top end of the range (e.g., White et al. 1995); in such models it is easy to quantify the benefit of raising \( \Omega_b \) and we shall show how to do this later.

For our investigation, we shall therefore treat as our three main parameters \( \Omega_\gamma \), \( n \) and \( h \), and in addition allow the incorporation of a gravitational wave component though the space of that parameter shall not be as extensively explored. Early investigations typically only varied \( \Omega_\gamma \), but were followed by treatments by Schaefer & Shafi (1993), Liddle & Lyth (1993b) and Schaefer & Shafi (1994) who investigated the \( \Omega_\gamma - n \) plane, both with and without gravitational waves but concentrating only on \( n < 1 \). Pogosyan & Starobinsky (1993) carried out an analogous investigation of the \( \Omega_\gamma - h \) plane, fixing \( n = 1 \). More recently, Pogosyan & Starobinsky (1995a) made a study of the full \( \Omega_\gamma - n - h \) parameter space, concluding that \( |n-1| \) should not exceed 0.1 for any \( h \) or \( \Omega_\gamma \). Dvali, Schaefer & Shafi (1995) have analyzed the \( \Omega_\gamma - n \) plane for three values of \( h \) and found the same trends evident in Pogosyan & Starobinsky (1995a), although the limits on \( n \) were somewhat dependent on \( h \) as 0.80 (h/0.5)^1/2.1.15. Note though that Pogosyan & Starobinsky (1995a) and Dvali et al. (1995) used a COBE normalization significantly lower than is usually recommended from the two year data (Görski et al. 1994; Bunn, Scott & White 1995). An analysis solely of microwave anisotropies on various scales applied to tilted CHDM models (de Gasperis, Muciaccia & Vittorio 1995) favours low \( n \) values.

Our present paper is closest in spirit to the Pogosyan & Starobinsky (1995a) and Dvali et al. (1995) analyses, so it is worth indicating here where we differ. We have made an entirely new calculation of the transfer functions for our models, including the incorporation of the baryonic component (not included by them) which is significant especially for low \( h \) values. In addition to using the more modern COBE normalization, we have made a recalculation of constraints from cluster abundance, which are now more conservative. We include a treatment of damped Lyman alpha system abundance, which has been seen as problematic for some versions of the CHDM scenario. We shall also use results from the POTENT analysis of velocity fields (Bertschinger & Dekel 1989; Dekel 1994), which includes detailed modeling of the effects of cosmic variance (as included in Schaefer & Shafi 1994). Also, only \( h > 0.4 \) has been considered previously. Regardless of one's view regarding constraints from direct measurement it is worth extending this to lower values to investigate the 'volume' of favoured parameters. Further motivation for this arises as models with extra massless species can mimic low values of \( h \) while keeping the actual \( h \), as would be directly measured, higher (Dodelson, Gyuk & Turner 1994).

An important subset of the parameter space which we shall also explore is the case of pure CDM models; that is, the case \( \Omega_\gamma = 0 \). These have seldom been studied in the context of permitting full variation of \( n_c \), \( h \) and the gravitational wave amplitude, and it has recently been suggested (White et al. 1995) that claims that all such models are ruled out may be premature. Our results support the assertion that there remains some viable parameter space for CDM models, without one having to modify the dark matter content or change the number of massless species.

It is useful to have a fiducial model to make comparisons with. We shall adopt the usual practice of taking this to be the standard CDM (SCDM) model, even though this is known not to be a good fit to the data. To be explicit about our assumptions, the parameters of this model are \( n = 1 \), no gravitational waves, \( h = 0.5 \), \( \Omega_b = 0.0165^{-2} = 0.064 \), \( \Omega_\gamma = 0 \) and the amplitude of the spectrum normalized to match the COBE observations with expected quadrupole \( Q_{\ell = 2} = 19.9 \mu K \) (Bunn et al. 1995).

The layout of the paper is as follows. In Section 2 we briefly outline the derivation of the power spectra we use to make the comparison with the observations, along with some discussion of our use of Press-Schechter theory. In Section 3 we shall provide a detailed account of the observations we have selected in order to make comparison with the theoretical predictions. Our procedure is not to use the power spectrum itself, but instead to concentrate on the spectrum filtered by a top-hat window, which represents the variance of fluctuations on a given scale. This quantity has several advantages, and in Section 3 we shall describe how we interpret our chosen observations in terms of this quantity. Section 4 will then provide the confrontation of the theoretical predictions with the observations.

2 THE THEORETICAL INPUT

2.1 Transfer functions

Inflation generates Gaussian density perturbations, which implies that their stochastic properties can be completely described by the power spectrum. In almost all inflationary models, the power spectrum \( P(k) \) can be accurately parametrized across observable scales by a power-law \( P(k) \propto k^n \), where \( k \) is the comoving wavenumber (see Lid-
Primarily we are interested in the spectrum of perturbation $f$ as (Liddle & Lyth 1993a)

$$P_f(k) = 4\pi (Lk/2\pi)^3 \langle |f|^2 \rangle,$$

(1)

where $L$ is the comoving size of the periodic box introduced to allow a Fourier expansion of $f$ into its comoving modes $f_k$, and the angled brackets indicate an averaging over a small region of $k$-space to make the spectrum a smooth function. We have used statistical isotropy to say that the spectrum can only be a function of the magnitude of $k$, and not its direction. The prefactor is chosen to guarantee that the mean square perturbation is given by

$$\sigma_f^2 = \int_0^\infty P_f(k) \frac{dk}{k}.$$  

(2)

Primarily we are interested in the spectrum $P_\delta$ of the density contrast $\delta$, which is related to the usual $P(k)$ by

$$P_\delta \propto k^3 P(k).$$

The initial spectrum generated by inflation will be modified as the universe evolves, since the growth of density perturbations is affected by the properties of the matter in the universe and also the value of the Hubble parameter. This modification is quantified by the transfer function $T(k, z)$, which measures the amount of growth a perturbation on scale $k$ receives by a redshift $z$ relative to the infinite wavelength $k = 0$ mode (thus $T(k, z) \to 1$ on large scales). With a power-law initial spectrum from inflation, at a given redshift one has

$$P_\delta(k, z) \propto k^{3+n} T^2(k, z),$$

(3)

where the constant of proportionality is to be fixed by observations.

For reasons discussed in the next Section, we choose not to try and place constraints directly on the power spectrum. Instead, we choose the dispersion $\sigma(R)$ of the density contrast smoothed on a scale $R$ as our fundamental quantity. The smoothing is carried out via a top-hat window function, defined by

$$W(kR) = \left[ \sin(kR) - \frac{\sin(3kR)}{3kR} \right]^3,$$

(4)

which filters out modes with $k^{-1} \ll R$. The variance of the smoothed field is

$$\sigma^2(R) = \int_0^\infty W^2(kR) P_\delta(k) \frac{dk}{k}.$$  

(5)

Often the spectrum is increasing towards short scales, in which case the variance is dominated by modes with $k^{-1} \sim R$.

It is often useful to associate a mass with the top-hat filter, which one gets by integrating the filter over a uniform density. Assuming critical density, this yields

$$M(R) = 1.16 \times 10^{12} h^{-1} \left( \frac{R}{h^{-1}\text{Mpc}} \right)^3 \text{M.}$$  

(6)

We have calculated the transfer functions numerically using the techniques described by Schaefer & de Laix (1995). The procedure can be summarized as follows. Starting from adiabatic initial conditions deep within the radiation dominated epoch, the gauge-invariant linear evolution equations for each of the components are numerically integrated up to the present time via a Flaming type Predictor–Corrector. We keep 1000 moments of the photon and relativistic neutrino distribution up until well into the matter dominated epoch redshift $z = 250$, at which time we set their amplitudes equal to zero. At this time they have a negligible influence on the growth of the matter perturbations. The massive neutrinos require special treatment. In this case we expand the neutrino distribution function in terms of angular moments of the cosine of the angle between the particle momentum and the wavevector, keeping 200 angular moments. The massive neutrino distribution function must be integrated over momentum at every integration step, and this is done with a point Gauss-Laguerre integration which is accurate to better than one part in $10^6$. The photons and baryons are treated using the tight coupling approximation until the temperature drops below 6000 K, at which point we switch to the full equations for the two coupled components.

We calculated transfer functions for $\Omega_\nu = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5$ for $h = 0.3, 0.4, 0.5, 0.6, 0.7$ using a value of the baryon fraction consistent with nucleosynthesis, $\Omega_B h^2 = 0.0125$. We calculated them for $z = 0, 3, 3.5, 4$. We have fit them with coefficients in a form somewhat similar to the Bardeen et al. (1986) CDM transfer functions; however, the coefficients are not smooth functions of $\Omega_\nu$, which proved to be inconvenient for testing. We note that there already exists a 'universal' transfer function for the CHDM models in universes with no baryons added (Pogosyan & Starobinsky 1995a), which we shall adapt to models with baryons. Their transfer function begins with the Bardeen et al. (1986) fit to standard CDM, which is

$$T_{\text{CDM}}(q) = \frac{\ln (1 + 2.34q)}{2.34q} \times$$

$$[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{-1/4},$$

(7)

where the scaled wavenumber $q$ is related to the usual Fourier wavenumber $k$ as $q = k/h$. Pogosyan & Starobinsky (1995a) then constructed a formula for the factor which describes the damping of the massive neutrino component:

$$D(q, z) = \left[ 1 + (Bq)^2 + \alpha_q (1 + z)(1 - \Omega_\nu)^{1/3} (Bq)^4 \right]^{1/4},$$

(8)

where

$$\beta = \frac{5}{4} (1 - \sqrt{1 - 24\Omega_\nu/25});$$

$$A = \frac{17.266 (1 + 10.912\Omega_\nu) \sqrt{\Omega_\nu (1 - 0.9465\Omega_\nu)}}{1 + (9.259\Omega_\nu)^2};$$

* We carried out these tests using the old nucleosynthesis value of Walker et al. (1991), before our decision to adopt the higher value $\Omega_B h^2 = 0.016$ (Copi et al. 1995a, 1995b) which is that used to obtain all results in this paper. This change does not affect our tests of the fitting quality.
\[ B = 2.6823 \frac{1.1435}{H_\infty + 0.1435} \cdot \frac{\Omega_B}{h^2} \]

\[ \alpha_{eq} = \frac{4.212 \times 10^{-5}}{h^2}. \]  

(9)

A widely used empirical formula for adding the effect of baryons to the standard CDM transfer function is to generalize the formula for \( q = k/H \) where the 'shape parameter' \( \Gamma \) is defined by \( \Gamma = \exp(-2\Omega_B) \). This is known to work well for CDM provided \( \Omega_B \) is not too large (Peacock & Dodds 1994). We then tested whether or not this worked with the Pogosyan & Starobinsky transfer functions. We found that this replacement works extremely well provided \( \Omega_B < 0.5 \) and \( \Omega_B > 0.1 \). As \( \Gamma = 0.21 \) the decoupling time approaches the time of matter-radiation equality, so the damping of fluctuation growth by the baryons recedes in importance. We have found that, for \( \Gamma > 0.21 \), a better replacement is to use

\[ \Gamma = \exp(-2(1 - (0.21/h^3))\Omega_B) \],

(10)

which is very accurate for \( \Omega_B > 0.1 \). This relation also holds for pure CDM transfer functions. If \( \Omega_B > 0.1 \), the baryons then become dynamically significant and impose a steep drop at the decoupling length scale, a feature which cannot be adequately described by simply shifting the scales in the transfer function. For \( \Gamma = 0.3 \), we have the central value \( \Omega_B = 0.139 \) and notice significant departures of the scaled Pogosyan & Starobinsky transfer function from our computed functions. We can compare values of the dispersion \( \sigma(R) \) calculated using the real and the scaled transfer functions. In this case we find the scaled functions overestimate the amplitude \( \sigma(R) \) by as much as 10 per cent on small scales \( \sim 1h^{-1}\text{Mpc} \) and underestimate it on scales \( \sim 200h^{-1}\text{Mpc} \). For comparison, when \( \Gamma = 0.4 \) implying \( \Omega_B = 0.078 \), the error in \( \sigma(R) \) is less than about 2 per cent when \( R > 0.1 h^{-1}\text{Mpc} \). At larger values of the Hubble constant the fits are even better.

Using the value \( \Omega_B h^2 = 0.016 \) we adopt to obtain results in this paper, the accuracy of the scaled Pogosyan & Starobinsky transfer function becomes worse for small values of \( h \) a good fit to our computed functions across the range of scales we are interested in can only be achieved for \( h > 0.4 \), instead of the previous limit \( h > 0.35 \). However, for Hubble constant values as low as these we find that using the exact transfer functions leads to slightly stronger constraints, so adopting the fitting function as above is a conservative choice.

Putting all this information together, the redshift-dependent transfer function for CHDM models is given by

\[ T(k, z) = T_0 \text{CDM}(k)D(k, z) \quad q = k/h\Gamma, \]

(11)

where \( \Gamma \) is given by equation (10).

Some of the observations we use apply at moderate redshift rather than redshift zero. In a cold dark matter dominated universe this can easily be accounted for using the scale-independent linear growth law \( \sigma(R) \propto (1 + z)^{-2} \); implying a redshift independent transfer function at late times. By contrast, in CHDM models the growth rate becomes scale-dependent with suppression on short scales due to neutrino free-streaming. Fig. 1 illustrates the redshift dependence of the transfer function for two choices of HDM density. We see that on scales greater than \( 3h^{-1}\text{Mpc} \), the CDM growth law is an excellent approximation from moderate redshift even when a sizeable HDM component is present.

### 2.2 Gravitational waves from inflation

In addition to generating a power-law spectrum of density perturbations, inflation generates a power-law spectrum of gravitational wave modes (Starobinsky 1979; Liddle & Lyth 1993a). The only observation we discuss that these are capable of influencing is the COBE observation, where a possible gravitational wave contribution to microwave background anisotropies (Abbott & Wise 1984; Starobinsky 1985) will add in quadrature to that from density perturbations.

Within the usual slow-roll inflation models, the amplitude of gravitational waves on COBE scales is another free parameter, independent of the spectral index of density perturbations\(^1\) (Liddle & Lyth 1992). We shall treat the amplitude as given independently; the independent choice of \( n \) and the gravitational wave amplitude is then the most general outcome of slow-roll inflation for any choice of potential for the scalar field driving inflation (Liddle & Lyth 1993b).

If one were to be more specific in the choice of inflation model, then the spectral index and gravitational wave amplitude could be related. For example, power-law inflation yields \( n < 1 \) and \( \sigma \approx 2\pi(1 - n) \), where \( \sigma \) is the relative contribution of gravitational waves to density perturbations to large angle microwave background anisotropies\(^1\), as defined by Liddle & Lyth (1992). Almost all known inflation models have gravitational wave contributions sandwiched between zero and that of a power-law inflation model yielding the same spectral index. We shall concentrate on these two options for \( n < 1 \), and ignore gravitational waves for \( n > 1 \) since it is hard to make inflationary models with \( n > 1 \) and significant gravitational waves.

### 2.3 Press–Schechter theory

The standard comparisons that we make between theory and observations, based on the spectrum integrated with a top-hat filter, are well established in the literature. The exception is the calculations based on object abundance, which contain greater theoretical uncertainties than other measures, and so we shall discuss in depth the way we carry this out. The standard technique is Press–Schechter theory (Press & Schechter 1974), which has been compared in depth with \( \Lambda \)-CDM simulations (e.g. Lacey & Cole 1993, 1994), and we shall use it to obtain constraints on the abundances of each of damped Lyman alpha systems, quasars and galaxy clusters.

When one applies a smoothing window with a given radius to a Gaussian random field, one obtains the corresponding smoothed density field which is also a Gaussian random field provided its dispersion is smaller than one. It is then straightforward to obtain the fraction of space in

\(^1\) However, the spectral index of the gravitational wave spectrum is then related to the amplitude via a 'consistency relation'.

\(^1\) In some papers, the relative amplitude of gravitational waves and density perturbations is given as \( 7(1 - n) \). This refers to the relative contributions to the quadrupole, which has a correction from the curvature of the last scattering surface. The version we give is appropriate to higher multipoles, and since the COBE normalization is most sensitive around the tenth multipole it is the best version to use in this context.
the universe occupied by regions where the linearly evolved smoothed density contrast exceeds some given threshold value. The insight of Press and Schechter was to assume that for the correct threshold value this fraction could be identified with the fraction of matter in the universe which is part of gravitationally bound objects with a certain minimum mass, the relation between the size of the regions and the minimum mass of the bound objects depending on the smoothing window applied to the underlying density field.

A problem with this assumption is that in linear theory half the volume of the universe is always composed of regions with a negative smoothed density contrast, and therefore only half of all the matter in the universe is available to form bound structures, which clearly does not happen in the real universe. This problem arises because one is not taking into account the matter in the regions whose linearly evolved density contrast does not exceed the threshold value, and thus are not considered to be bound according to the above criterion, but which are part of bigger regions whose linearly evolved density contrast does exceed the threshold value, and are therefore bound. The original Press-Schechter derivation tries to allow for the matter in those regions simply by assuming they contain as much matter as the matter contained within the regions which are bound according with the original criterion. Though this assumption makes some sense if one thinks in terms of the statistics of a Gaussian random field, the main motivation was that it is the simplest way of allowing all the matter in the universe to be available to form gravitationally bound structures. This is less than satisfactory, and since then many people have tried in all sorts of ways to determine if this assumption has any validity. The conclusion reached from $N$-body simulations is that it depends on the smoothing window, being a reasonable assumption for a window which is a top-hat in $k$-space, known as a sharp-$k$ window (for which Peacock & Heavens (1990) and Bond et al. (1991) have proven using excursion-set theory that the factor two correction is exact) and for the real space top-hat window we use, but not so good for a Gaussian window (Lacey & Cole 1994). We use the top-hat window as the relation between the size of a region and its mass is then straightforward, which is not the case for the sharp-$k$ window.

The density in collapsed objects above a given mass at a redshift $z$ is then given simply by integrating over the tail of the Gaussian with the additional factor two multiplier, yielding

$$\Omega(M(R), z) = \text{erf} \left( \frac{\mu}{\sqrt{2} \sigma(R, z)} \right),$$

(12)

where $\mu$ is the threshold value, $\sigma(R, z)$ the dispersion smoothed on scale $R$ at redshift $z$ and 'erf' is the complementary error function.

The choice of threshold is crucially important, as typically it is one of the main sources of uncertainty. The literature features a wide range of values, but it is vital to note that this is primarily because different types of smoothing window require different thresholds. Once a specific choice of window is made the uncertainty is not so great. In the original manifestation of the Press-Schechter theory, a threshold $\mu$ of 1.7 was motivated by the spherical collapse model for a top-hat perturbation. However, this is a highly idealized model which assumes that the collapsing perturbation is not under any external influence; that is, it does not possess shear. This should be an increasingly good assumption the less evolved the smoothed density field is (Bernardeau 1994) and the less relative large-scale power there is. The influence and relative importance of shear on the time a perturbation takes to collapse depends critically on one's definition of collapse. If one identifies collapse of a perturbation with collapse along the first collapsing axis then shear decreases the timescale of the collapse, but if one identifies collapse of a perturbation with complete collapse along three normal axes then shear increases this timescale (Monaco 1995). The first case relates to the formation of pancakes, and for example can be useful in the study of the objects which give rise to the Lyman alpha forest lines in the spectra of quasars. However, if one is interested in completely virialized objects like quasars or clusters then the second definition of collapse should be used. The damped Lyman alpha systems are likely to lie somewhere in between these two extremes.

*Recently Yano, Nagashima & Gouda (1995) have recovered this result using a different technique, first proposed by Jedamzik (1995), which relies on the use of the integral equation of the mass function.*

*In this case it is possible that the Press-Schechter correction slightly underestimates the fraction of matter in more massive structures relative to less massive ones, at least when one considers hierarchical models (Jain & Bertschinger 1994). This is because although the Press-Schechter assumptions allow for merging and fragmentation of regions within an already bound object, they do not allow for merging or fragmentation of bound objects which are not part of bigger ones. Merging should be increasingly important the more relative small-scale power there is, distorting the Press-Schechter mass function towards relatively more massive objects. The opposite change in the mass function should happen the more relative large-scale power there is, as then fragmentation should become increasingly important. Nevertheless, for reasonable power spectra one expects these effects not to be very influential as long as the dispersion of the smoothed density field remains less than one, which in any case is the validity condition for the Gaussian assumption.*
3 THE OBSERVATIONAL DATA

To effectively constrain the density perturbation spectrum, one requires a compilation of estimates of its amplitude at a variety of different scales by a variety of different methods. To some extent, this occurs naturally as different types of observations are best suited to estimating the power spectrum on different scales. For example, only microwave background data are presently capable of providing information on the largest scales, and only the abundance of objects at high redshift allows access to presently nonlinear scales at a time when they may still be addressed using quasi-linear theory. It is only the intermediate scales, running from perhaps $8\,h^{-1}\,\text{Mpc}$ up to $100\,h^{-1}\,\text{Mpc}$, that have been simultaneously constrained by a number of different types of measurements, from abundance of clusters to galaxy correlation functions to peculiar velocity flows; in the near future reliable microwave background experiments should also extend down into this region.

Our general strategy is to impose constraints not on the power spectrum $P(k)$ directly (where $k$ is the comoving wavenumber), but instead to impose on them the dispersion of the density field filtered through a top-hat window function, denoted $\sigma(R)$, whose radius $R$ is varied in order to pick out different scales. This method is useful because the bulk of the observational data are obtained in this form, and typically the conversion of such data into power spectrum form introduces systematic errors. In contrast, a theoretical calculation of the filtered variance is very simple to make given a theoretical power spectrum. Further, observations on short scales connected with object formation at high redshift have no interpretation at all in terms of the power spectrum, and this result can be used to generate the required normalization as a function of $n$.

However, more recently it has been noted that the assumption of a power-law spectrum of anisotropies, corresponding to the Sachs-Wolfe contribution, is not a perfect one, because the ‘Doppler peak’ extends to small multipoles and invades part of the region COBE samples. Consequently, one should fit the amplitude to full anisotropy spectra, and this has been carried out by Bunn et al. (1995). For CDM spectra with no gravitational waves, they find conditional likelihoods yielding

$$Q_{\text{rms-PS}}(n) = (19.9 \pm 1.5) \exp [0.69(1 - n)] \mu K,$$

and this is the normalization we shall adopt. Note that the $n$ dependence in the fit given was calculated only taking into account the Sachs-Wolfe effect; it should nevertheless provide a very good approximation. They note that this result is more or less independent of the nature of any dark matter, of $\Omega_0$ and of $\Omega_0$, so we are justified in using it for all models without gravitational waves. Although the full anisotropy spectra are needed for performing the fit to the COBE data, it is fine to compute the perturbation spectrum normalization corresponding to a given quadrupole using the Sachs-Wolfe formula, since the quadrupole is least affected by the Doppler peak.

As we are concentrating on interpreting data in terms of the filtered dispersion $\sigma(R)$, it is interesting to ask what sort of scales this normalization is sampling. One way to do this is to normalize a set of CDM models with different $n$ and see where the curves cross. One finds that the lines more or less cross (with an accuracy of a few percent, doing less well as $h$ is varied) at a scale of $4000\,h^{-1}\,\text{Mpc}$. We shall occasionally use this to schematically represent the COBE data, with the main purpose of indicating the size of the COBE error; however, in all cases we shall calculate using the precise normalization of the power spectrum given above rather than this approximate data point.

To be completely accurate, if gravitational waves are included one should add their radiation power spectrum to that of the density perturbations and perform a full model fit. However, as long as the gravitational wave contribution is not too significant, one can approximate it as having the same functional form as the density perturbations over the COBE range and simply normalize down the density perturbation power spectrum as appropriate. As discussed by Liddle & Lyth (1993b), the relative contribution of gravitational waves to density perturbations to the microwave anisotropies, $\tau$ as defined in Subsection 2.2, leads to a reduction in the amplitude of the COBE normalized dispersion $\sigma(R)$ by a factor $1/\sqrt{1 + \tau}$.

Although we will normalize $\sigma(R)$ to the Bunn et al. (1995) COBE central value, we shall allow for their $15\%$ uncertainty at the $2\sigma$ level by adding it in quadrature to the relative errors of those other observations which also constrain the amplitude of $\sigma(R)$.

3.1 COBE

Recently there has been considerable activity concerning the interpretation of the anisotropies detected by COBE (Smoot et al. 1992; Bennett et al. 1994; Wright et al. 1994). The methods used have become sufficiently sophisticated that simple normalization methods, such as to the ten degree variance of the anisotropies as widely used in the two years following the COBE announcement, are no longer appropriate, since they make inadequate use of the full COBE data set. Instead, it is better to rely on the normalizations published in the literature which do take the full data set into account.

The first development in this regard was a very elegant pair of papers by Górski and collaborators (Górski 1994; Górski et al. 1994) who fit power-law spectra to the observations, thus obtaining likelihoods in the $n$-$Q_{\text{rms-PS}}$ plane, where $Q_{\text{rms-PS}}$ is the expected quadrupole (over an ensemble of independent observers). The quantity which is actually desired in order to constrain theoretical models is not the full likelihood or the marginalized one, but rather the conditional likelihood on $Q_{\text{rms-PS}}$ for fixed $n$, given as a function of $n$. Although they only provided this for $n = 1$, they noted that regardless of the fitted $n$ the preferred amplitude of the ninth multipole remains unchanged to excellent accuracy, and this result can be used to generate the required normalization as a function of $n$.

Critical density dark matter models

The deficiencies in the shape of the standard CDM spectrum are most apparent in surveys of galaxy correlations span-
ning the range from a few megaparsecs up to tens of megaparsecs. A variety of surveys such as QDOT, CfA, APM and 1.2 Jansky provide information in this region. In an attempt to evade systematics particular to the types of analysis provided, one can combine data from a variety of different sources, hoping to demonstrate consistency between the different data sets, and this has been achieved in an impressive analysis by Peacock & Dodds (1994). The cost is that the formal errors are somewhat larger than those one sees in individual surveys, and it is not easy to see whether or not one is unfairly penalizing the most accurate data sets rather than uncovering overoptimistically small error bars across all data sets.

Another problem with using galaxy data is that although it determines the shape of the spectrum very well, the overall normalization is less certain due to the expectation that galaxy correlations are biased relative to the underlying matter, multiplying the power spectrum by a (hopefully scale-independent at least over the limited range of scales considered) bias parameter. One can attempt to determine the bias parameter from the surveys themselves by using redshift distortions and/or nonlinear effects, or instead by utilizing an entirely separate method such as peculiar velocity flows. Or alternatively one can allow the normalization of the galaxy correlation data to ‘float’, with its best amplitude determined by the other types of data under consideration, which amounts to throwing away information on the bias.

Peacock & Dodds (1994) quote their final results in terms of the power spectrum \( P(k) \) (\( \Delta^2(k) \) in their notation). However, the original data are provided in a mixture of the power spectrum, the dispersion \( \sigma(R) \) and the correlation function \( \xi(R) \). They switch between them using an analytic prescription

\[
\sigma(R) = P_1^{1/2}(kR),
\]

\[
\xi(R) = P_2^{1/2}(\sqrt{3}kR),
\]

where

\[
k_R = \left[ \frac{1}{2} \pi \left( \frac{m+3}{2} \right) \right]^{1/(m+3)} \frac{\sqrt{5}}{R},
\]

and \( m = (k/\tau)(dP_1/dk) \) is the effective spectral index. These formulae are obtained by assuming \( m \) constant over the range of \( k \) modes contributing, and using the approximation

\[
W(kR) = \exp(-k^2 R^2/10),
\]

which is exact for \( kR \ll 1 \).

In making the conversion, one needs to specify a power spectrum in order to calculate the effective spectral index. This is best done by choosing a model spectrum which fits the data; we use the best-fitting CDM spectrum, specified by a shape parameter \( \Gamma \). Since the raw data are provided in a variety of forms, there is no reason to think that expressing them in terms of \( \sigma(R) \) is any less accurate than expressing them via the power spectrum, and we shall use both. The shape parameter provides a good indication of the quality of fit to the galaxy correlation data and we shall occasionally use this language.

Recently some doubt has been cast over the assumption by Peacock and Dodds that the bias parameter is scale-independent down to the smallest scales, around \( 4h^{-1}\) Mpc, considered in their analysis (John Peacock, private communication). As it seems that it is for scales below around \( 8h^{-1}\) Mpc that the bias parameter starts becoming nonlinear, we have excluded from our final data, presented in Table 1 of Peacock & Dodds (1994), the four points corresponding to the smallest scales and re-calculated the best \( \Gamma \) fit to their remaining data. For \( 0.7 < n < 1.2 \) we find \( \Gamma = 0.23 - 0.28(1 - 1/n) \), where the 2\( \sigma \) relative error is plus 18 per cent and minus 15 per cent. This compares with the central value (for \( n = 1 \)) from the full data set of 0.255 (Peacock & Dodds 1994).

When utilizing data of this form in a statistical analysis, as we do below, it is vital to ensure that the points used are taken suitably far apart as to be independent, and in general one needs the full correlation matrix to determine this (which has only been calculated for the QDOT survey power spectrum (Feldman, Kaiser & Peacock 1994)). If one is not careful as regards this point, then statistical tests are biased and, depending on the form of test used, this can make bad models look good or, much more seriously, make good models look bad. For a statistical treatment there is the further problem that the errors are systematic as well as statistical, and hence will not be normally distributed; unfortunately in the absence of a detailed understanding of an experiment there is no way to counter this other than to treat results with mild skepticism.

After the exclusion of the four points corresponding to the smallest scales, a chi-squared analysis of the remaining data in Table 1 of Peacock & Dodds (1994), where \( n, h, \Omega_c \) and the normalization are the fitting parameters, has 8 degrees of freedom. Performing this analysis we find a very low minimum chi-square of around 4; although it is perfectly reasonable that this occurred by chance, it may also indicate weak residual correlations of neighbouring data points. In the present case this typically makes models seem much better in relation to the data than they really are. The best we found of avoiding this problem is to calculate not the absolute exclusion level of each model against the data, but the relative confidence limits in the three-dimensional space formed by the parameters \( n, h \) and \( \Omega_c \) (Press et al. 1992). This is achieved by calculating the difference between the chi-square obtained for each model characterized by a fixed set of values for \( n, h \) and \( \Omega_c \), where the normalization is calculated so as to minimize the chi-square, and the minimum chi-square obtained by varying the four parameters. This difference still has a chi-square distribution, now with 3 degrees of freedom. The 68 per cent and 95 per cent confidence limits are then defined in the \( n, h, \Omega_c \) space by the chi-square difference being respectively smaller than 3.508 and 7.815. We will plot cross-sections of the region in the \( n, h, \Omega_c \) space defined by the 95 per cent confidence limit.

### 3.3 Peculiar velocities

#### 3.3.1 POTENT

Peculiar velocities directly sample the matter power spectrum and so are unaffected by clustering bias. However, measurements of the peculiar velocity field are much harder to obtain. The best measurements using velocities alone come from the POTENT method (Bertschinger & Dekel 1989), the most recent version available being the Mark III PO-
TENT data (Dekel 1994), which supplies an estimate of the velocity smoothed on various length scales around us. This can be used as an estimator for $\sigma(R)$ on a particular scale, as follows.

Firstly, we restrict ourselves to using a single piece of data, the velocity on a $40\, h^{-1}\, \text{Mpc}$ sphere. Although measurements exist for a range of scales, they are highly correlated because the window function for the peculiar velocities samples a wide range of scales and in particular is more sensitive to longer scales than the density dispersion. We choose this particular value as it is in the centre of the supplied range.

In making a theoretical comparison, one needs a two stage smoothing, since POTENT involves first smoothing the observed peculiar velocities with a $12\, h^{-1}$ Gaussian before the velocity reconstruction can be undertaken and the $40\, h^{-1}$ top-hat smoothing applied to obtain $v(40\, h^{-1}\, \text{Mpc})$. The appropriate formula for the dispersion of the velocity is

$$
\sigma^2_v(R) = H_0^2 \int_0^\infty W^2(kR) \exp\left(-\left(12\, h^{-1} k\right)^2\right) \frac{\rho}{k^2} \frac{d k}{k^3}.
$$

(18)

As with COBE above, one can then ask what scales in the filtered dispersion $\sigma(R)$ of the density field correspond to a fixed observed velocity. This can again be addressed by plotting $\sigma(R)$ for a set of CDM models with different $n$, each normalized to yield the same $\sigma_v(40\, h^{-1}\, \text{Mpc})$. It turns out that such curves cross, extremely accurately, at a scale of $117\, h^{-1}$ Mpc. As stated above, the velocities sample considerably longer scales than the smoothing length by itself suggests.

This crossing point remains quite accurate even if one goes to CHDM models, and this fact coupled with the much larger observational errors as compared to COBE means that we can represent the POTENT data as a single constraint on $\sigma(117\, h^{-1}\, \text{Mpc})$.

The Mark III POTENT analysis gives for the bulk flow in a $40\, h^{-1}$ Mpc sphere (Dekel 1994)

$$
v_{\text{POTENT}}(40\, h^{-1}\, \text{Mpc}) = 373 \pm 50 \, \text{km s}^{-1},
$$

(19)

where the error arises from different ways of dealing with sampling-gradient bias and can thus be thought of as reflecting the systematic uncertainty in the POTENT analysis. Additionally there is an intrinsic uncertainty in the POTENT calculation due to random distance errors, which at the 1σ level is $\pm 15$ per cent (Dekel 1994). Note that COBE normalized standard CDM yields

$$
v_{\text{SCDM}}(40\, h^{-1}\, \text{Mpc}) = 385 \, \text{km s}^{-1},
$$

(20)

suggesting that SCDM produces about the right answer using the modern COBE normalization. The observational error is dominated by cosmic variance, resulting from the POTENT observation being a single measurement from a random field. Since each velocity component separately has a Gaussian distribution, the velocity squared has a chi-squared distribution with three degrees of freedom. From this one can calculate the range of theoretical values for which the observed value would not lie in the tail of the distribution, and the probabilities corresponding to 68 per cent confidence yield an upward error of 89 per cent and a downward error of 24 per cent on the estimator for $\sigma(R)$. At the 95 per cent confidence level the error bars are $+273$ per cent and $-43$ per cent. The asymmetry of the errors originates in the asymmetry of the chi-squared distribution. We can now convolve the systematic and random errors arising from the POTENT calculation with the cosmic variance error. Assuming the error in expression (19) corresponds to something like 95 per cent confidence (though as it is the smallest error this assumption is insignificant), we then obtain the total error in using the Mark III POTENT bulk flow calculation as an estimator of the normalization of the dispersion of the density contrast: at the 68 per cent confidence level, $+98$ per cent and $-25$ per cent; at the 95 per cent confidence level, $+295$ per cent and $-47$ per cent. Clearly, only the lower limits are of use for us. At a level corresponding to 95 per cent confidence, the bulk flow constraint can then be written as

$$
\sigma_{\text{POTENT}}(117\, h^{-1}\, \text{Mpc})
\sigma_{\text{SCDM}}(117\, h^{-1}\, \text{Mpc}) = 0.97^{+2.295}_{-0.47} \text{pc per cent}.
$$

(21)

3.3.2 Velocities versus densities

An alternative use of velocity data is through the comparison with the density field obtained via galaxy surveys. Present technology focuses on an estimate of a single parameter $\Omega_0^b$, $b$ is the bias parameter appropriate to whatever type of galaxies are being studied, normally IRAS galaxies with bias $b$. The degenerate combination of $\Omega_0$ and $b$ arises through the inability to distinguish slow velocities due to a slowing of the perturbation growth rate in low density universes from having a high irregularity in the galaxy distribution relative to that of the matter distribution actually generating the velocities. However, we are only considering critical density models, so these methods directly estimate the bias. This information can then be used in conjunction with the galaxy number counts dispersion to supply constraints on the variance in the density. Note though that there seems no good way to quantify the errors arising from the inadequacy of a single bias parameter to explain the difference between the galaxy and density variances.

We shall not utilize the range of bias found by Peacock & Dodds (1994), the reason being that there remains widespread disagreement in the literature between values obtained by different methods (for instance, see Dekel 1994 for a compilation). Consequently, the true uncertainty appears much greater than advertised by any single study, and if one attempts to take a more realistic view the amplitude becomes so uncertain as to provide no useful constraint.

3.4 Abundance of galaxy clusters

The typical mass of large galaxy clusters, about $10^{15} \, \text{M}_\odot$, corresponds to a linear scale of around $8\, h^{-1}$ Mpc. Observation indicates that large clusters are relatively rare, indicating that this scale is still in the quasi-linear regime. The usual technique of Press–Schechter theory calibrated by N-body simulations can therefore be used to impose constraints. A variety of estimates of the cluster mass function exist in the literature; some authors (e.g. Lilje 1992; White, Efstathiou & Frenk 1993) aim to reproduce the observed number density at a single mass scale whilst others (e.g. Evrard 1989; Henry & Arnaud 1991; Hattori & Matsuzawa 1995) more ambitiously aim to fit the shape of the cluster mass function. The analysis we perform belongs to the first type. Typically,
the number density of a given type of cluster is quite well
known, at least at low redshift — most of the uncertainty
comes from poor knowledge of the mass of individual
clusters. This can be estimated in a variety of ways, the most
common being the virial theorem, the X-ray temperature
distribution as a tracer of the gravitational potential and,
most recently, weak shear lensing of background objects.
All these methods suffer from several problems, though the
one which at the present seems most likely to give the best
results is the use of X-ray temperature observations.

The observed number density of clusters per unit tempera-
ture at $z = 0$ about a mean X-ray temperature of 7 keV
was calculated by Henry & Arnaud (1991) to be
\begin{equation}
\pi(7 \text{ keV}, 0) = 2.0^{+2.0}_{-1.0} \times 10^{-7} \text{ h}^3 \text{ Mpc}^{-3} \text{ keV}^{-1}.
\end{equation}

The comoving number density of clusters with virial mass $M_v$ per mass interval $dM_v$ at a redshift $z$ is obtained by
differentiating equation (12) with respect to $z$ and multiplying it by $\rho_v / M_v$, where $\rho_v$ is the comoving back-
ground density (a constant during matter domination), thus giving
\begin{equation}
\pi(M_v, z) dM_v = -\sqrt{\frac{2}{\pi}} \rho_v \frac{\delta_c}{M_v} \frac{d\Delta(z)}{dM_v} \exp \left[ -\frac{\delta_c^2}{2\Delta^2(z)} \right] dM_v,
\end{equation}
where $\Delta \equiv \sigma^2(\tau_L)$ with $\tau_L$ the comoving linear scale associated with $M_v$, $\tau_L^2 = 3M_v / 4\pi\rho_v$. Traditionally the cluster abundance is used to constrain the present-day dispersion at $8 h^{-1} \text{ Mpc}$, $\delta_8 \equiv \sigma(8 h^{-1} \text{ Mpc}, 0)$, and the quantity $\Delta$ is specified by an analytic approximation to the power spectrum in the vicinity of this scale. Generally, one can write
\begin{equation}
\Delta(z) = \sigma_8(z) \left( \frac{\tau_L}{8 h^{-1} \text{ Mpc}} \right)^{\gamma(\tau_L)}.
\end{equation}

In Liddle et al. (1995) we adopted the form
\begin{equation}
\gamma(\tau_L) = (0.3\Gamma + 0.2) \left[ 2.92 + \log \left( \frac{\tau_L}{8 h^{-1} \text{ Mpc}} \right) \right],
\end{equation}
where $\Gamma$ is a shape parameter. Though this fit is strictly only
correct for scale-invariant pure CDM models, it can also be
used as a fitting function for the dispersion of the observed
linear power spectrum on some restricted range of scales,
which for our purposes means within a factor of 1.5 of $8 h^{-1} \text{ Mpc}$. The values used for $\Gamma$ will then be those allowed by
observations, i.e., $\Gamma \in [0.19, 0.27]$ at the 2$\sigma$ confidence level.\footnote{Using the Peacock & Efstathiou (1994) 2$\sigma$ interval, $\Gamma \in [0.22, 0.29]$, does not change the final results.}

Note that, unlike with pure CDM models, the shape of the power spectrum for CHDM models is not redshift independent since the growth of perturbations at a given scale depends on the mean random peculiar velocities of the massive neutrinos at the scale in question which in turn are redshift dependent. However for the scales of interest for clusters, in the CHDM models we consider the redshift evolution of the shape of the power spectrum is very small in the redshift interval where most clusters form in these models, i.e., $z < 0.5$.

Using expression (24) to calculate the derivative in
equation (23), we therefore get
\begin{equation}
\frac{\pi(M_v, z) dM_v}{\sqrt{\frac{2}{\pi}} \rho_v \frac{\delta_c}{M_v} \frac{d\Delta(z)}{dM_v} \exp \left[ -\frac{\delta_c^2}{2\Delta^2(z)} \right] dM_v}.
\end{equation}

As we are considering clusters massive enough so that
at the corresponding scale the density field is not yet well
developed into the nonlinear regime, according with the discussion on Subsection 2.3 we can therefore ignore the influ-
ence of shear on their formation and assume that they col-
'lapse spherically. Nevertheless, to be conservative we shall include an assumed 1$\sigma$ dispersion of $\pm 0.1$ in the value of $\delta_c$, i.e., $\delta_c = 1.7 \pm 0.1$.

Using self-similar evolution arguments (e.g., Hanami 1993), which have been shown to be in good agreement
with hydrodynamical N-body simulations (Navarro, Frenk & White 1995), one obtains the following relation between
the cluster virial mass, $M_v$, its mean X-ray temperature, $k_T T$, and its redshift of virialization, $z_v$,
\begin{equation}
M_v \propto (1 + z_v)^{-3/2} (k_T T)^{3/2}.
\end{equation}

We begin by considering the case of a CDM universe.
In order to normalize equation (27) we use results from the
hydrodynamical N-body simulations for a $\Omega_0 = 1.0$ CDM
model performed by White et al. (1993b). From a cata-
logue of 12 simulated clusters with a wide range of X-ray
temperatures they estimated that a cluster with a present
mean X-ray temperature of 7.5 keV corresponds to a mass
within one Abell radius (1.5 $h^{-1}$ Mpc) of the cluster centre
of $M_v = (1.10 \pm 0.22) \times 10^{15} h^{-1} \text{ M}$. The error arises from
the dispersion in the catalogue and is supposed to represent
the 1$\sigma$ significance level. White et al. (1993b) also found
that the simulated clusters had a density profile in their
outer regions approximately described by $\rho_v(\tau) \propto \tau^{-2.4 \pm 0.1}$. This same result was obtained by Metzler & Evrard (1994)
and Navarro et al. (1995). Bearing in mind that the clus-
ter virial radius in a $\Omega_0 = 1.0$ universe encloses a density
175 times the background density, it is then straightforward
to calculate the cluster virial mass from $M_\Delta$. Through a
Monte Carlo procedure, where we assume the errors in $M_\Delta$
and in the exponent of $\rho_v(\tau)$ to be normally distributed, we
find $M_v = (1.23 \pm 0.32) \times 10^{15} h^{-1} \text{ M}$ for a cluster with
a present mean X-ray temperature of 7.5 keV in a $\Omega_0 = 1.0$
universe. Assuming that such a cluster virialized at a red-
shift of $z_v \simeq 0.05 \pm 0.05$ (e.g., Metzler & Evrard 1994; Navarro
et al. 1995), we can now normalize equation (27)
\begin{equation}
M_v = (3.22 \pm 0.34) \times 10^{15} \times
(1 + z_v)^{-3/2} \left( \frac{k_T T}{7.5 \text{ keV}} \right)^{3/2} h^{-1} \text{ M}.
\end{equation}

This result is in very close agreement with the one obtained
by Evrard (1990) from his own hydrodynamical N-body
simulations. Hence the virial mass $M_v$ for a cluster with
a present mean X-ray temperature of 7 keV is given by
\begin{equation}
M_v = (1.2 \pm 0.3) \times 10^{15} (1 + z_v)^{-3/2} h^{-1} \text{ M}.
\end{equation}

According to Press–Schechter theory, the comoving
number density of clusters per mass interval $dM_v$ about
virial mass $M_v$, which virialize in an interval $dz$ about some
redshift $z$ and survive until the present, is given by (Sasaki
where for the type of models we are considering we have
\[ \sigma_8(z) = \sigma_8(1 + z)^{-1}. \]  

In equation (30) the expression within the square brackets gives the formation rate of clusters with virial mass \( M_v \) at redshift \( z \), whereas the fraction outside gives the probability of these clusters surviving until the present. If one now assumes that at each redshift \( z \) the cluster virial mass \( M_v \) in equation (30) is determined by expression (29) with \( z_c = z \), then equation (30) gives the comoving number density of clusters per unit mass which virialize at each redshift \( z \) and survive up to the present such that they have a mean X-ray temperature of 7 keV at the present. Through the chain rule we can then determine the comoving number density of clusters per unit mass which virialize at each redshift \( z \) and survive up to the present such that they have a mean X-ray temperature of 7 keV at the present

\[ N(M_v, z) dM_v dz = \left[ - \frac{\delta_c^2}{\Delta^2(z)} \frac{n(M_v, z)}{\sigma_8(z)} \right] \sigma_8(z) dM_v dz, \]  

where for the type of models we are considering we have
\[ \sigma_8(z) = \sigma_8(1 + z)^{-1}. \]  

In equation (30) the expression within the square brackets gives the formation rate of clusters with virial mass \( M_v \) at redshift \( z \), whereas the fraction outside gives the probability of these clusters surviving until the present. If one now assumes that at each redshift \( z \) the cluster virial mass \( M_v \) in equation (30) is determined by expression (29) with \( z_c = z \), then equation (30) gives the comoving number density of clusters per unit mass which virialize at each redshift \( z \) and survive up to the present such that they have a mean X-ray temperature of 7 keV at the present. Through the chain rule we can then determine the comoving number density of clusters per unit mass which virialize at each redshift \( z \) and survive up to the present such that they have a mean X-ray temperature of 7 keV at the present

\[ N(k_B T, z) d(k_B T) dz = \frac{dM_v}{d(k_B T)} N(M_v, z) d(k_B T) dz \]  
\[ = \frac{3}{2} \frac{M_v}{k_B T} N(M_v, z) d(k_B T) dz, \]  

where the second equality uses equation (27). We therefore have
\[ N(k_B T, z) d(k_B T) dz = \frac{3}{2} \frac{M_v}{k_B T} \Delta^2(z) (1 + z)^2 d(k_B T) dz. \]

Numerically integrating this expression from \( z = 0 \) to \( z = \infty \) then gives the present comoving number density of clusters per unit mass with a mean X-ray temperature of 7 keV as a function of the present value of \( \sigma_8 \). Comparing with the observational value given by equation (22) we then find to a good approximation that
\[ \sigma_8 = 0.60^{+0.18}_{-0.14}. \]  

The errors in equation (34) represent 95 per cent confidence levels and arise from the dispersions in the observational value of \( T \), in the assumed value for \( \xi_c \), and in expressions (22) and (29). They were estimated via a Monte Carlo procedure, the full details of which are given by Vianna & Liddle (1995).

However this result only applies to models where all the dark matter is cold. We would now like to know how this result is affected if one substitutes part of the cold dark matter by massive neutrinos.

In galaxy clusters the X-ray emission comes mainly from a nearly isothermal core, and thus strongly depends on the depth and width of its gravitational potential. Outside the core the shape of the gravitational potential is of much less importance to the total X-ray emission. It is then possible to have galaxy clusters with the same mean X-ray temperature at virialization but slightly different virial masses. Though for a given cosmological model this dispersion should be quite small, the differences in virial mass between galaxy clusters with the same mean X-ray temperature at virialization in two different cosmological models could be significantly higher.

Whilst the dependence of cluster density profiles on the slope of the power spectrum at the cluster scale has been studied quite thoroughly (Crone, Evrard & Richstone 1994), the consequences of changing the nature of some of the dark matter have not been so extensively studied. In the case of interest to us, where only one neutrino species has a cosmologically significant mass, the typical cluster density profile has been determined only for a pure neutrino model (Cen 1994) and for a model with \( \Omega_\nu = 0.3 \) (Kofman et al. 1995). As the fraction of massive neutrinos is increased at the expense of the same amount of cold dark matter, the depth and width of the gravitational potential at the nearly isothermal core, and therefore the core mass and radius, should remain approximately the same for clusters with equal mean X-ray temperature at virialization. However we now have a component which clusters less, therefore leading to a more extended mass distribution. For a power-law density profile \( \rho \propto r^{-\alpha} \), this corresponds to a smaller \( \alpha \). It can then easily be shown that the cluster virial mass increases. This increase will be greater either if more CDM is substituted by HDM or if the neutrino free-streaming length is increased by making them lighter**. In reality these two effects oppose each other as the neutrino mass increases with \( \Omega_\nu \). In the limit where all the dark matter is composed of massive neutrinos, these are sufficiently massive that at the scales corresponding to high mass clusters, the clustering behaviour of the massive neutrinos seems to closely resemble that of cold dark matter (Cen 1994). If \( \Omega_\nu \) is between 0 and 1, then for high mass galaxy clusters we have very little information about the clustering properties of massive neutrinos on the scales we are interested in, and therefore about the virial masses one should expect in such models.

To our knowledge there is only one hydrodynamical N-body simulation study (Bryan et al. 1994) which has tried to relate \( \sigma_8 \) to the abundance of X-ray clusters for a CHDM model. Though their resolution is insufficient to determine the internal density distribution of the galaxy clusters they obtain, we can use the Press-Schechter approximation to re-normalize their simulation. First we need to calculate the present-day cluster virial mass \( M_v \) which corresponds to a present mean X-ray temperature of 7 keV by using their normalization of the power spectrum, \( \sigma_8 = 0.60 \), and the cluster number densities they obtain for that X-ray temperature. Using expression (33) and assuming \( \xi_c = 1.7 \pm 0.1 \), through a Monte Carlo procedure as before we get \( M_v = (1.30^{+0.36}_{-0.29}) \times 10^{15} h^{-1} \) M at the 1σ confidence level for a mean X-ray temperature of 7 keV, which we have read the cluster number density from Fig. 1 of Bryan et al. (1994) to be \( n(7 \text{ keV}, 0) = (1.6^{+1.6}_{-0.6}) \times 10^{-7} h^3 \text{ Mpc}^{-3} \text{ keV}^{-1} \). We can now use the calculated \( M_v \) to obtain the normalization

** If the neutrino mass is so small that the neutrinos are unable to cluster at the scales we are considering, around \( 2 h^{-1} \) Mpc, the cluster virial mass will not increase as the neutrinos would not be gravitationally bound to the cluster. However, the effect on the relationship between \( \sigma_8 \) and the cluster number density would be exactly the same as if the cluster virial mass had increased in reality, as will be become clear in the Section dealing with damped Lyman alpha systems.
which corresponds to the observed abundance of present-day galaxy clusters with mean X-ray temperature of 7 KeV, which is given by equation (22). Again using $\delta_c = 1.7 \pm 0.1$ and a Monte Carlo procedure, we obtain $\sigma_8 = 0.62^{+0.17}_{-0.14}$. The errors represent 95 per cent confidence limits, and hence both the central value and the size of the uncertainty are very similar to those we got for a pure CDM model, where $\sigma_8 = 0.60^{+0.14}_{-0.18}$ at 95 per cent confidence. It is encouraging that two rather different calculations give such similar answers. We shall use the relative errors obtained for a pure CDM model as they are slightly more conservative, and model the shift in the central value due to a change in $\Omega_c$ by a simple linear fit

$$\sigma_8 = (0.60 + 0.2\Omega_c/3)^{+0.3}_{-0.2} \text{ per cent},$$  \hspace{1cm} (35)

where the uncertainty is 95 per cent confidence. This relation will hold well for the models we are interested in (indeed, it would be satisfactory just to employ the CDM result and ignore the slight shift in central value brought on by the hot component).

### 3.5 Abundance of high redshift objects

To constrain the present-day power spectrum on scales around $1/Mpc$ requires detailed numerical simulations as those scales are well into the nonlinear regime. However, a convenient alternative exists in the abundance of objects at high redshifts, which can sample the spectrum on those scales while they were still in the quasi-linear regime. The constraints on the linear power spectrum can then be evolved to the present day. In this context it is vital to recall that when a hot dark matter component is introduced perturbations on these scales can have their growth affected, typically growing more slowly than in a CDM model which has the effect of making the constraints weaker than naive expectations. It is common to use analytic treatments based on rather nebulously defined neutrino Jeans’ masses to make this correction. Although this is often fine (since the corrections are typically small), we shall instead use direct calculations of the transfer functions at the appropriate redshift.

The most important objects for our purpose are quasars and damped Lyman alpha systems, and we shall place particular emphasis on the latter as they provide stronger constraints. Uncertainties as to the efficiency of quasar formation and the number of quasar generations mean that only a lower bound on the power spectrum can be obtained from them at present. Damped Lyman alpha systems, on the other hand, in principle also offer an upper limit (though to our knowledge one has never been quoted) and indeed the evolution of the amount of gas in such systems as a function of redshift may well imply significant constraints on star formation.

For each object type, it is important to be as conservative as possible in supplying limits; the standard strategy is to obtain a rigid constraint that all models are compelled to satisfy, rather than a number with an error bar which can be subjected to a statistical test.

#### 3.5.1 Quasars

The type of power spectra we are considering flatten towards short scales, so that when one studies short scales, the relative influence of perturbations from larger scales becomes more important. Thus, in accordance with the discussion in subsection 2.3, we should expect shear to become more important, and therefore the relative time efficiency for the formation of bound objects to decrease, as one considers the formation of increasingly smaller objects. This seems to have some support from the few $N$-body simulations with sufficiently high resolution to be able to study the problem (Jain & Bertschinger 1994). Bearing this in mind, one should then expect the virialized dark halos associated with the formation of galaxies to assemble more slowly than those for clusters due to the presence of a relatively stronger shear field††. Even if these suppositions turn out to be correct it is difficult to quantify precisely both the strength of the shear field for a given scale at a certain epoch and its relation with the value one should consider for $\delta_c$. It is due to this limitation that we will use in our analysis the abundance of the most luminous quasars at a redshift of $z = 4$, when the density field at the scale associated with the virialized dark halos in which this type of quasars is embedded, which we will assume to have masses in excess of $10^{12} h^{-1} M_{\odot}$, is still not well developed and consequently shear can be ignored to a good approximation. We can therefore assume that these dark halos collapsed nearly spherically and accordingly use the $\delta_c$ associated with spherical collapse. We will also assume that the time lag between halo virialization and quasar ignition is negligible. Adopting the most conservative result given by Haehnelt (1993) as corresponding to 95 per cent confidence, we have

$$\sigma(M = 10^{12} h^{-1} M_{\odot}, z = 4) \geq 0.26,$$  \hspace{1cm} (36)

for an assumed quasar number density of around $5 \times 10^{-8} h^3 Mpc^{-3}$. The corresponding comoving scale is $R = 0.95 h^{-1} Mpc$. However, this constraint is always weaker than that coming from damped Lyman alpha systems.

#### 3.5.2 Damped Lyman alpha systems

At low and intermediate redshifts, $z \leq 2$, the most popular view is that the vast majority of the damped Lyman alpha lines which appear in the spectra of quasars are produced by neutral hydrogen present in quiescent large-scale disks, similar to those presently found in spiral galaxies. However, these disk systems would have to be 2 to 3 times bigger in size than present spiral galaxies in order to explain the apparent increase in filling factor with redshift if one assumes the comoving number density of these systems to remain constant (Lanzetta, Wolfe & Turnshek 1995). An alternative explanation would be that this increase in filling factor

†† Some studies (Antonuccio-Delogu & Colafrancesco 1994) suggest that the presence of substructure within a collapsing object can increase its timescale of collapse through dynamical friction. Though this effect, similarly to shear, delays collapse, its dependence on the shape of the power spectrum is the opposite, thus effectively diminishing the overall dependence of the timescale of collapse on it. However, this effect seems not to be nearly as important as shear (Monaco 1995).
with redshift is instead due at least partially to an increase in the comoving number density of disk systems with redshift, in particular for $1 < z < 2$. The excess number of systems would then disappear by merging, possibly giving rise to some of the presently observed elliptical galaxies. At higher redshifts, $z \geq 2$, there are some hints that these lines may be produced in objects more akin to turbulent protospheroids, from the apparent short timescales of consumption of the neutral gas by star formation and the discrepancy between the observed low metallicities associated with the lines at those redshifts and the expected higher metallicity of the gas if it is to be the material from which disk stars in present spiral galaxies formed (Lanzetta et al. 1995). These protospheroids are the natural progenitors of galaxies, and the indication would then be that the transition between turbulent collapsing halos and quiescent rotationally supported disks occurred at $z \sim 2$.

Instead of the widely quoted data of Lanzetta et al. (1995), we use the more recent data of Storrie-Lombardi et al. (1995) which revises downwards\(^{\dagger}\) the estimated abundances at a redshift of around 3 and provides a new estimate at redshift 4. The strongest constraint comes from the redshift 4 data, though it is not significantly weakened if the redshift 3 data is used instead. We will present constraints from both.

Following the discussion in the previous subsection, we are interested in the amount of matter associated with damped Lyman alpha systems at redshifts 3 and 4. As we have seen, at these redshifts the systems are probably collapsing protospheroids massive enough to give rise to rotationally supported gaseous disks. The minimum total mass needed for that to happen seems to be around $10^{10} h^{-1}$ M (Haehnelt 1995), which corresponds to a circular velocity of 77 $\mathrm{km \; s^{-1}}$. It is not clear how far these systems have collapsed gravitationally. A reasonable, and for our purposes conservative, hypothesis is that they have just collapsed along the first two collapsing axes, i.e. 'filament' formation, though the baryonic fraction of the collapsing material would have collapsed further through radiative cooling (e.g. Katz et al. 1994). Numerical studies indicate that a value of $\xi$ around 1.5 is associated with the timescale of gravitational collapse along the first two collapsing axes (Monaco 1995), and accordingly we shall use it in the Press-Schechter calculation. This gives a more conservative bound than $\xi = 1.7$.

In Storrie-Lombardi et al. (1995), the fraction of the critical density in the form of neutral gas associated with damped Lyman alpha systems at redshifts 3 and 4 is observed to be

$$\Omega_{\text{gas}}(z = 3) = (0.0017 \pm 0.0003) h^{-1},$$

$$\Omega_{\text{gas}}(z = 4) = (0.0011 \pm 0.0002) h^{-1}.$$  \hspace{1cm} (37)

The total amount of matter which was involved in the formation of damped Lyman alpha systems at these redshifts as a fraction of the critical density is then given by

$$\Omega_{\text{DLAS}}(z) = \frac{\Omega_{\text{gas}}(z)}{f_{\text{gas}}} \Omega_{\text{b}},$$

where $f_{\text{gas}}$ is the neutral fraction of the gas in these systems, which conservatively we will assume to be 1, and $\Omega_{\text{b}} = 0.016 h^{-2}$ is the cosmological baryon density given by standard nucleosynthesis. We now have to be careful in deciding which is the characteristic comoving mass scale associated with these systems. If one takes $10^{10} h^{-1}$ M to be the minimum mass of damped Lyman alpha systems then, because we do not expect massive neutrinos within the mass range we are considering to be able to cluster on this mass scale at $z \geq 2$, the characteristic comoving mass scale involved in the formation of these systems is given by

$$M = 10^{10} (1 - \Omega_{\nu})^{-1} h^{-1} \mathrm{M.}$$

That is, it is originally perturbations on this larger mass scale which begin to collapse, but at some point during the collapse of the perturbations there will be a segregation between the massive neutrinos and the cold dark matter, the former remaining in an oscillatory mode roughly at the scale of segregation (approximately equal to the neutrino Jeans scale) and the latter collapsing further eventually leading to the formation of $10^{10} h^{-1}$ M virialized objects.

All this therefore implies that the fraction $f(M, z)$ of the total mass which is involved in the formation of damped Lyman alpha systems at redshifts 3 and 4 is given by

$$f(M, z = 3) > (0.106 \pm 0.033) h,$$

$$f(M, z = 4) > (0.069 \pm 0.021) h,$$

where $M = 10^{10} (1 - \Omega_{\nu})^{-1} h^{-1} \mathrm{M}$. A 25 per cent uncertainty in the baryon fraction, corresponding loosely to 1$\sigma$, has been added in quadrature to the observational uncertainty. Since we want a lower bound on the density perturbation we take the 2$\sigma$ lower end of the error bar. Using equation (12), we then have to a good approximation the 95 per cent confidence limits

$$\sigma(R, z = 3) > 0.54 \pm 0.2 h;$$

$$\sigma(R, z = 4) > 0.50 \pm 0.2 h,$$

for $0.3 < h < 0.7$, where $R = 0.2 (1 - \Omega_{\nu})^{-1/3} h^{-1} \mathrm{Mpc}$. In fact, the constraint is quite insensitive to the confidence limit chosen. We shall use the redshift 4 point as it provides the stronger constraint; although numerically the constraint is similar, it applies at a higher redshift.

3.6 Compilation

Fig. 2 shows all the data we have discussed, plotted at the present epoch. The data on short scales, which is obtained at moderate redshift, is scaled to the present epoch assuming a pure CDM model (though in later analysis we shall apply the high redshift transfer function). The COBE point is represented schematically as discussed. We have plotted the Peacock & Dodds points assuming a bias parameter of 1.1, which is the best fit for $f_0 = 1$; the errors shown on the individual points correspond to the errors on their relative location, and the uncertainty in bias, $\pm 0.2$ (not illustrated in this Figure), then allows the entire data set to be shifted up or down.

This Figure shows that the data follow a more or less

\[^{\dagger}\text{Note that this still ignores the effect of gravitational lensing, which it is claimed can reduce the estimated abundance by a further 50 per cent (Bartelmann & Loeb 1995).}\]
continuous path, across a range of roughly four orders of magnitude both in linear scale and in the size of the dispersion. However, this large range makes the individual error bars very small, and were one to attempt to plot theoretical predictions on this figure it would be very hard to discern which were the best fit to the data.

In order to overcome this, we can use the knowledge that the standard CDM model, while unable to fit the observational data in detail, is certainly able to fit all of them to within a factor two or so. Consequently, we can greatly improve the graphical representation by plotting the observational data divided by the prediction of the COBE normalized standard CDM model. The choice of this particular model as the fiducial one is governed by history; it does not indicate any preference for this model over any other but rather is simply a graphical convenience. The data normalized to the standard CDM model are shown in Fig. 3.

As anticipated, the data all lie within a factor two or so of this canonical model, with the short scale data falling below the prediction of COBE normalized standard CDM. The Peacock & Dodds data (1994)\(^{56}\) are represented by a band, and the error bars on the end indicate the overall normalization uncertainty.

When we make comparisons of theory and observations, one of the aspects we have to take into account is that the data are available at different redshifts. When one has a hot dark matter component, the rate of growth of perturbations on short scales is slower than in a CDM model, and this must be taken into account. Rather than impose an analytic approximation to the different growth rate, we directly use transfer functions calculated at the appropriate redshift of around \(z = 3.5\).

It is fortunate that the data available at the present day are on scales large enough that the growth rate is the same as in the CDM model for these moderate redshifts, as seen in Fig. 1. This means that one can shift this data back to a redshift 3.5 in a model-independent way\(^{35}\). Consequently, the simplest approach is to consider all the data as given at redshift 3.5. Had we plotted this, it would look exactly as Fig. 2, but with the vertical axis divided by a factor 4.5 in accordance with the CDM growth law \(\sigma(R) \propto 1/(1 + z)\). Fig. 3 remains exactly the same, and when interpreted at this redshift one doesn’t have to worry about perturbation growth suppression corrections in models with an HDM component.

4 CONFRONTATION

Our three primary parameters are \(n, h\) and \(\Omega_c\). Let us first specialize our discussion to varying single parameters of the standard CDM model. Although there is no clear motivation for adopting either \(h = 0.5\) or \(n = 1\) as standard values, this is the most common strategy in the literature.

4.1 Single parameter variations

4.1.1 Scale-invariant CDM models

Since it was recognized that the CDM model could be fixed by lowering the shape parameter, considerable attention has been directed towards achieving this end by lowering the density parameter \(\Omega_0\), usually retaining spatial flatness via the introduction of a cosmological constant. The alternative strategy to achieve this is to lower the Hubble parameter. With a slightly different motivation, such a strategy has been long advocated by Shanks (e.g. 1985). It was mentioned by Liddle & Lyth (1993a) and proposed as a possibility more vigorously by Bartlett et al. (1995). However, neither of those papers took advantage of the effect of the baryon content in these models, which can play a significant role in reducing the shape parameter, as given by equation (10). Consequently, it seems that the proposed \(h \leq 0.3\) may be too strict and we find that one can get away with \(h \leq 0.35\). This is still a long way from the values currently discussed via direct observation (Freedman et al. 1994; Schmidt et al. 1994). However, the preferred baryon density may prove yet higher than the value we have adopted, say at the top or beyond the range given by recent analyses of nucleosynthesis (Copi et al. 1995a, b), which would help alleviate worries about the high baryon abundance in clusters if \(\Omega_0\) does turn out to be one (White et al. 1993b; White & Fabian 1995). Then high baryon density may become an increasingly attractive solution to the problems of standard CDM.

We remind the reader in passing that altering the number of massless species provides a way of mimicking the low \(h\) power spectrum (Dodelson et al. 1994) while retaining a higher true value of \(h\).

4.1.2 Scale-invariant CHDM models

The idea of introducing a hot dark matter component to reduce the short scale power relative to CDM has a long history (Shafi & Stecker 1984; Bonometto & Valdarnini 1984; Fang et al. 1984; Vlaklinini & Bonometto 1985; Holzman 1989; Schaefer et al. 1989; van Dalen & Schaefer 1992) and it quickly received a lot of attention (Schaefer & Shafi 1992; Davis, Summers & Schlegel 1992; Taylor & Rowan-Robinson 1992; Holzman & Primack 1993; Schaefer & Shafi 1993) after the COBE observations. The most widely explored versions of the CHDM model assume both \(n = 1\) and \(h = 0.5\). As far as detailed simulation is concerned, the bulk of attention has gone to the choice \(\Omega_0 = 0.3\) (Davis et al. 1992; Klypin et al. 1994), usually with a normalization of the spectrum rather lower\(^{55}\) than that currently recommended (Bunn et al. 1995). Simulations have investigated a wide range of properties of this model (Davis et al. 1992; Klypin et al. 1994; Jing et al. 1994; Nolthenius, Klypin & Primack 1994; Yepes et al. 1994; Bryan et al. 1994; Klypin, Nolthenius & Primack 1995).

The alternative approach, as adopted in this paper, is

\(^{55}\) We have left out from this Figure the two points corresponding to the largest scales in order to obtain a clearer picture of the observations as these points are very close to the POTENT point.

\(^{35}\) This is only true for CHDM models, and would not hold for open or cosmological constant models.

\(^{55}\) This may be ascribed to a modest gravitational wave contribution (e.g. Primack et al. 1995), though it then becomes hard to justify retaining the scale-invariant density perturbation spectrum.
to investigate the parameter space more widely by concentrating on linear theory, and this has been done in many recent papers (Taylor & Rowan-Robinson 1992; Lidelle & Lyth 1993b; Pogosyan & Starobinsky 1993; Schaefer & Shafi 1994; Pogosyan & Starobinsky 1995a). Our aim here is to examine the widest possible parameter space using the most up-to-date linear theory constraints.

Should the recently claimed detections of the muon neutrino mass be confirmed, then it corresponds to a particular region of our parameter space. Assuming the standard abundance calculation, the possible LSND detection (Caldwell 1994; Primack et al. 1995) corresponds, with considerable uncertainty, to $\Omega_\nu = 0.1(h/0.5)^{-2}$; if $h$ becomes smaller this corresponds to a greater fraction of the total density.

A much advertised drawback of the $\Omega_\nu = 0.3$ CHDM model is the possibility that it may not reproduce the observed abundance of damped Lyman alpha systems (Mo & Miralda-Escudé 1994; Kauffmann & Charlot 1994; Ma & Bertschinger 1994). This led some to favour the reduction of $\Omega_\nu$ to 0.25 or 0.20 (Klypin et al. 1995a). We find that, were we to make the same assumptions as they do concerning the COBE normalization and the damped Lyman alpha system abundance, we more or less reproduce the constraint of Klypin et al. (1995a). Since their calculation is considerably more sophisticated than ours, being simulation based, one could regard this as a calibration of our calculation, though we have not had to do any tuning. Anyway, since their calculation was made, things have generally gone in the direction of weakening the early galaxy formation constraint on the CHDM model; the COBE normalization has gone up, and more recent damped Lyman alpha system abundance observations (Starrie-Lombardi et al. 1995) have produced lower results than those of Lanzetta et al. (1995). Consequently, we find that the constraint from damped Lyman alpha systems on COBE normalized CHDM models has weakened somewhat, back to $\Omega_\nu 0.30$ for the $n = 1$, $h = 0.5$ version.

However, a more important question is what values of $\Omega_\nu$ are preferred when one brings other data into play. Fig. 4 shows that there are already indications from the shape of the galaxy correlation function that the $\Omega_\nu = 0.30$ model is subtracting too much short scale power. A better eyeball fit to the correlation function data from Fig. 4 is $\Omega_\nu = 0.20$. This sort of value has been criticized on alternative grounds, that it may overproduce clusters (Primack et al. 1995), a problem made worse by the higher normalization but slightly improved by our view that the cluster constraint is weaker than usually advertised. Fig. 5 illustrates our allowed parameter region as a function of both $\Omega_\nu$ and $h$. The combination of cluster and damped Lyman alpha system abundance appears sufficient to at least marginally exclude all $h = 0.5$ models, though systematic errors may allow one to evade this conclusion, as discussed in Section 5 below. For values of $\Omega_\nu 0.3$ the cluster constraint alone is what limits the values of $h$ to be less than 0.5. [The model predictions of the cluster amplitude would however be compatible with the constraint for $h = 0.5$ models if $\Omega_\nu$ is comprised of more than one degenerate mass flavour (Primack et al. 1995; Babu et al. 1995).]

Fig. 5 makes it clear that there is also a lot of extra freedom to be gained via fairly modest decreases in $h$, with a wide band of allowed values opening up. If the LSND detection corresponds to a single neutrino, giving $\Omega_\nu \sim 0.1(h/0.5)^{-2}$, it cuts across the allowed region around $h = 0.4$, with considerable uncertainty.

Overall, our analysis suggests that, largely due to the higher COBE normalization, the parameter space of scale-invariant CHDM is not too large. However, we shall see that when one allows $n$ to vary and includes the possibility of gravitational waves, the freedom becomes greater.

### 4.2 Tilted CDM models

Let us now extend the discussion to take in the general class of inflation-based cold dark matter Models. Naturally, one chooses $\Omega_\nu$ to be zero, but the parameters $n$ and $h$ are to be freely varied and gravitational waves added if desired.

In the context of an arbitrary choice for the initial perturbation spectrum, the possibility of choosing a spectral index other than $n = 1$, which has now become known as 'tilt', has often been discussed. The modern context, where the origin of the tilt is identified as inflation and a connection made to the desired slope of the galaxy correlation function, was discussed by Bond (1992), Lidelle, Lyth & Sutherland (1992), Cen et al. (1992), Adams et al. (1993) and Lidelle & Lyth (1993a). After the original COBE result came out (Smoct et al. 1992) the prognosis for such models was not particularly good; the necessary tilt to explain the galaxy correlation function, especially as witnessed by the APM survey (Maddox et al. 1990), left a perceived deficit of short scale power when adjusted to the COBE data. Since then the situation has improved somewhat, due to the higher normalization of the current COBE data (Goiński et al. 1994; Bunn et al. 1995). It appears from Fig. 4 that for $h = 0.5$, even tilting to $n = 0.7$, a commonly discussed number, is not sufficient to get the slope of the galaxy correlation function right, and one has to go even lower. Substantial gravitational waves, as there would be in a power-law inflation model, would make things yet worse, so for values of $h$ near 0.5 the implementation must be in a model such as natural inflation (Adams et al. 1993) which predicts negligible gravitational waves.

Fig. 6 shows contour plots of the constraining observations in the $n$–$h$ plane. The top panel is with no gravitational waves; the lower panel adopts the power-law inflation amplitude of gravitational waves for $n < 1$ and zero otherwise, as discussed in Section 2.3. This Figure confirms the viability of scale-invariant CDM provided $h$ is low enough. More importantly, it shows that there may still be a reasonable amount of parameter space available for CDM models. Provided $n$ is lowered sufficiently, these can work for values of $h$ up to about 0.5, but not for any higher values. The parameter space widens out to low values of $h$, so no useful lower limit on $h$ can be obtained this way. The incorporation of gravitational waves reduces the favoured area.

Although we have fixed the baryon density, in the regime where the spectrum is well described via the shape parameter defined by equation (10) one can account for a variation by defining an effective $h$ value. For the range of $\Omega_B$ allowed by nucleosynthesis the change is always small, and via a small parameter expansion one can write $h_{eff} \approx h(1 - 2\Delta \Omega_B)$, where $\Delta \Omega_B = \Omega_B - \Omega_B^{0.5}$. This allows one to interpret a point in Fig. 5 as representing a range of models with slight simultaneous variation in $\Omega_B$ and $h$ satisfying...
this relation. However, within the nucleosynthesis range only small changes can be made.

White et al. (1995) analysed two particular versions of CDM models. In their favoured models, the desired reduction in short scale power is brought about by accumulating small changes from all sources. They considered a slightly higher baryon density which can be incorporated as just discussed. Without gravitational waves, they favoured $n = 0.8$ and $h = 0.45$ (their higher $\Omega_B$ giving $h_{\text{eff}} = 0.43$), which is at the edge of our favoured parameter range. However, with gravitational waves they preferred $n = 0.9$, still with $h = 0.45$, which our results do not favour; our treatment of the shape of the galaxy correlation function is more stringent than theirs and a smaller value of $h$ would be required.

4.3 Tilted CHDM models

The full parameter space is best investigated by running slices through the volume of $n-h-\Omega_\Lambda$ volume. Following Pogosyan & Starobinsky (1995a), we show cuts of constant $n$ and of constant $h$. We do each of these both for the case without gravitational waves and the case with power-law inflation gravitational waves.

Figs. 7 and 8 show the slicings for the case of no gravitational waves. Fig. 7 includes a reproduction of the scale-invariant CHDM case shown in Fig. 5. As anticipated from all the special cases we have already examined, there is a fairly reasonable parameter space available which explains all the observational data. Unless $h$ is below 0.5, a component of HDM appears to be required. Larger values of $h$, up to around 0.70 in the most extreme cases, are then permitted provided one introduces a strong tilt as well the hot component.

As regards $\Omega_\Lambda$, it seems that the highest it can reach is 0.30 in the rather extreme case of $h = 0.4$ and $n = 1.15$. For $n$, the lowest working value is $n = 0.6$, while concerning high values $n = 1.2$ is possible provided a very low value of $h$ is tolerated. In this regard, our conclusions favour those of Pogosyan & Starobinsky (1995a) rather than Lucchin et al. (1995) in that we see no particular advantage in adopting a blue ($n > 1$) spectrum and find values above 1.1 can only be maintained via a dubiously low Hubble parameter. Borgani et al. (1995) have suggested that the adoption of a blue spectrum helped alleviate worries about damped Lyman alpha system abundance; while in isolation this is certainly true we find that this strategy is not favoured by other data.

Figs. 9 and 10 show equivalent slicings for the case where power-law inflation gravitational waves are included for $n < 1$. For $n \geq 1$ they are the same as Figs. 7 and 8. They show that $n$ has to be at least 0.8 before any sort of fit to the data is available. This is part of a fairly general result that the incorporation of gravitational waves reduces the total available parameter space. Only a few narrow slivers of extra parameter space (e.g., slightly larger $n$ values for $h = 0.5$) are opened up by the inclusion of gravitational waves. This is because of the strength of the shape parameter constraint, which forces a fairly dramatic reduction in short scale power. Models able to fit this are typically not able to lose much more of this power by permitting some of the COBE signal to be soaked up by gravitational waves.

5 DISCUSSION

Despite the recent attention directed at low density models of structure formation, either open or with a cosmological constant, the possibility of working models with the critical density remains an attractive one. We have been able to show that the current observational constraints continue to allow a substantial amount of parameter space for these models.

Our intent here has been to determine what parameter space is allowed, assuming that the unknown systematic errors are negligible. The possibilities for such uncertainties to creep in are many and varied. As an illustration of how this can happen, let us consider the case of scale-invariant CHDM models. From Fig. 7, it is apparent that values of $h > 0.5$ are excluded mainly by the cluster constraint. We have tried to be conservative in estimating the implied amplitude of the cluster constraint, so we believe that our limits on $\sigma_8$ are reliable. However, if the model predictions are off due to systematic errors, the positions of the constraint lines in the Figures will shift to new values. If, for example, the primordial $^4$He abundance is underestimated by 5 per cent, the true $^4$He abundance combined with other BBN constraints would then allow values of the baryon abundance as high as $\Omega_B h^2 \sim 0.025$ (Hata et al. 1995; Copi, Schramm & Turner 1995a) which would reduce the predicted $\sigma_8$ by 5–10 per cent. Similarly, although we have used what we believe to be the cleanest estimate of the COBE amplitude (Görski et al. 1994; Bunn et al. 1995) there are estimates using other techniques (Bennett et al. 1994; Tegmark & Bunn 1994) which give normalizations towards the lower end of our 2$\sigma$ range. Other possible systematic effects, for example mis-estimation of the power spectrum in the procedure of Peacock & Dodds (1994), may also cause trouble (e.g., an erroneous value of the normalization will lead to an altered shape of the power spectrum when turning non-linear into linear data, though we have excluded the four points in their data most likely to suffer from this effect.) It is therefore prudent to point out that the Figures represent our present best guess for the allowed region of parameter space, and that the shapes of the allowed regions will change if any systematic effects are found to be comparable to the statistical errors.

An important sub-class of the models we have discussed is generalized CDM models. Here all the dark matter is assumed to be cold, and one attempts to fit the data by permitting variation of the Hubble parameter and the initial conditions (tilt and gravitational waves) coming from inflation. We illustrated the constraints in Fig. 6; they demonstrate that there is still an area of parameter space available for CDM models. The most plausible situation requires a tilt to $n < 1$ and not too high a gravitational wave amplitude. The biggest drawback these models face is that they require a low value of the Hubble parameter; a strong tilt just permits $h = 0.5$, but that is the highest in any region of parameter space. The bulk of available parameter space, where the tilt is not so strong, requires $h$ some way below this. This sits uncomfortably with recent direct measurements of the Hubble constant (Freeland et al. 1994; Schmidt et al. 1994), though we remind the reader again that power spectra mimicking low values of $h$ can be achieved by introducing extra massless species.

In the case of the scale-invariant CHDM models, we find
that recent observations (especially the rise in the COBE normalization when one moves to the 2 year data) have exerted some pressure on the model and it does not seem viable for $h_0 = 0.5$ whatever the choice of $\Omega_\Lambda$; as in the CDM case, one is forced to lower values of $h$. However, when one introduces the additional freedom of varying $\eta$, a much more substantial parameter region opens up which does allow fits at higher values of $h$. A value $h = 0.6$, consistent with recent direct measurements, seems easy to achieve provided one introduces a strong enough tilt along with the HDM component, and perhaps even $h = 0.7$ is possible if one pushes right to the corner of the parameter space (though the age of the universe in such a model would be problematic).

We have not investigated the introduction of gravitational waves as thoroughly as the other parameters, but we have looked at the case where the amplitude is that given by power-law inflation. What we find is that the gravitational waves are not very helpful; they lead to a reduction in the allowed parameter 'volume' and only in very limited regions do they allow working models for values of $\eta$, $h$ and $\Omega_\Lambda$ that would fail without gravitational waves. However, that said, even with this rather large gravitational wave component there are still significant allowed regions. Large scale structure is therefore not able to exclude the possibility of such gravitational waves.

To conclude, we have presented an extensive comparison of critical density models for structure formation against linear and quasi-linear observational data. We have calculated new transfer functions and provided an improved fitting formula for them which gives the power spectra as a continuous function of all of $\eta$, $h$, $\Omega_\Lambda$, $\Omega_M$ and $z$. We have interpreted the data in terms of the dispersion of the density contrast smoothed on some scale $R$. We have found a substantial allowed parameter space for CDM and CHDM models, which at least in the latter case seems likely to survive for some time to come. Critical density models continue to offer a viable and aesthetically simple basis for understanding structure formation.

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Figure Captions

Figure 1: We plot the dispersion $\sigma(R,z)$ for two different CHDM models, $\Omega_\lambda = 0.2$ and 0.3, at redshifts of zero (solid lines) and 3.5 (dashed lines). The lower curves correspond to $\Omega_\Lambda = 0.3$. The curves have been normalized onto each other at large scales, which is achieved by using the CDM growth law $\sigma(R,z) \propto (1+z)^{-1}$. Except for the shortest scales, the transfer functions are redshift independent indicating that the CDM growth law holds for $R \geq 3h^{-1}$ Mpc.

Figure 2: The observational data we consider, interpreted in terms of $\sigma(R)$. Error bars are 1σ and lower limits are 95 per cent confidence. We represent the COBE data schematically at 4000$h^{-1}$ Mpc as discussed in text; it is indicated by a filled square whose size roughly represents the uncertainty. The Peacock & Dodds data are shown by circles (as discussed, we omit the leftmost four points); there is an uncertainty in overall normalization which has not been illustrated. The bulk flow constraint is a star, and the cluster abundance constraint a cross. The lower limits on the left hand side correspond to damped Lyman alpha systems (leftmost, values for redshifts 3 and 4 overlap) and quasars (right). Although the data clearly show a smooth trend, they cover such a range in $\sigma(R)$ values that one cannot use a Figure of this form to compare models by eye.

Figure 3: As Fig. 2, but plotting $\sigma(R)$ relative to its value in the COBE normalized standard CDM model. This greatly improves clarity. The data points are as in Fig. 2 [COBE, filled square; bulk flows, star; cluster abundance, cross; damped Lyman alpha systems (now shown at two different redshifts) and quasar abundances, lower limits] except that we now show the Peacock & Dodds data as a
band representing the 1σ errors about the (unplotted) central values. The error bars on the end of the band indicate their estimate of the uncertainty in overall normalization of this data set. We see that the data are not well fit by the standard CDM model, which possesses too much short scale power. Although to a reasonable accuracy this Figure applies at any epoch, it is most accurately applied at redshift 3.5 corresponding to the quasar and damped Lyman alpha system abundances, so that one need not worry about the suppressed perturbation growth rate in models with an HDM component.

**Figure 4:** The data plotted as in Fig. 3, with some illustrative theoretical curves overlaid for comparison. These curves are those appropriate to redshift 3.5, as discussed in the text. We have only shown examples where a single parameter of the standard CDM scenario has been modified. The solid line is the standard CDM model; the others modify one parameter from this fiducial model, as indicated in the key. All models are precisely COBE normalized; the COBE point at 4000 h⁻¹ Mpc is illustrative. Remembering that one can shift the entire Peacock & Dodds data set vertically, reasonable eyeball fits to the data are possible via any of the following: lowering h to about 0.3, lowering n to about 0.7 assuming no gravitational waves, introducing a hot dark matter component at about the Ω₀ = 0.2 level.

**Figure 5:** Scale-invariant CHDM models. The lines shown are from the galaxy correlation data (dotted), cluster abundance (dashed) and damped Lyman alpha systems (solid). Shading indicates the favoured area. All constraints are plotted at 95 per cent confidence.

**Figure 6:** Contour plots of constraining observations for CDM models. The upper panel is without gravitational waves, the lower panel includes power-law inflation gravitational waves for n < 1. The lines shown are galaxy correlations (dotted), cluster abundance (dashed), damped Lyman alpha system abundance (solid) and POTENT (dot-dashed). Shading indicates the favoured area and all data are plotted at 95 per cent confidence.

**Figure 7:** Four slices through the Ω₀–h plane at different n, with no gravitational waves included. The values of n are as indicated, and the data are plotted as in Fig. 6.

**Figure 8:** Four slices through the Ω₀–n plane at different h, with no gravitational waves included. The values of h are as indicated, and the data are plotted as in Fig. 6.

**Figure 9:** Four slices through the Ω₀–h plane at different n, with gravitational waves included. The values of n are as indicated, and the data are plotted as in Fig. 6.

**Figure 10:** Four slices through the Ω₀–n plane at different h, with gravitational waves included. The values of h are as indicated, and the data are plotted as in Fig. 6.