Lattice Schwinger Model with Interpolated Gauge Fields

Christof Gattringer*

Max-Planck-Institut für Physik, Werner-Heisenberg-Institut
Föhringer Ring 6, 80805 Munich, Germany

and

Institut für Theoretische Physik der Universität Graz
Universitätsplatz 5, 8010 Graz, Austria

Abstract
We analyze the Schwinger model on an infinite lattice using the continuum definition of the fermion determinant and a linear interpolation of the lattice gauge fields. The possible class of interpolations for the gauge fields, compatible with gauge invariance is discussed. The effective action for the lattice gauge field is computed for the Wilson formulation as well as for non-compact lattice gauge fields. For the non-compact formulation we prove that the model has a critical point with diverging correlation length at zero gauge coupling \( e \). We compute the chiral condensate for \( e > 0 \) and compare the result to the N-flavor continuum Schwinger model. This indicates that there is only one flavor of fermions with the same chiral properties as in the continuum model, already before the continuum limit is performed. We discuss how operators have to be renormalized in the continuum limit to obtain the continuum Schwinger model.

* e-mail: chg@mppmu.mpg.de
1 Introduction

Although the study of lattice gauge theories with interpolated gauge fields has a long history [1]-[4], the interest in this concept has considerably increased [5]-[7] recently. This increased interest is due to 't Hooft's paper [8] where it is argued that interpolating the lattice gauge fields and using a carefully regularized continuum fermion determinant provides an effective lattice gauge theory carrying chiral fermions. Some aspects of this approach have been discussed both in the Schwinger model and the chiral Schwinger model [1, 2, 3, 6]. However, no systematic treatment of the (chiral) lattice Schwinger model with interpolated gauge fields and continuum prescription of the fermion determinant can be found in the literature. This paper tries to close this gap for the standard Schwinger model with vectorlike coupling.

The fact that the Schwinger model provides a testbed for problems involving chiral properties of fermions and species doubling is not the only motivation for this analysis. The model is one of the few instances where good analytic control over a lattice theory with not too poor physical features can be obtained. In particular the approach to the critical point and thus to the continuum model can be analyzed.

The paper is organized as follows. We start with fixing the action for the lattice gauge fields, where we discuss both the Wilson and the non-compact formulation. Once the lattice action and thus the corresponding lattice gauge transformation are fixed, they restrict the class of possible interpolations. In particular a lattice gauge transformation for the lattice gauge fields should give rise to a continuum gauge transformation for the interpolated fields. This condition is sufficient for the effective lattice action to be gauge invariant. From this restriction, we derive a condition for the possible interpolations of U(1)-lattice gauge fields. A linear interpolation which obeys this condition is then used to compute the effective action. Both the Wilson and the non-compact model can be quantized by integrating over the field strength of the lattice fields, but only the latter gives rise to a Gaussian measure and thus is easy to analyze. In what follows then we restrict ourselves to the non-compact formulation. From the exponential falloff of the two point function of the field strength we compute the correlation length for small gauge coupling \( e \). It turns out that the correlation length diverges as \( e \) approaches zero, giving rise to a critical point at \( e = 0 \). In Section 5 we compute the chiral condensate for \( e > 0 \). The result is compared to the chiral condensate of the continuum Schwinger model with N flavors. This indicates that the lattice model involves only one flavor degree of freedom with the same chiral properties as in the continuum model already for finite correlation length. Finally we discuss how operators have to be renormalized when performing the continuum limit \( e \rightarrow 0, et = \text{const} \), where \( t \) is the distance in lattice units. We explicitly compute the wave function renormalization constants for the field strength and the chiral densities.

2 Interpolation and gauge invariance

The lattice under consideration is \( \mathbb{Z}^2 \), i.e. the lattice spacing is set to one. Functions defined in the continuum can be identified by their arguments \( x, y \in \mathbb{R}^2 \), while lattice quantities have arguments \( n, m, k, r, s \in \mathbb{Z}^2 \). We consider two types of actions for the U(1)-gauge fields. The non-compact action

\[
S_{NC} := \frac{1}{4} \sum_{n \in \mathbb{Z}^2} F_{\mu\nu}(n) F_{\mu\nu}(n) = \frac{1}{2} \sum_{n \in \mathbb{Z}^2} F_{12}(n)^2 ,
\]  

(1)
with
\[ F_{\mu\nu}(n) := \left( A_{\nu}(n + \hat{e}_\mu) - A_{\nu}(n) - A_{\mu}(n + \hat{e}_\nu) + A_{\mu}(n) \right). \] (2)

\( A_{\mu}(n) \) may assume values in \((-\infty, +\infty)\) for all \( n \in \mathbb{Z}^2 \) and \( \mu = 1, 2 \). \( \hat{e}_\mu \) is the unit vector in \( \mu \)-direction. We also consider the Wilson formulation of the gauge field action
\[ S_W := \frac{1}{e^2} \sum_{n \in \mathbb{Z}^2} \left[ 1 - \Re \left( U_1(n)U_2(n+\hat{e}_1)U_1(n+\hat{e}_2)U_2(n) \right) \right] = \frac{1}{e^2} \sum_{n \in \mathbb{Z}^2} \left[ 1 - \Re e^{iF_{12}(n)} \right]. \] (3)

In the last step the gauge transporters were expressed as
\[ U_\mu(n) := \exp(i\epsilon A_\mu(n)) \quad \text{with} \quad \epsilon A_\mu(n) \in [0, 2\pi]. \] (4)

Although we use the same symbol, it has to be kept in mind that the \( A_\mu(n) \) are restricted to the principal branch \([0, 2\pi/e]\) in the case of the Wilson action.

Both gauge field actions (1), (3) are invariant under the lattice gauge transformation
\[ A_\mu(n) \longrightarrow A_\mu(n) + \Lambda(n + \hat{e}_\mu) - \Lambda(n), \] (5)

where \( \Lambda(n) \) is some scalar lattice field. The lattice discretization of the derivatives in (1)-(5) could have been chosen in a different way, e.g. symmetric with respect to the sites. This would modify the equation for the lattice gauge transformation and thus the class of possible interpolations.

Having fixed the form of the gauge transformation of the lattice fields we can start to think of an interpolation to the interior of the lattice cells. In order to study the class of possible interpolations, we introduce the notation [6] of an interpolation kernel \( D(x; n) \), and denote the interpolated gauge field as (\( \mu \) not summed)
\[ A_\mu(x) := \sum_{n \in \mathbb{Z}^2} D_\mu(x; n)A_\mu(n) \quad x \in \mathbb{R}^2. \] (6)

We require the interpolation to be equivariant, i.e. to transform the lattice gauge transformation (5) into a continuum gauge transformation
\[ A_\mu(x) \longrightarrow A_\mu(x) + \frac{\partial}{\partial x_\mu}\Lambda(x), \] (7)

for the interpolated vector field \( A_\mu(x) \), which makes use of a continuum scalar field \( \Lambda(x) \) obtained from an interpolation of the lattice scalar \( \Lambda(n) \)
\[ \Lambda(x) := \sum_{n \in \mathbb{Z}^2} D^*(x; n)\Lambda(n) \quad x \in \mathbb{R}^2. \] (8)

The interpolation kernel \( D^* \) for the scalar field is not necessarily the same as the kernel \( D_\mu \) for the vector field. The requirement that a lattice gauge transformation gives rise to a continuum gauge transformation in the ‘interpolated world’ is a sufficient condition for the gauge invariance of the effective lattice model, since a proper definition of the continuum fermion determinant is invariant under continuum gauge transformations. This requirement restricts the class of possible interpolation kernels. Combining (5)-(8) gives rise to
\[ \left( D_\mu(x; n - \hat{e}_\mu) - D_\mu(x; n) \right) \Rightarrow \frac{\partial}{\partial x_\mu}D^*(x; n). \] (9)
This restriction which assumes the form (9) for U(1)-gauge fields also in higher dimensions has to be obeyed by the interpolation kernels $D_\mu, D^*$. A further restriction of possible solutions to (9) is that the interpolation should be local in some sense; e.g. $A_\mu(x) = A_\mu(n)$ for $x = n$. This is not a physical requirement, but only a technical assumption in order to remain close to the spirit of an interpolation. Here we do not discuss the class of all possible interpolation kernels that are solutions to (9), although this would be an interesting challenge by itself. We interpolate the fields as follows (here we denote the interpolation explicitly instead of using the kernels $D_\mu$ and $D^*$ in order to simplify the notation)

\begin{align}
A_1(x) &:= A_1(n) [1 - t_2] + A_1(n + \hat{e}_2) t_2, \quad A_2(x) := A_2(n) [1 - t_1] + A_2(n + \hat{e}_1) t_1, \quad (10) \\
\Lambda(x) &:= (\Lambda(n)[1 - t_1] + \Lambda(n + \hat{e}_1) t_1) [1 - t_2] + (\Lambda(n + \hat{e}_2)[1 - t_1] + \Lambda(n + \hat{e}_1 + \hat{e}_2) t_1) t_2, \quad (11)
\end{align}

for $x = n + t$ and $t_1, t_2 \in (0, 1]$. It is easy to check that the interpolation (10), (11) obeys the requirement of transforming a lattice gauge transformation into a continuous one. The interpolation (10) was already obtained [1, 3] from another motivation, namely the definition of a winding number for lattice gauge fields [9]. It has to be remarked, that the interpolation of the gauge fields $A_\mu$ is not differentiable on the links and in $\mu$-direction even not continuous. In particular the 1-component $A_1$ is discontinuous with finite step height on the links parallel to 2-direction and vice versa. Thus the derivatives of the $A_\mu$ have to be computed in the sense of distributions giving rise to $\delta$-distributions times the step height at the lines of discontinuity. However, things work out simpler if one considers gauge invariant operators only, i.e. functions of $F_{12}(n)$. Since $A_1$ is continuous in 2-direction and vice versa, the field strength $F_{12}$ picks up no contributions involving $\delta$-distributions

\begin{align}
F_{12}(x) &= \left( A_2(n + \hat{e}_1) - A_2(n) - A_1(n + \hat{e}_2) + A_1(n) \right) = F_{12}(n), \quad (12)
\end{align}

for $x = n + t$ and $t_1, t_2 \in (0, 1]$. The interpolated field strength is constant inside the lattice cells, but discontinuous along the links. For later use we quote the continuum Fourier transform $\overline{F_{12}}(p)$ of the interpolated field strength given by (12)

\begin{align}
\overline{F_{12}}(p) := \int_{-\infty}^{\infty} d^2 x \ F_{12}(x) e^{-i p \cdot x} \\
= \sum_{n \in \mathbb{Z}^2} F_{12}(n) e^{-i p \cdot n} \int_0^1 d^2 t \ e^{-i p \cdot t} = -\overline{F_{12}}(p) \ \frac{1 - e^{-i p_1 n}}{p_1} \ \frac{1 - e^{-i p_2 n}}{p_2}, \quad (13)
\end{align}

where we introduced the lattice Fourier transform $\overline{F_{12}}(p)$ as

\begin{align}
\overline{F_{12}}(p) := \sum_{n \in \mathbb{Z}^2} F_{12}(n) e^{-i p \cdot n}. \quad (14)
\end{align}

$\overline{F_{12}}(p)$ is periodic in $p$ with respect to the Brillouin zones.

3 Effective lattice action and quantization

The announced strategy is to quantize the fermions by a formal Berezin path integral in the continuum [10]. Integrating out the continuum fermions gives rise to the fermion determinant
It has to be remarked, that the falloff condition corresponds to zero topological charge. In the continuum, when restricting to gauge invariant observables, the model can be quantized where the periodicity of $\det[\beta - i e A]$ is introduced. The fermion determinant is only defined when an ultraviolet and infrared cutoff (for instance a finite space-time lattice also for the fermions, \cite{11}) is introduced. The determinant can then be normalized to 1 for $\epsilon = 0$, by replacing it with $\det[1 - K(A)]$ where $K(A) = i e A \beta^{-1}$. In two dimensions this determinant can be computed explicitly, using the idea of regularized fermion determinants (see e.g. \cite{12}). If we assume that the interpolated vector potential $A_\mu(x)$, and thus the lattice gauge field $A_\mu(n)$ satisfies (see below) some mild regularity and falloff conditions at infinity to make it square integrable \cite{12}, the answer is

$$\det_{reg}[1 - K(A)] = \exp \left( -\frac{e^2}{2\pi} \int_{-\infty}^{\infty} \frac{d^2p}{(2\pi)^2} \tilde{F}_{12}(-p) \frac{1}{p^2} \tilde{F}_{12}(p) \right).$$

(15)

It has to be remarked, that the falloff condition corresponds to zero topological charge. Expression (15) makes sense only if $\tilde{F}_{12}(p)$ vanishes at zero momentum which in turn requires $0 = \tilde{F}_{12}(0) = \sum_n [A_2(n + \hat{e}_1) - A_2(n) - A_1(n + \hat{e}_2) + A_1(n)]$. The last equation is always fulfilled if the lattice gauge fields $A_\mu(n)$ are absolutely summable over $\mathbb{Z}^2$. This is the announced falloff condition expressed in terms of the lattice gauge fields. It can e.g. be imposed by restricting the $A_\mu(n)$ to a finite rectangle in $\mathbb{Z}^2$. Inserting (13) one obtains for the contribution of the fermion determinant to the effective gauge field action

$$S_F := \frac{e^2}{2\pi} \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \tilde{F}_{12}(-p) \tilde{F}_{12}(p) \sum_{k \in \mathbb{Z}^2} \frac{1}{(p + 2\pi k)^2} \frac{2 - 2 \cos(p_1)}{(p_1 + 2\pi k_1)^2} \frac{2 - 2 \cos(p_2)}{(p_2 + 2\pi k_2)^2},$$

(16)

where the periodicity of $\tilde{F}_{12}(p)$ was used, in order to restrict the integration to the first Brillouin zone.

Fourier transforming the non-compact gauge field action (1) and adding it to (16) gives the effective lattice action

$$S_{EFF} := \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} \tilde{F}_{12}(-p) \tilde{F}_{12}(p) \left[ 1 + \frac{e^2}{2\pi} \sum_{k \in \mathbb{Z}^2} \frac{2 - 2 \cos(p_1)}{(p_1 + 2\pi k_1)^2} \frac{2 - 2 \cos(p_2)}{(p_2 + 2\pi k_2)^2} \right],$$

(17)

The effective action for non-compact gauge fields is a quadratic form in the field strength $F_{12}$. As in the continuum, when restricting to gauge invariant observables, the model can be quantized by defining a path integral for $F_{12}$ (see e.g. \cite{12}) and no extra gauge fixing term has to be taken into account. As it stands, the effective action (17) is restricted to $\tilde{F}_{12}(p)$ which are $L^2$ in the first Brillouin zone and vanish at zero momentum. In order to define a proper Gaussian measure for $F_{12}$, the latter requirement has to be abolished by the introduction of a cutoff. However as in the continuum the resulting measure converges in the weak sense (i.e. moments and characteristic function converge) to a Gaussian measure $d\mu_{C}[F_{12}]$ with lattice covariance

$$\tilde{C}(p) := \left[ 1 + \frac{e^2}{2\pi} \sum_{k \in \mathbb{Z}^2} \frac{2 - 2 \cos(p_1)}{(p_1 + 2\pi k_1)^2} \frac{2 - 2 \cos(p_2)}{(p_2 + 2\pi k_2)^2} \right]^{-1}. $$

(18)

The Gaussian measure $d\mu_{C}[F_{12}]$ with covariance $C$ is the starting point for quantizing the effective lattice gauge theory in the non-compact formulation.

A measure for the model using the Wilson formulation (3) can be defined by integrating over the link variables $U_\mu(n)$. In two dimensions for $U(1)$ gauge fields this can be done by integrating each $F_{12}(n)$ over $[-\pi/e, \pi/e]$. However, the resulting measure is not Gaussian and thus much more involved. In the following we will restrict ourselves to the non-compact formulation.
4 The correlation length for small gauge coupling

In order to compute the correlation length of the effective lattice gauge theory, we evaluate the two point function of the field strength. It will be shown that for small $\epsilon$ the two point function falls off exponentially for large time-like separation, thus defining a correlation length. The two point function of $F_{12}(n)$ is simply given by the inverse Fourier transform of the covariance (18)

$$\left< F_{12}(0,t) F_{12}(0,0) \right> = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{-ip_2t} \tilde{C}(p) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{-ip_2t} \left[ 1 - \frac{e^2/\pi}{\sigma(p_1,p_2)^{-1} + e^2/\pi} \right]$$

$$= \delta_{0,0}\delta_{0,t} - \frac{e^2}{\pi} \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{-ip_2t} \frac{1}{\sigma(p_1,p_2)^{-1} + e^2/\pi}. \quad (19)$$

In the last expression the contact term which is also known from the continuum was split off. We furthermore introduced the abbreviation

$$\sigma(p_1,p_2) := \sum_{k \in \mathbb{Z}^2} \frac{1}{(p + 2\pi k)^2} \frac{2 - 2 \cos(p_1)}{(p_1 + 2\pi k_1)} \frac{2 - 2 \cos(p_2)}{(p_2 + 2\pi k_2)}.$$

In order to extract the exponential falloff of the two point function, the singularities of

$$f(p_1,p_2) := \frac{1}{\sigma(p_1,p_2)^{-1} + e^2/\pi} \quad (21)$$

in the complex $p_2$-plane have to be computed. For small $|p|$, $\sigma(p_1,p_2)^{-1}$ behaves as $p_1^2 + p_2^2$, (as should be for any proper dispersion function). Thus one indeed can expect to find a mass gap. Unfortunately, due to the form of $\sigma(p_1,p_2)$, the explicit computation of the singularities requires the solution of transcendental equations. However for small gauge coupling $\epsilon$, the pole structure can be analyzed perturbatively. $\sigma(p_1,p_2)$ has the following symmetry properties

$$\sigma(-p_1,p_2) = \sigma(p_1,-p_2) = \sigma(p_1,p_2),$$

$$\sigma(p_1 + 2\pi j, p_2) = \sigma(p_1, p_2) = \sigma(p_1, p_2 + 2\pi j) \quad \forall j \in \mathbb{Z},$$

$$\sigma(p_1, p_2) = \sigma(p_1, p_2), \quad (22)$$

where $p_2$ is now a complex variable. As can be seen from (21) those symmetry properties carry over to $f(p_1,p_2)$. Since $(2 - 2 \cos(z))/z^2$ is an analytic function, the only singularities in $\sigma(p_1,p_2)$ come from the $(p + 2\pi k)^{-2}$ terms in (20). This implies that the singularities of $\sigma(p_1,p_2)$ in the complex $p_2$-plane are situated at

$$p_2 := -2\pi j \pm i(p_1 + 2\pi l) \quad j, l \in \mathbb{Z}. \quad (23)$$

$\sigma(p_1,p_2)^{-1}$ for fixed $p_1$ is a meromorphic function in the complex $p_2$-plane with zeroes given also by (23). Thus for small $\epsilon$, the poles of $f(p_1,p_2)$ lie in the vicinity of the $p_2$ given by (23). We expand $\sigma(p_1,p_2)^{-1}$ around the zero $p_2 = ip_1$ by setting

$$p_2 := ip_1 + \epsilon + i\delta \quad \text{with} \quad \epsilon, \delta \in \mathbb{R}, \quad |\epsilon|, |\delta| \ll 1. \quad (24)$$
Due to the symmetry properties (22) of \(\sigma(p_1, p_2)\) it is sufficient to study the expansion around the zero \(p_2 = ip_1\). All the other zeroes can be obtained by applying the symmetry transformations (22). The integrand \(f(p_1, p_2)\) can be rewritten identically to (using (24))

\[
f(p_1, p_2) = \left(\frac{2 - 2 \cos(p_1)}{p_1^2} \frac{2 - 2 \cos(ip_1 + \varepsilon + i\delta)}{(ip_1 + \varepsilon + i\delta)^2} + \left[\varepsilon^2 - \delta^2 - 2p_1 \delta + i2\varepsilon(p_1 + \delta)\right] \sigma'(p_1, ip_1 + \varepsilon + i\delta)\right) \times \left(\varepsilon^2 - \delta^2 - 2p_1 \delta + i2\varepsilon(p_1 + \delta)\right) + \frac{e^2}{\pi} \left(\frac{2 - 2 \cos(p_1)}{p_1^2} \frac{2 - 2 \cos(ip_1 + \varepsilon + i\delta)}{(ip_1 + \varepsilon + i\delta)^2} + \left[\varepsilon^2 - \delta^2 - 2p_1 \delta + i2\varepsilon(p_1 + \delta)\right] \sigma'(p_1, ip_1 + \varepsilon + i\delta)\right) \right)^{-1},
\]

(25)

where \(\sigma'(p_1, p_2)\) is obtained from \(\sigma(p_1, p_2)\) by omitting the \(k = (0, 0)\) term in the sum (20). \(\sigma'(p_1, p_2)\) has no more singularities in the first Brillouin zone. Now we restrict ourselves to small \(\varepsilon\), in particular we assume

\[
\varepsilon \ll 1 \quad \text{and} \quad \varepsilon, \delta \sim O(\varepsilon).
\]

(26)

It has to be remarked, that we consider small but fixed \(\varepsilon\) and extract the exponential falloff for \(t \to \infty\). Later the joint limit \(\varepsilon \to 0\), \(\varepsilon t = \text{const}\) which leads to the continuum Schwinger model will be discussed. The denominator of \(f(p_1, p_2)\) is given by

\[
\varepsilon^2 - \delta^2 - 2p_1 \delta + \frac{e^2}{\pi} \frac{2 - 2 \cos(p_1)}{p_1^2} \frac{2 \cosh(p_1) - 2}{p_1^2} + i2\varepsilon(p_1 + \delta) + O(\varepsilon^3).
\]

(27)

To compute the pole up to order \(O(\varepsilon^2)\) the denominator (27) has to vanish up to this order. This gives two equations for the real and the imaginary part, which are used to compute \(\varepsilon\) and \(\delta\). The equation for the imaginary part has the two solutions \(\varepsilon = 0\) (\(\delta\) arbitrary) and \(\delta = -p_1\) (\(\varepsilon\) arbitrary). When inserting the latter into the equation for the real part, it does not allow for a real \(\varepsilon\) and thus is ruled out. Inserting the other solution \(\varepsilon = 0\) into the equation for the real part gives

\[
\delta^2 + 2p_1 \delta - \frac{e^2}{\pi} \frac{2 - 2 \cos(p_1)}{p_1^2} \frac{2 \cosh(p_1) - 2}{p_1^2} = 0.
\]

(28)

In case that \(p_1 \sim O(\varepsilon)\) this is an equation of \(O(\varepsilon^2)\) only, in case of \(p_1 \sim 1\) it is an equation of \(O(\varepsilon)\) with all corrections up to \(O(\varepsilon^3)\) included. Anyway we will solve (28) as a quadratic equation for \(\delta\) and thus obtain the correct result up to \(O(\varepsilon^2)\). This gives rise to

\[
\delta = -p_1 \pm \beta(p_1),
\]

(29)

where we introduced the abbreviation

\[
\beta(p_1) = \sqrt{p_1^2 + \frac{e^2}{\pi} \frac{2 - 2 \cos(p_1)}{p_1^2} \frac{2 \cosh(p_1) - 2}{p_1^2}}.
\]

(30)

The solutions with the minus sign in front of the square root is related to the plus-solution by the transformations (22). Inserting \(\varepsilon = 0\), and the solutions (29) for \(\delta\) into (24), and applying the transformations (22) to the result gives the positions of all poles up to \(O(\varepsilon^2)\)

\[
p_2 = \left[2\pi j \pm i \left(2\pi l + \beta(p_1)\right)\right] \left(1 + O(\varepsilon)\right) \quad j, l \in \mathbb{Z}.
\]

(31)
Note that $p_1$ is restricted to $[-\pi, \pi]$. The exponential decay of the two point function is determined by the pole with $j, l = 0$ and the plus sign in (31), which is given by $p_2 = i\beta(p)\frac{[1+ O(e)]}{\varepsilon}$. In order to apply the residue theorem to the complex $p_2$-integration we also need the residue at $p_2^0$. A simple computation gives

$$\text{Res}_{p_2 = p_2^0} = e^{itp_2} f(p_1, p_2) = -\frac{ie^{-\beta(p_1)\frac{[1+ O(e)]}{\varepsilon}}}{2\beta(p_1)^2} \frac{2 - 2\cos(p_1)}{p_1^2} \frac{2\cosh(\beta(p_1)) - 2}{\beta(p_1)^2} \left(1 + O(e)\right).$$

Now the $p_2$ integration can be solved easily by applying the residue theorem to a contour integral in the complex $p_2$-plane. The contour is a rectangle in the upper half plane with a piece from $\pi$ to $\pi$ along the real axis, and a piece from $\pi + i\pi R$ to $\pi + i\pi R$ parallel to the real axis, where $R$ is some large, real and positive number. These two pieces are connected by two straight lines parallel to the imaginary axis. Due to the symmetry properties (22) the contributions along those two lines cancel each other. Furthermore the contribution from the piece parallel to the real axis decreases exponentially for large $R$. Thus the integral along the real axis from $-\pi$ to $\pi$ is $i2\pi$ times the sum over the residues of all poles inside the contour. The exponential falloff is dominated by the pole $p_2^0$. The contributions from the other poles decay exponentially stronger and their residues can be summed. One obtains

$$\langle F_{12}(0, t) F_{12}(0, 0) \rangle = \delta_{0,0} \delta_{0,t} - \frac{\varepsilon^2}{\pi} \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} e^{-ip_2 t} f(p_1, p_2)$$

$$= \delta_{0,0} \delta_{0,t} - \frac{\varepsilon^2}{\pi} \int_{-\pi}^{\pi} \frac{dp_1}{\pi} \frac{e^{-\beta(p_1)\frac{[1+ O(e)]}{\varepsilon}}}{\beta(p_1)} \frac{2 - 2\cos(p_1)}{p_1^2} \frac{2\cosh(\beta(p_1)) - 2}{\beta(p_1)^2} \left(1 + O(e) + O(e^{-t\pi})\right).$$

The integrand of the remaining $p_1$-integration has no more singularities. Since $\beta(p_1)$ is bounded from below by $\varepsilon/\sqrt{\pi}$, one concludes

$$\left| \langle F_{12}(0, t) F_{12}(0, 0) \rangle \right| \leq \frac{\varepsilon^2}{\pi} C(e) \exp \left(-\frac{t}{\xi} [1 + O(e)] \right),$$

where the correlation length $\xi$ is given by

$$\xi = \frac{\sqrt{\pi}}{\varepsilon}.$$ 

Thus we have established exponential decay of the two-point function of $F_{12}$. The correlation length diverges at $\varepsilon = 0$ giving rise to a critical point, where the continuum limit can be taken.

Beside the factor $\varepsilon^2$ on the right hand side of (34) a factor $C(e)$, constant in $n$ shows up, which behaves as $\ln(e)$ for small $\varepsilon$. However, in Section 6 we will show that when performing the continuum limit $\varepsilon \rightarrow 0$, $t/\xi = \text{const}$, no such extra logarithmic divergence emerges.

5 Chiral condensate

The chiral condensate is a rather instructive expectation value in the Schwinger model. In particular for the N-flavor continuum Schwinger model one obtains [13, 14]

$$\lim_{|\pi - y| \rightarrow \infty} \left( \prod_{a=1}^{j} \langle \bar{\psi}^{(\alpha)}(x) P_+ \psi^{(\alpha)}(x) \bar{\psi}^{(\alpha)}(y) P_- \psi^{(\alpha)}(y) \rangle \right) = \left\{ \begin{array}{ll} \left(-\frac{\varepsilon^2 N}{16\pi^2} e^{2y}\right)^N & \text{for } j = N \\ 0 & \text{for } j < N \end{array} \right.,$$

(36)
where $a$ is the flavor index, $P_\pm$ denote the projectors on left and right handed chirality, $\gamma$ is Euler's constant and finally $e_c$ denotes the gauge coupling of the continuum model, where we introduced the subscript $c$ for conceptual hygiene. The chiral condensate of the N-flavor model involves chiral densities of all flavors, and thus is sensitive to the number of flavors. One can learn about the doubling problem and the chiral properties of the lattice model by comparing it's condensate to the condensate of the N-flavor continuum model. On the lattice we will evaluate

$$ \langle \bar{\psi}(n) P_+ \psi(n) \bar{\psi}(m) P_- \psi(m) \rangle = - \int d\mu [F_{12}] G_{12}(n, m; A_\mu) G_{21}(m, n; A_\mu). \quad (37) $$

$G$ is the continuum fermion propagator in an external field given by [15]

$$ G(n, m; A_\mu) = \frac{1}{2\pi} \frac{\gamma(n - m)}{(n - m)^2} e^{i[\Phi(n) - \Phi(m)]}, \quad (38) $$

with

$$ \Phi(n) = - \int d^2 x D(n - x) \left( \partial_\mu A_\mu(x) + i \gamma_5 \epsilon_{\mu\nu} \partial_\nu A_\mu(x) \right) =: \theta(n) + i \gamma_5 \chi(n), \quad (39) $$

and $D$ denotes the Green's function of $-\triangle$. In the last step we introduced the longitudinal part $\theta = \triangle^{-1} \partial_\mu A_\mu$ which cancels in gauge invariant expectation values like (37), and the gauge invariant part

$$ \chi(n) := \triangle^{-1} F_{12}(n) = - \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} F_{12}(p) e^{ipn} \sum_{k \in \mathbb{Z}^2} \frac{1}{(p + 2\pi k)^2} \frac{1 - e^{-ip1}}{p_1 + 2\pi k_1} \frac{1 - e^{-ip2}}{p_2 + 2\pi k_2}, \quad (40) $$

where the Fourier transform (13) of the interpolated $F_{12}$ was inserted. The Gaussian integral in (37) can be solved, giving rise to (for simplicity we choose the space-time arguments to be $n = (0, t), m = (0, 0)$)

$$ \langle \bar{\psi}(0, t) P_+ \psi(0, t) \bar{\psi}(0, 0) P_- \psi(0, 0) \rangle = - \frac{1}{(2\pi)^2} \frac{1}{t^2} e^{2E(t,e)}, \quad (41) $$

where we defined

$$ E(t, e) = e^2 \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \left[ 2 - 2 \cos(p_2 t) \right] \frac{\rho(p_1, p_2)}{1 + \frac{\sigma^2}{\sigma(p_1, p_2)}}, \quad (42) $$

$\rho(p_1, p_2)$ is given by

$$ \rho(p_1, p_2) := \sum_{\tau, \epsilon \in \mathbb{Z}^2} \frac{1}{(p + 2\pi \tau)^2} \frac{1}{(p + 2\pi \epsilon)^2} \frac{2 - 2 \cos(p_1)}{(p_1 + 2\pi \tau_1)(p_1 + 2\pi \tau_1)} \frac{2 - 2 \cos(p_2)}{(p_2 + 2\pi \epsilon_2)(p_2 + 2\pi \epsilon_2)}. \quad (43) $$

The condensate is now being formed by $E(t, e)$ which contains a term proportional to $\ln(t)$ which cancels the $1/t^2$ factor in (41), plus a term which approaches a finite constant for $t \rightarrow \infty$. Thus we have to split off the logarithm. The integrand in (42) can be rewritten identically to give

$$ E(t, e) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d^2 p \left[ 1 - \cos(p_2 t) \right] \frac{1}{p_1^2 + p_2^2} - \frac{1}{2\pi} \int_{-\pi}^{\pi} d^2 p \left[ 1 - \cos(p_2 t) \right] I(p), \quad (44) $$

with

$$ I(p) := \frac{1 + \frac{e^2}{\pi} \left[ \sigma'(p_1, p_2) - p^2 \rho'(p_1, p_2) \right]}{\frac{2 - 2 \cos(p_1)}{p_1^2} - \frac{2 - 2 \cos(p_2)}{p_2^2}} + \frac{e^2}{\pi} \left[ 1 + \frac{e^2}{\pi} \sigma'(p_1, p_2) \right]. \quad (45) $$
\( \rho'(p_1, p_2) \) is obtained from \( \rho(p_1, p_2) \) by omitting the term with \( r = s = (0, 0) \) in the sum (43). \( p^2 \rho'(p_1, p_2) \) and \( \sigma'(p_1, p_2) \) have no more poles in the first Brillouin zone. By applying the Riemann-Lebesgue Lemma, the second integral can easily be shown to approach a finite constant for \( t \to \infty \). Thus one is left with the problem of extracting the logarithm from the first integral in (44). This integral can again be treated by applying the residue theorem in the complex \( p_2 \)-plane. When using the same contour as for the correlation length, the contribution along \( x + iR, x \in [-\pi, \pi] \) falls off exponentially with \( R \). The contributions along the lines parallel to the imaginary axis do not cancel each other this time, but give rise to the term \( (R \to \infty) \)

\[
R(t) := 2 \int_{-\pi}^{\pi} dp_1 \int_0^\infty dy \frac{y[1 - (-1)^t e^{-iy}]}{(p_1^2 + \pi^2 - y^2)^2 + 4\pi^2 y^2}, \tag{46}
\]

which approaches a constant for \( t \to \infty \). Thus when applying the residue theorem to the first term in (44) one obtains

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} d^2p [1 - \cos(p_2t)] \frac{1}{p_1^2 + p_2^2} = \int_0^\pi dp_1 \frac{1 - e^{-ip_1}}{p_1} + R(t). \tag{47}
\]

The logarithm from the \( p_1 \) integral can be made explicit by using the exponential integral (see e.g. [16] for the expansion)

\[
E_1(x) = -Ei(-x) = \int_\infty^\infty d\tau \frac{e^{-\tau}}{\tau} = -\gamma - \ln(x) - \sum_{l=1}^{\infty} \frac{(-x)^l}{l \cdot l!}. \tag{48}
\]

One obtains

\[
\lim_{\epsilon \to 0} \int_0^\pi dp_1 \frac{1 - e^{-ip_1}}{p_1} = \lim_{\epsilon \to 0} \left( \ln(\pi) - \ln(\epsilon) - E_1(\epsilon t) + E_1(\pi t) \right) = \ln(t) + \gamma + \ln(\pi) + E_1(\pi t), \tag{49}
\]

where \( \gamma \) denotes the Euler constant. Putting things together, the exponent \( E(t, \epsilon) \) reads

\[
E(t, \epsilon) = \ln(t) + R(t, \epsilon), \tag{50}
\]

where \( R(t, \epsilon) := R(t) + \gamma + \ln(\pi) + E_1(\pi t) + E_2(t, \epsilon) \) collects all terms, which were already shown to approach a finite (\( \epsilon \)-dependent) constant, giving \( \lim_{t \to \infty} R(t, \epsilon) := r(\epsilon) \). We end up with the following result for the condensate at \( \epsilon > 0 \)

\[
\lim_{t \to -\infty} \langle \bar{\psi}(0, t) P_+ \psi(0, t) \bar{\psi}(0, 0) \psi(0, 0) \rangle = -\frac{1}{(2\pi)^2} e^{2r(\epsilon)} \neq 0. \tag{51}
\]

The result (51) is exact for all \( \epsilon > 0 \) and contains no expansion in \( \epsilon \). The limit \( \epsilon \to 0 \) will be discussed in the next section.

The nonvanishing condensate for finite \( \epsilon \) is an important result. If there were doublers in the model under consideration, the chiral condensate would include higher powers of the chiral densities, and the right hand side of (51) would be zero. However since we used the continuum determinant no doublers can be expected and we conclude that the formation of the chiral condensate at \( \epsilon > 0 \) is a strong indication that there is only one flavor of fermions which has the chiral properties of the continuum already for finite correlation length. Also the continuum limit discussed in the next section supports this result.
6 Continuum limit

In Section 4 it was shown (compare Eq. (35)) that the correlation length is given by $\xi = \sqrt{\pi}/c$, where $c$ is the coupling of the lattice model. We now define our length scale $L_0$ to be proportional to the correlation length, i.e $L_0 := \lambda \xi$. A physical distance $|x|$ is measured in units of $L_0$ giving rise to $|x| = t/L_0$. The continuum gauge coupling $c_c$ which has the dimension of a mass is defined as $c_c = cL_0$. Thus we obtain for the ratio $t/\xi$

$$
\frac{t}{\xi} = t \frac{e}{\sqrt{\pi}} = \text{const} = |x| \frac{c_c}{\sqrt{\pi}}.
$$

(52)

Since (34) holds for all $t$ (and small $c$), the prescription (52) reproduces the exponential decay of the corresponding two-point function in the continuum ($K_0$ denotes the modified Bessel function)

$$
\langle F_{12}(x) F_{12}(0) \rangle = \delta^{(d)}(x) - \frac{c_c^2}{\pi} \frac{1}{2} K_0 \left( \frac{c_c}{\sqrt{\pi}} |x| \right) \quad |x| \gg 1, \quad e^{-\frac{c_c}{\sqrt{\pi}} |x|}.
$$

(53)

When sending $c \to \infty$, $ct = \text{const}$, in (34) the correct overall factor depending on $c$ has to be determined and included into a wave function renormalization $Z_{12}(c)$ for the field strength. At the end of Section 4 it was noticed, that the extra factor $C(c) \sim \ln(c)$ on the right hand side of (34) is not there when one considers the joint limit $c \to \infty, ct = \text{const}$. To establish this one has to prove that (compare (19))

$$
I(t, c) := \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{ip_2 t} \frac{1}{\sigma(p_1, p_2)^{-1} + e^2} \xrightarrow{c \to 0, ct = \text{const}} \text{const} < \infty.
$$

(54)

Define

$$
J(t, c) := \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{ip_2 t} \frac{1}{p^2 + e^2}.
$$

(55)

This gives rise to

$$
f(t, c) := I(t, c) - J(t, c) = \int_{-\pi}^{\pi} \frac{d^2p}{(2\pi)^2} e^{ip_2 t} \frac{2 \cos(p_1) \cos(p_2) - 1}{p_1^2 + p_2^2} \frac{1}{[1 + c^2 e^{\sigma(p_1, p_2)}]}(p_1^2 + e^2)^{1/2}.
$$

(56)

The integrand in (56) has no more infrared singularities, and the limit $c \to 0$ can be performed without letting $t \to \infty$ at the same time. Furthermore for all values of $c$ the limit $t \to \infty$ gives zero due to the Riemann-Lebesgue lemma. Thus $f(t, c)$ approaches zero when taking the joint limit in the sense of (52). Finally performing the change of variables $q := p\sqrt{\pi}/c$ and using (52) one obtains

$$
J(t, c) := \int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} e^{iq_2 t} \frac{1}{p^2 + 1} \frac{1}{2\pi} K_0 \left( \frac{c_c}{\sqrt{\pi}} |x| \right) + o(c).
$$

(57)

Comparing (57) and (53) we find that we have obtained much more than simply the correct power of $c$. Indeed we reproduced the functional form of the continuum two point function of the field strength. Defining the wave function renormalization constant as

$$
Z_{12}(c) := \frac{c_c^2}{e^2},
$$

(58)

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one ends up with (use (19), (56) and (57))

$$
\lim_{\epsilon \to 0} \frac{Z_{12}(\epsilon)}{Z_{12}(0, t) F_{12}(0, 0)} = \delta(x) - \frac{\epsilon_c^2}{2 \pi} K_0 \left( \frac{\epsilon_c}{\sqrt{\pi}} |x| \right),
$$

(59)

where even the contact term of the continuum is reproduced since $\delta_{0,t} \delta_{0,0} \epsilon_c^2 / \epsilon^2 \to \delta(x)$.

When performing the continuum limit for the two point function of the chiral densities, we did not get that far, since in (42) the Riemann-Lebesgue type argument cannot be applied (the exponent (42) is not a Fourier transform), which was used to obtain the functional form of the two point function of the field strength. However the necessary wave function renormalization for the chiral densities can be computed explicitly. In particular we will show that the exponent $E(t, \epsilon)$ in (41) remains bounded in the continuum limit. It can be rewritten to $E(t, \epsilon) = E_1(t, \epsilon) + E_2(t, \epsilon)$ where

$$
E_1(t, \epsilon) := \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \frac{2 - 2 \cos(p_\eta t)}{p^2} \frac{e^{2 - 2 \cos(p_1 t) 2 - 2 \cos(p_2 t)}}{1 + \frac{e^2}{\pi^2} \sigma(p_1, p_2)},
$$

(60)

and

$$
E_2(t, \epsilon) := \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \frac{2 - 2 \cos(p_\eta t)}{p^2} \frac{p^2 \rho(p_1, p_2)}{1 + \frac{e^2}{\pi^2} \sigma(p_1, p_2)}.
$$

(61)

Using the fact that $1 \geq (2 - 2 \cos(p_\eta t)) / p_\eta^2 > 1/5$ for $p_\eta \in [-\pi, \pi]$, one can estimate

$$
E_1(t, \epsilon) \leq \int_{-\pi/\epsilon}^{\pi/\epsilon} \frac{d^2 q}{(2\pi)^2} \frac{2 - 2 \cos(q_\eta t)}{q^2} \frac{1}{q^2 + \frac{1}{\pi^2} \sigma(q_1, q_2) q^2} < \infty,
$$

(62)

where the variable transformation $q = p/\epsilon$ was performed in the second step and (52) was inserted in the last step. $E_2(t, \epsilon)$ is even simpler shown to be bounded

$$
E_2(t, \epsilon) \leq e^2 \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \frac{p_\eta^2}{p^2} \frac{p^2 \rho(p_1, p_2)}{1 + \frac{e^2}{\pi^2} \sigma(p_1, p_2)} < \infty.
$$

(63)

Thus the exponent $E(t, \epsilon) = E_1(t, \epsilon) + E_2(t, \epsilon)$ remains bounded and the only factor that causes renormalization comes from the $1/t^2$ term in (41). The wave function renormalization $Z_\chi(\epsilon)$ for the chiral densities can be chosen equal to $Z_{12}(\epsilon)$ and the continuum limit of the two point function of the chiral densities is defined as

$$
\lim_{\epsilon \to 0} \frac{Z_\chi(\epsilon)}{Z_{12}(0, 0) \psi(0, t) \psi(0, 0) P_+ \psi(0, t) \psi(0, 0) P_-} = Z_\chi(\epsilon).
$$

(64)

This completes the discussion of the wave function renormalization. The continuum charge $\epsilon_c$ entering $Z_{12}(\epsilon)$ and $Z_\chi(\epsilon)$ is obtained from the rate at which the lattice coupling $\epsilon$ is driven to zero (compare (52)).
7 Concluding remarks

It has been demonstrated that the Schwinger model with interpolated gauge fields in combination with a continuum fermion determinant gives an interesting effective lattice gauge theory. It was shown that the model has a critical point where the continuum limit can be taken. It reproduces the Schwinger model in the continuum. The continuum model is well understood, and one knows that the model is OS-positive from its bosonization to free fields [17]. However, usually one wants to construct a continuum model from a lattice model at the critical point and no independent approach to the continuum theory is available. In particular one would like to prove OS-positivity already for the lattice model, in order to have it granted for the continuum limit. The standard proof for the Wilson formulation of lattice gauge theories [18] makes use of an explicit representation of the functional integral for the lattice fermions. This possibility is lost when one considers the effective lattice gauge model obtained from the continuum determinant. On the other hand, the effective gauge field action is highly nonlocal, as can already be seen from the Schwinger model. There the contribution $S_F$ from the fermion determinant (compare (15)) can be expressed in configuration space using the Greens function for $-\triangle$ which is given by $-1/4\pi \ln(x^2)$. Thus $S_F$ is a quadratic form $\sum_{n,m} F_{12}(n) F_{12}(m) M(n,m)$ for the lattice field strength $F_{12}(n)$, with a kernel $M(n,m)$ which behaves as $\ln((n-m)^2)$ for large distances. This nonlocal behaviour even rules out the strategy [19]. To prove OS-positivity one maybe has to go back beyond the integration of the fermions. However, this unresolved problem is a missing cornerstone of the approach using interpolated gauge fields and a continuum definition of the fermion determinant.

From computing the chiral condensate in the lattice model and comparing it to the N-flavor continuum Schwinger model it was demonstrated that there is no fermion doubling which is of course not surprising due to the usage of the continuum determinant. Also the chiral properties of the continuum model are reproduced already for finite correlation length.

Finally for the Schwinger model it was possible to entirely control the continuum limit. The necessary wave function renormalization constants for the field strength and the chiral densities were computed. Together with the result for the correlation length this sheds light on the problem of taking the continuum limit in an ab initio lattice calculation [20].

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